



Capacitary moduli for Lévy processes and intersections

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Abstract

We introduce the concept of capacitary modulus for a set $A \subseteq \mathbb{R}^d$, which is a function h that provides simple estimates for the capacity of A with respect to an arbitrary kernel f , estimates which depend only on the L^2 inner product (h, f) . We show that for a large class of Lévy processes, which include the symmetric stable processes and stable subordinators, a capacitary modulus for the range of the process is given by its 1-potential density $u^1(x)$, and a capacitary modulus for the intersection of the ranges of m independent such processes is given by the product of their 1-potential densities. The uniformity of estimates provided by the capacitary modulus allows us to obtain almost-sure asymptotics for the probability that one such process approaches within ε of the intersection of m other independent processes, conditional on these latter processes. Our work generalizes that of Pemantle et al. (1996) on the range of Brownian motion. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

For any decreasing kernel function $f: [0, \infty) \rightarrow [0, \infty]$, define the *capacity* of a Borel set $A \subset \mathbb{R}^d$ with respect to f by

$$\text{Cap}_f(A) = \left[\inf_{\nu \in \mathcal{P}_1(A)} \mathcal{E}_f(\nu) \right]^{-1},$$

where $\mathcal{E}_f(\nu)$, the *energy* of a Borel measure ν on \mathbb{R}^d with respect to f , is given by

$$\mathcal{E}_f(\nu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(|x - y|) d\nu(x) d\nu(y)$$

and $\mathcal{P}_1(A)$ denotes the set of Borel probability measures supported on A . When $f(|x|) = u^\alpha(x)$, the α -potential density of a symmetric Lévy process X , then $\text{Cap}_f(A)$ coincides with the *natural α -capacity* for A with respect to X of probabilistic potential theory.

In the sequel we assume that all kernel functions f considered are (weakly) decreasing and satisfy $\lim_{r \downarrow 0} f(r) = f(0)$ if this limit is finite.

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The following definition is central to this paper.

Definition 1. A function $h(x)$ on \mathbb{R}^d is a *capacitary modulus* for $A \subset \mathbb{R}^d$ if there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \int f(|x|)h(x) \, dx \leq \inf_{v \in \mathcal{P}_1(A)} \mathcal{E}_f(v) \leq C_2 \int f(|x|)h(x) \, dx \quad \text{for all } f,$$

or equivalently

$$\left[C_2 \int f(|x|)h(x) \, dx \right]^{-1} \leq \text{Cap}_f(A) \leq \left[C_1 \int f(|x|)h(x) \, dx \right]^{-1} \quad \text{for all } f.$$

We emphasize that the constants $C_1 \leq C_2$ in the above definition are required to be independent of the kernel function f . Clearly, the capacitary modulus of $A \subset \mathbb{R}^d$ is not unique. It depends only on the behaviour of $h(x)$ near $x = 0$. Two sets $A_1, A_2 \subset \mathbb{R}^d$ with the same capacitary modulus are *capacity equivalent* in the sense of Pemantle et al. (1996).

In a recent paper Pemantle et al. (1996) showed that almost surely $h(x) = |x|^{-(d-2)}$ is a capacitary modulus for the range

$$B[0, 1] = \{x \in \mathbb{R}^d : B_t = x \text{ for some } 0 \leq t \leq 1\}$$

of Brownian motion in \mathbb{R}^d when $d \geq 3$, while almost surely $h(x) = |\log|x||$ is a capacitary modulus for the range $B[0, 1]$ of Brownian motion in \mathbb{R}^2 . We can generalize this to a large class of Lévy processes X_t in \mathbb{R}^d , which includes the symmetric stable processes, stable subordinators and many processes in their domains of attraction. The collection of processes we consider is specified precisely at the end of this section and is referred to as class \mathcal{V} . We use

$$X[0, 1] = \{x \in \mathbb{R}^d : X_t = x \text{ for some } 0 \leq t \leq 1\}$$

to denote the range of X , and $u^1(x)$ to denote the 1-potential density of X .

Theorem 1.1. *Let X_t be a Lévy process of class \mathcal{V} in \mathbb{R}^d . Then almost surely $u^1(x)$ is a capacitary modulus for the range $X[0, 1]$ of X .*

Thus, for almost every path the range will have the same capacitary modulus $u^1(x)$, although the constants C_1, C_2 in the definition of capacitary modulus will depend on the path.

I first became interested in this subject when Yuval Peres asked if I could generalize the work of Pemantle et al. (1996) to intersections. Here is our generalization.

Theorem 1.2. *Let $X_t^{(i)}$, $i = 1, 2, \dots, k$ be k independent Lévy processes of class \mathcal{V} in \mathbb{R}^d which intersect almost surely. Let $u^{(i),1}(x)$ denote the 1-potential density of $X^{(i)}$. Then almost surely $\prod_{i=1}^k u^{(i),1}(x)$ is a capacitary modulus for the intersection $\bigcap_{i=1}^k X^{(i)}[0, 1]$ of $X_t^{(i)}$, $i = 1, 2, \dots, k$.*

Pemantle et al. (1996) also showed that almost surely the zero set $Z = \{0 \leq t \leq 1 : B_t = 0\}$ for linear Brownian motion has the capacitary modulus $x^{-1/2}$. This is an

immediate consequence of Theorem 1.1 once we recall that the zero set Z for linear Brownian motion is ‘essentially’ the range S of a stable subordinator of index $1/2$; more precisely, Z contains $S \cap [0, 1]$ as a dense subset with countable complement, which allows us to show that $\text{Cap}_f(Z) = \text{Cap}_f(S \cap [0, 1])$ for all f .

We also mention the work of Delmas (1998) who determined the capacity modulus of the support and range of super-Brownian motion.

Our main interest in capacity is that for many stochastic processes, particularly Markov processes (see Chung (1973) and Fitzsimmons and Salisbury (1989) and the references therein) and certain fractal percolation processes (see Pemantle and Peres, 1995), hitting probabilities of sets are equivalent to their capacities.

The next theorem exploits this equivalence, as well as the fact that our almost-sure capacity estimates hold uniformly over all kernels. Aizenman (1985) showed that if $[B]$ and $[B']$ are the traces of two independent d -dimensional Brownian motions started apart, then

$$P[\text{dist}([B], [B']) < \varepsilon] \asymp \begin{cases} \varepsilon^{d-4} & \text{if } d > 4, \\ (\log \frac{1}{\varepsilon})^{-1} & \text{if } d = 4, \end{cases}$$

as $\varepsilon \downarrow 0$. (Earlier, Lawler (1982) had obtained precise asymptotics for the analogous problem for two random walks on \mathbb{Z}^d . See Albeverio and Zhou (1994) for a recent refinement of Aizenman’s estimates.) Theorem 2.6 of Peres (1996) contains the following generalization of Aizenman’s result: if $[X^\alpha]$ and $[B]$ denote the traces of an independent α -stable process and Brownian motion, started apart, then

$$P[\text{dist}([B], [X^\alpha]) < \varepsilon] \asymp \begin{cases} \varepsilon^{d-\alpha-2} & \text{if } \alpha < d - 2, \\ (\log \frac{1}{\varepsilon})^{-1} & \text{if } \alpha = d - 2, \end{cases}$$

as $\varepsilon \downarrow 0$.

Pemantle et al. (1996) derived an almost-sure version of these estimates, uniform over α , conditional on the Brownian motion B . Here is our generalization. Let P_x^X be the law of X in \mathbb{R}^d started at x . Write $[X] = X[0, \infty)$ and $\bigcap_{i=1}^k [Y^{(i)}] = \bigcap_{i=1}^k Y^{(i)}[0, 1]$.

Theorem 1.3. *Let X_t be a symmetric Lévy processes of class \mathcal{V} in \mathbb{R}^d with a monotone 1-potential density $u^1(x)$, and let $Y_t^{(i)}$, $i = 1, 2, \dots, k$, be k independent Lévy processes of class \mathcal{V} in \mathbb{R}^d started at $x = 0$ with 1-potential densities $u^{(i),1}(x)$. Set*

$$m(x, Y) = \inf_{y \in \bigcap_{i=1}^k [Y^{(i)}]} |x - y|,$$

$$M(x, Y) = \sup_{y \in \bigcap_{i=1}^k [Y^{(i)}]} |x - y|.$$

Then for some constants $c_d, c'_d > 0$ the following is true: For a.e. $Y^{(1)}, \dots, Y^{(k)}$ and all $x \in \mathbb{R}^d$, there exists $\varepsilon_0 = \varepsilon_0(Y, x)$ such that, for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} c_d u^1(M(x, Y)) &\leq \zeta(\varepsilon) P_x^X \left[\text{dist} \left([X], \bigcap_{i=1}^k [Y^{(i)}] \right) < \varepsilon \mid Y^{(1)}, \dots, Y^{(k)} \right] \\ &\leq c'_d u^1(m(x, Y)) \end{aligned}$$

where $\zeta(\varepsilon) = \varepsilon^d u^1(\varepsilon) \prod_{i=1}^k u^{(i),1}(\varepsilon)$.

Remark. Note the uniformity in X in the statement above. Even for a fixed X , the proof of Theorem 1.3 requires estimating the capacity of the intersection $\bigcap_{l=1}^k [Y^{(l)}]$ for a fixed sample path in infinitely many kernels simultaneously.

As detailed below, Theorems 1.1–1.3 will follow once we have established the next two theorems, which are of some interest in their own right. We formulate things in terms of intersections, but point out that our results also apply to the case of a single process.

Let $X_t^{(l)}$, $l = 1, \dots, m$, denote m independent Lévy processes of class \mathcal{V} in \mathbb{R}^d with 1-potentials $u^{(l),1}(x)$.

The Lévy sausage $S_\varepsilon^{(l)}(t)$ of radius ε for $X^{(l)}$ is defined by

$$S_\varepsilon^{(l)}(t) = \{y \in \mathbb{R}^d \mid \inf_{0 \leq s \leq t} |y - X_s^{(l)}| \leq \varepsilon\}.$$

Let $\lambda^{(1)}, \dots, \lambda^{(m)}$ denote independent mean-1 exponential times. If A is any set in \mathbb{R}^d we use $|A|$ to denote the Lebesgue measure of A .

Theorem 1.4. *We can find a random variable $\mathcal{S} \in L^2$ such that*

$$\lim_{\varepsilon \rightarrow 0} \left(\prod_{l=1}^m u^{(l),1}(\varepsilon) \right) \left| \bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)}) \right| = \mathcal{S} \quad \text{a.s.} \tag{1.1}$$

Remark. With more effort, \mathcal{S} can be identified with the intersection local time $\alpha(\lambda^{(1)}, \dots, \lambda^{(m)})$ which we now define.

We first define the approximate intersection local time

$$\alpha_\delta(B) = \int_B \int \prod_{l=2}^m f_\delta(X_t^{(l)} - X_t^{(1)}) \prod_{l=1}^m dt_l \tag{1.2}$$

where B is any bounded Borel set in \mathbb{R}_+^m , $f_\delta(x) = \delta^{-d} f(x/\delta)$, and $f(x)$ is any fixed continuous symmetric function on \mathbb{R}^d supported in the unit ball. When $B = \times_{l=1}^m [0, T_l]$ we will write $\alpha_\delta(T_1, \dots, T_m)$ for $\alpha_\delta(B)$.

It is not hard to show, see e.g. Rosen (1986), that $\{\alpha_\delta(T_1, \dots, T_m); T_l \leq M\}$, for any $M < \infty$, converges uniformly a.s. and in all L^p spaces as $\delta \rightarrow 0$ to a limit which we denote by $\{\alpha(T_1, \dots, T_m); T_l \leq M\}$. Consequently $\{\alpha(T_1, \dots, T_m); T_1, \dots, T_m\}$ will be continuous and monotone increasing a.s. and the measures $\alpha_\delta(\cdot)$ converge weakly to a limit which we denote by $\alpha(\cdot)$.

Let $\pi : \mathbb{R}_+^m \mapsto \mathbb{R}^d$ be defined by $\pi(s_1, \dots, s_m) = X_{s_1}^{(1)}$, and define the (random) measure μ_{T_1, \dots, T_m} on \mathbb{R}^d by $\mu_{T_1, \dots, T_m} = \pi_*(\alpha|_{\times_{l=1}^m [0, T_l]})$, i.e.

$$\int g(x) d\mu_{T_1, \dots, T_m}(x) = \int_{\times_{l=1}^m [0, T_l]} g(\pi(s_1, \dots, s_m)) d\alpha(s_1, \dots, s_m)$$

for bounded continuous functions g on \mathbb{R}^d . Let

$$\Delta_\varepsilon = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |x - y| \leq \varepsilon\}.$$

Let $\lambda^{(1)}, \dots, \lambda^{(m)}$ denote independent mean-1 exponential times. We shall use the abbreviation μ_λ for $\mu_{\lambda^{(1)}, \dots, \lambda^{(m)}}$.

Theorem 1.5. *We can find a random variable $\mathcal{F} \in L^2$ such that*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^d \prod_{l=1}^m u^{(l),1}(\varepsilon)} (\mu_\lambda \times \mu_\lambda)(\Delta_\varepsilon) = \mathcal{F} \quad a.s. \tag{1.3}$$

Remark. With more effort, \mathcal{F} can be identified with $\alpha(\lambda^{(1)}, \dots, \lambda^{(m)})$.

We now describe the class \mathcal{V} of Lévy processes considered in this paper. This class contains all symmetric Lévy process in \mathbb{R}^d and subordinators with 1-potential density $u^1(x)$ regularly varying at 0 of index $\beta - d < 0$, bounded outside any neighborhood of the origin and satisfying

$$|u^1(x + a) - u^1(x)| \leq C \frac{|a|^\rho}{|x|^{d-\beta+\rho}} \quad \forall |a| \leq |x|/4 \tag{1.4}$$

for some $\rho > 0$. Finally, we also include planar Brownian motion in \mathcal{V} .

It is clear what the regular variation of $u^1(x)$ means for subordinators. For symmetric processes, since $u^1(x)$ depends only on $|x|$, we can write $u^1(x) = g(|x|)$ for some $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$. By abuse of notation we shall also write u^1 for g , and it is this u^1 which is assumed to be regularly varying, and which appears as $u^1(\varepsilon)$ in the statement of our theorems.

We expect that our results can be generalized to a large class of Lévy processes in the domain of attraction of general strictly stable processes in \mathbb{R}^d . We have restricted ourselves to the symmetric and subordinator case to avoid getting bogged down in details. For a similar reason we did not attempt to work with processes in the domain of attraction of planar Brownian motion.

Theorem 1.4 is proven in Section 2, and Theorem 1.5 is proven in Section 3. In the brief Section 4 we show how Theorems 1.1–1.3 follow from Theorems 1.4 and 1.5.

2. Lévy sausages

We begin by recalling some notions from probabilistic potential theory, see Blumenthal and Gettoor (1968). If X is a Lévy process in \mathbb{R}^d of class \mathcal{V} , $\alpha \geq 0$ and $B \subseteq \mathbb{R}^d$ a compact set we can define the *natural α -capacity* of B as

$$C^\alpha(B) = \sup_{\mu \subseteq B} \left\{ \mu(B) : \int u^\alpha(y - x) d\mu(y) \leq 1 \quad \forall x \right\} \tag{2.1}$$

where the supremum runs over all measures $\mu \subseteq B$, i.e. supported in B . When X is a symmetric Lévy process we have that $C^\alpha(B) = \text{Cap}_{u^\alpha}(B)$.

Let now X be a fixed Lévy process of class \mathcal{V} in \mathbb{R}^d with 1-potential density $u^1(x)$ regularly varying at 0 of index $\beta - d < 0$. We will use the abbreviation $c(\varepsilon) = C^1(B(0, \varepsilon))$. We will need the asymptotics of $c(\varepsilon)$.

Lemma 2.1.

$$\lim_{\varepsilon \rightarrow 0} u^1(\varepsilon)c(\varepsilon) = u_\beta^0(1)C_\beta^0(B(0, 1)) \tag{2.2}$$

where $u_\beta^0(x)$ and $C_\beta^0(B(0, 1))$ denote, respectively, the 0-potential density and natural 0-capacity of the unit ball for the strictly stable process of index β in \mathbb{R}^d , symmetric if X is symmetric and a subordinator if X is a subordinator.

Proof. We have

$$\begin{aligned} c(\varepsilon) &= \sup_{\mu \subseteq B(0, \varepsilon)} \left\{ \mu(B(0, \varepsilon)): \int u^1(y - x) d\mu(y) \leq 1 \quad \forall x \right\} \\ &= \sup_{\mu \subseteq B(0, 1)} \left\{ \mu(B(0, 1)): \int u^1(\varepsilon(y - x)) d\mu(y) \leq 1 \quad \forall x \right\}. \end{aligned} \tag{2.3}$$

We claim that for small ε and any $\mu \subseteq B(0, 1)$

$$\int u^1(\varepsilon(y - x)) d\mu(y) \leq 1 \quad \forall |x| \leq 1 \Rightarrow \int u^1(\varepsilon(y - x)) d\mu(y) \leq 1 \quad \forall x. \tag{2.4}$$

For X symmetric (and for all ε) this is just a special case of the maximum principle, see Blumenthal and Gettoor (1970), but we can give a simple proof which covers our subordinators along the lines of the proof given in Blumenthal and Gettoor (1970) for stable subordinators. By our assumptions for class \mathcal{V} we can find $1 < r < \infty$ such that $u^1(x)$ is monotone decreasing in $(0, r]$, and since u^1 is bounded (say by M) outside this interval we can find ε_0 such that $u^1(2\varepsilon_0) > M$. It is then easy to check that for any $|y| \leq 1, |x| > 1$ and $\varepsilon \leq \varepsilon_0$ we have $u^1(\varepsilon(y - x/|x|)) > u^1(\varepsilon(y - x))$, and noting that we can assume that we are dealing with non-atomic μ completes the proof of our claim.

Hence by (2.4) for small enough ε

$$u^1(\varepsilon)c(\varepsilon) = \sup_{\mu \subseteq B(0, 1)} \left\{ \mu(B(0, 1)): \int \frac{u^1(\varepsilon(y - x))}{u^1(\varepsilon)} d\mu(y) \leq 1 \quad \forall |x| \leq 1 \right\}. \tag{2.5}$$

We recall the Potter bounds (Bingham et al., 1987). For any $\delta > 0$ we can find $\varepsilon_\delta > 0$ so that

$$(1 - \delta) \frac{1}{|x|^{d-\beta-\delta}} \leq \frac{u^1(\varepsilon x)}{u^1(\varepsilon)} \leq (1 + \delta) \frac{1}{|x|^{d-\beta+\delta}} \quad \forall \varepsilon \leq \varepsilon_\delta \tag{2.6}$$

and all $|x| \leq 2$ in the symmetric case, and $0 \leq x \leq 2$ for subordinators. We can combine this as

$$(1 - \delta) \frac{u_{\beta+\delta}^0(x)}{u_{\beta+\delta}^0(1)} \leq \frac{u^1(\varepsilon x)}{u^1(\varepsilon)} \leq (1 + \delta) \frac{u_{\beta-\delta}^0(x)}{u_{\beta-\delta}^0(1)} \quad \forall \varepsilon \leq \varepsilon_\delta, |x| \leq 2. \tag{2.7}$$

From (2.5) we then have

$$\begin{aligned} &u^1(\varepsilon)c(\varepsilon) \\ &\geq \sup_{\mu \subseteq B(0, 1)} \left\{ \mu(B(0, 1)): \int (1 + \delta) \frac{u_{\beta-\delta}^0(y - x)}{u_{\beta-\delta}^0(1)} d\mu(y) \leq 1 \quad \forall |x| \leq 1 \right\} \\ &= \frac{u_{\beta-\delta}^0(1)}{(1 + \delta)} \sup_{\mu \subseteq B(0, 1)} \left\{ \mu(B(0, 1)): \int u_{\beta-\delta}^0(y - x) d\mu(y) \leq 1 \quad \forall |x| \leq 1 \right\} \\ &= \frac{u_{\beta-\delta}^0(1)}{(1 + \delta)} C_{\beta-\delta}^0(B(0, 1)). \end{aligned}$$

$C_\beta^0(B(0, 1))$ is a known constant which depends on whether we are in the symmetric or subordinator case, but in either event both $C_\beta^0(B(0, 1))$ and $u_\beta^0(1)$ are continuous in β , (Bliedtner and Hanson, 1986; Hawkes, 1970).

We thus see that

$$\liminf_{\varepsilon \rightarrow 0} u^1(\varepsilon)c(\varepsilon) \geq u_\beta^0(1)C_\beta^0(B(0, 1)).$$

Similarly, using the other half of (2.7) we see that

$$\limsup_{\varepsilon \rightarrow 0} u^1(\varepsilon)c(\varepsilon) \leq u_\beta^0(1)C_\beta^0(B(0, 1))$$

which completes the proof of Lemma 2.1. \square

Proof of Theorem 1.4. Let now $X_t^{(l)}, l=1, \dots, m$, denote m independent Lévy processes of class \mathcal{V} in \mathbb{R}^d with 1-potentials $u^{(l),1}(x)$ which are regularly varying at $x=0$ of index $d - \beta_l, l=1, \dots, m$. We assume for now that all $\beta'_l \equiv d - \beta_l > 0$ and that $\zeta \equiv \sum_{l=1}^m \beta'_l = \sum_{l=1}^m (d - \beta_l) < d$. At the end of this section we shall describe the simple modifications necessary for Brownian motion in \mathbb{R}^2 .

We will use the abbreviation $c^{(l)}(\varepsilon) = \text{Cap}_{u^{(l),1}}(B(0, \varepsilon))$. We intend to show that for all $0 < \varepsilon' \leq \varepsilon \leq 1$

$$\left\| \frac{|\bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)})|}{\prod_{l=1}^m c^{(l)}(\varepsilon)} - \frac{|\bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)})|}{\prod_{l=1}^m c^{(l)}(\varepsilon')} \right\|_2 \leq \varepsilon^\rho \tag{2.8}$$

for some $\rho > 0$ and our theorem will follow easily from this and Lemma 2.1. It suffices to prove (2.8) for all $0 < \varepsilon/2 < \varepsilon' \leq \varepsilon \leq 1$, since the general case can be obtained from this by using a telescoping sum. The proof of (2.8) will be accomplished in a series of lemmas. Before plunging into the details we present a short outline of our strategy.

Eq. (2.8) involves the expectation of the square of a difference. By expanding this square as a sum of four terms, it suffices to show that for all $0 < \varepsilon/2 < \varepsilon' \leq \varepsilon \leq 1$

$$E \left(\frac{|\bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)})|}{\prod_{l=1}^m c^{(l)}(\varepsilon)} - \frac{|\bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)})|}{\prod_{l=1}^m c^{(l)}(\varepsilon')} \right)^2 = A + O(\varepsilon^\rho) \tag{2.9}$$

for some $\rho > 0$ and the same constant A . (This constant is identified in Lemma 2.4.)

Let

$$T_{y,\varepsilon}^{(l)} = \inf\{s > 0 \mid X_s^{(l)} \in B(y, \varepsilon)\}.$$

Note that

$$\left| \bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)}) \right| = \int \prod_{l=1}^m 1_{\{T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\}} dx \tag{2.10}$$

so that

$$\begin{aligned} E \left(\left| \bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)}) \right| \left| \bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)}) \right| \right) &= E \left(\int \prod_{l=1}^m 1_{\{T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\}} dx \int \prod_{l=1}^m 1_{\{T_{y,\varepsilon'}^{(l)} < \lambda^{(l)}\}} dy \right) \\ &= \int \int \prod_{l=1}^m P(T_{x,\varepsilon}^{(l)} < \lambda^{(l)}; T_{y,\varepsilon'}^{(l)} < \lambda^{(l)}) dx dy. \end{aligned} \tag{2.11}$$

Our first lemma shows that up to the error term allowed in (2.9) we can restrict the last integration to a particularly convenient subset.

Lemma 2.2.

$$\begin{aligned}
 & E \left(\left| \bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)}) \right| \left| \bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)}) \right| \right) \\
 &= \int_{\Theta_\varepsilon} \int \prod_{l=1}^m P(T_{x,\varepsilon}^{(l)} < \lambda^{(l)}; T_{y,\varepsilon'}^{(l)} < \lambda^{(l)}) dx dy + O \left(\varepsilon^d \prod_{l=1}^m c^{(l)}(\varepsilon) \right) \tag{2.12}
 \end{aligned}$$

where

$$\Theta_\varepsilon = \{(x, y) \mid |x| \geq 4\varepsilon, |y| \geq 4\varepsilon, |x - y| \geq 4\varepsilon\}.$$

This lemma will be proven shortly. Lemma 2.3 will then show that up to the error term allowed in (2.9) we can replace each $P(T_{x,\varepsilon}^{(l)} < \lambda^{(l)}; T_{y,\varepsilon'}^{(l)} < \lambda^{(l)})$ appearing in (2.12) by the sum of $P(T_{x,\varepsilon}^{(l)} + T_{y,\varepsilon'}^{(l)} \circ \theta_{T_{x,\varepsilon}^{(l)}} < \lambda^{(l)})$ and a corresponding term in which the x, ε and y, ε' are interchanged. These are then evaluated using the strong Markov property. This is used in the proof of Lemma 2.4 to complete the proof of (2.9) and hence of Theorem 1.4. We now present the details.

Proof of Lemma 2.2. Recall (Blumenthal and Gettoor, 1970), that for a Lévy process X in \mathbb{R}^n of the type considered here with 1-potential density $u^1(x)$ and for any compact set $B \subseteq \mathbb{R}^n$ there exists a unique measure ν_B supported in B , the 1-capacitary measure for B with respect to X , which satisfies

$$E^x(e^{-T_B}) = \int u^1(y - x) d\nu_B(y) \quad \forall x \tag{2.13}$$

where T_B is the first hitting time of B , and $\nu_B(B) = \text{Cap}^1(B)$.

Let $B(y, \varepsilon)$ denote the ball centred at y of radius ε . The 1-capacitary measure $\nu_{y,\varepsilon}^{(l)}$ for $B(y, \varepsilon)$ with respect to $X^{(l)}$ is supported in $B(y, \varepsilon)$ and density $\nu_{y,\varepsilon}^{(l)}(dx) = \nu_{0,\varepsilon}^{(l)}(dx + y)$. From (2.13) we have

$$\begin{aligned}
 \int u^{(l),1}(x, z) \nu_{y,\varepsilon}^{(l)}(z) dz &= E^x(e^{-T_{y,\varepsilon}^{(l)}}) \\
 &= E^x(T_{y,\varepsilon}^{(l)} < \lambda^{(l)}) \begin{cases} \leq 1 \\ = 1 \text{ if } x \in B(y, \varepsilon) \end{cases} \tag{2.14}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int \left(\int u^{(l),1}(x, z) \nu_{y,\varepsilon}^{(l)}(dz) \right) dy \\
 &= \int \left(\int u^{(l),1}(x, z + y) \nu_{0,\varepsilon}^{(l)}(dz) \right) dy = \int \nu_{0,\varepsilon}^{(l)}(dz) = c^{(l)}(\varepsilon). \tag{2.15}
 \end{aligned}$$

Note from Lemma 2.1 that $\prod_{l=1}^m c^{(l)}(\varepsilon) \sim \varepsilon^{\check{s}} s(\varepsilon)$ as $\varepsilon \rightarrow 0$ for some slowly varying $s(\varepsilon)$.

We now establish (2.12). If, for example, $|x - y| \leq 4\epsilon$, then using (2.14) and (2.15) we have the bound

$$\begin{aligned}
 & \int \int_{|x-y| \leq 4\epsilon} \prod_{l=1}^m P(T_{x,\epsilon}^{(l)} < \lambda^{(l)}; T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) \, dx \, dy \\
 & \leq \int \prod_{l=1}^m P(T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) \left(\int_{|x-y| \leq 4\epsilon} dx \right) dy \\
 & \leq C\epsilon^d \int \prod_{l=1}^m P(T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) \, dy \\
 & \leq C\epsilon^d \int \prod_{l=1}^m \left(\int u^{(l),1}(z_l) v_{y,\epsilon'}^{(l)}(dz_l) \right) dy \\
 & \leq C\epsilon^d \int \left(\int \prod_{l=1}^m u^{(l),1}(y + z_l) \, dy \right) \prod_{l=1}^m v_{0,\epsilon'}^{(l)}(dz_l) \\
 & \leq C\epsilon^d \int \prod_{l=1}^m v_{0,\epsilon'}^{(l)}(dz_l) \leq C\epsilon^d \prod_{l=1}^m c^{(l)}(\epsilon')
 \end{aligned} \tag{2.16}$$

where we have used the multiple Holder inequality

$$\int \prod_{l=1}^m u^{(l),1}(y + z_l) \, dy \leq \prod_{l=1}^m \|u^{(l),1}\|_{\zeta/\beta'} < \infty. \tag{2.17}$$

The other cases are bounded similarly, completing the proof of Lemma 2.2. \square

Let us define the first-order hitting operator of $X^{(l)}$ for $B(x, \epsilon)$

$$H_{x,\epsilon}^{(l)} f(z) = E^z \left(e^{-T_{x,\epsilon}^{(l)}} f \left(X_{T_{x,\epsilon}^{(l)}}^{(l)} \right) \right)$$

and note that

$$H_{x,\epsilon}^{(l)} 1(z) = E^z (e^{-T_{x,\epsilon}^{(l)}}) = \int u^{(l),1}(z, y) v_{x,\epsilon}^{(l)}(dy). \tag{2.18}$$

Lemma 2.3. *For some $\rho > 0$*

$$\begin{aligned}
 & E \left(\left| \bigcap_{l=1}^m S_\epsilon^{(l)}(\lambda^{(l)}) \right| \left| \bigcap_{l=1}^m S_{\epsilon'}^{(l)}(\lambda^{(l)}) \right| \right) \\
 & = \int_{\Theta_\epsilon} \int \prod_{l=1}^m (H_{x,\epsilon}^{(l)} H_{y,\epsilon'}^{(l)} 1(0) + H_{y,\epsilon'}^{(l)} H_{x,\epsilon}^{(l)} 1(0)) \, dx \, dy + O(\epsilon^{2\zeta+\rho}).
 \end{aligned} \tag{2.19}$$

Proof. Note that for $(x, y) \in \Theta_\epsilon$

$$P(T_{x,\epsilon}^{(l)} < \lambda^{(l)}; T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) = P(T_{x,\epsilon}^{(l)} < T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) + P(T_{y,\epsilon'}^{(l)} < T_{x,\epsilon}^{(l)} < \lambda^{(l)})$$

and

$$\begin{aligned}
 & P(T_{x,\epsilon}^{(l)} < T_{y,\epsilon'}^{(l)} < \lambda^{(l)}) \\
 & = P(T_{x,\epsilon}^{(l)} + T_{y,\epsilon'}^{(l)} \circ \theta_{T_{x,\epsilon}^{(l)}} < \lambda^{(l)}) - P(T_{y,\epsilon'}^{(l)} < T_{x,\epsilon}^{(l)} < T_{x,\epsilon}^{(l)} + T_{y,\epsilon'}^{(l)} \circ \theta_{T_{x,\epsilon}^{(l)}} < \lambda^{(l)})
 \end{aligned}$$

$$\begin{aligned}
 &= P\left(e^{-T_{x,\varepsilon}^{(l)} + T_{y,\varepsilon'}^{(l)} \circ \theta_{T_{x,\varepsilon}^{(l)}}}\right) - P\left(T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)}; e^{-T_{x,\varepsilon}^{(l)} + T_{y,\varepsilon'}^{(l)} \circ \theta_{T_{x,\varepsilon}^{(l)}}}\right) \\
 &= E\left(e^{-T_{x,\varepsilon}^{(l)}} E^{X_{T_{x,\varepsilon}^{(l)}}^{(l)}}\left(e^{-T_{y,\varepsilon'}^{(l)}}\right)\right) - E\left(T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)}; e^{-T_{x,\varepsilon}^{(l)}} E^{X_{T_{x,\varepsilon}^{(l)}}^{(l)}}\left(e^{-T_{y,\varepsilon'}^{(l)}}\right)\right) \\
 &= H_{x,\varepsilon}^{(l)} H_{y,\varepsilon'}^{(l)} 1(0) - E\left(H_{y,\varepsilon'}^{(l)} 1\left(X_{T_{x,\varepsilon}^{(l)}}^{(l)}\right); T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\right). \tag{2.20}
 \end{aligned}$$

Using (2.18) and then (2.20) together with $|x - y| \geq 4\varepsilon$ and $|x| \geq 4\varepsilon$ we see that

$$\begin{aligned}
 H_{x,\varepsilon}^{(l)} H_{y,\varepsilon'}^{(l)} 1(0) &= E\left(e^{-T_{x,\varepsilon}^{(l)}} E^{X_{T_{x,\varepsilon}^{(l)}}^{(l)}}\left(e^{-T_{y,\varepsilon'}^{(l)}}\right)\right) \\
 &= E\left(e^{-T_{x,\varepsilon}^{(l)}} \int u^{(l),1}\left(\left(X_{T_{x,\varepsilon}^{(l)}}^{(l)}\right), z\right) v_{y,\varepsilon'}^{(l)}(dz)\right) \\
 &\leq C c^{(l)}(\varepsilon') u^{(l),1}(x, y) P\left(e^{-T_{x,\varepsilon}^{(l)}}\right) \\
 &\leq C c^{(l)}(\varepsilon') u^{(l),1}(x, y) c^{(l)}(\varepsilon) u^{(l),1}(x) \\
 &\leq C (c^{(l)}(\varepsilon))^2 u^{(l),1}(x, y) u^{(l),1}(x) \tag{2.21}
 \end{aligned}$$

so that by (2.20)

$$P\left(T_{x,\varepsilon}^{(l)} < T_{y,\varepsilon'}^{(l)} < \lambda^{(l)}\right) \leq C (c^{(l)}(\varepsilon))^2 u^{(l),1}(x, y) u^{(l),1}(x). \tag{2.22}$$

Similarly, and using (2.22)

$$\begin{aligned}
 &E\left(H_{y,\varepsilon'}^{(l)} 1\left(X_{T_{x,\varepsilon}^{(l)}}^{(l)}\right); T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\right) \\
 &= E\left(\int u^{(l),1}\left(\left(X_{T_{x,\varepsilon}^{(l)}}^{(l)}\right), z\right) v_{y,\varepsilon'}^{(l)}(dz); T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\right) \\
 &\leq C c^{(l)}(\varepsilon') u^{(l),1}(x, y) P\left(T_{x,\varepsilon}^{(l)} < T_{y,\varepsilon'}^{(l)} < \lambda^{(l)}\right) \\
 &\leq C (c^{(l)}(\varepsilon))^3 u^{(l),1}(x, y)^2 u^{(l),1}(x) \tag{2.23}
 \end{aligned}$$

so that by interpolation

$$E\left(H_{y,\varepsilon'}^{(l)} 1\left(X_{T_{x,\varepsilon}^{(l)}}^{(l)}\right); T_{y,\varepsilon'}^{(l)} < T_{x,\varepsilon}^{(l)} < \lambda^{(l)}\right) \leq C (c^{(l)}(\varepsilon))^{2+\rho} (u^{(l),1}(x, y))^{1+\rho} u^{(l),1}(x) \tag{2.24}$$

which can then be used to establish our lemma.

Lemma 2.4.

$$\begin{aligned}
 &E\left(\left|\bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)})\right| \left|\bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)})\right|\right) = \left(\prod_{l=1}^m c^{(l)}(\varepsilon') c^{(l)}(\varepsilon)\right) \\
 &\quad \times \int_{\Theta_\varepsilon} \int \prod_{l=1}^m (u^{(l),1}(x) u^{(l),1}(x, y) + u^{(l),1}(y) u^{(l),1}(y, x)) dx dy + O(\varepsilon^{2\zeta+\rho}). \tag{2.25}
 \end{aligned}$$

Proof. This follows easily from Lemma 2.3, the first three lines of (2.21) and (1.4). □

Proof of Theorem 1.4. Eq. (2.8) then follows easily from Lemma 2.4 which completes the proof of our Theorem 1.4. □

We now describe the simple modifications necessary to handle Brownian motion in \mathbb{R}^2 so that $c(\varepsilon) \sim \pi(\log(1/\varepsilon))^{-1}$ as $\varepsilon \rightarrow 0$. For simplicity of exposition we assume that our m independent processes are all Brownian motions in \mathbb{R}^2 . With this in mind, in place of (2.8) we show that for all $0 < \varepsilon' \leq \varepsilon \leq 1$

$$\left\| \frac{|\bigcap_{l=1}^m S_\varepsilon^{(l)}(\lambda^{(l)})|}{(c(\varepsilon))^m} - \frac{|\bigcap_{l=1}^m S_{\varepsilon'}^{(l)}(\lambda^{(l)})|}{(c(\varepsilon'))^m} \right\|_2 \leq C(\log(1/\varepsilon))^{-1} \tag{2.26}$$

and our theorem will follow easily from this.

We now note that it suffices to prove (2.26) for all $0 < \varepsilon^2 < \varepsilon' \leq \varepsilon \leq 1$, since the general case can be obtained from this by using a telescoping sum. The proof of (2.6) for all $0 < \varepsilon^2 < \varepsilon' \leq \varepsilon \leq 1$ then follows as before.

3. Intersections of Lévy processes

Proof of Theorem 1.5. Let $X_t^{(l)}, l = 1, \dots, m$, denote m independent Lévy processes of class \mathcal{V} in \mathbb{R}^d with 1-potentials $u^{(l),1}(x)$ which are regularly varying at $x = 0$ of index $d - \beta_l, l = 1, \dots, m$. We assume for now that all $\beta_l'' \equiv d - \beta_l > 0$ and that $\zeta \equiv \sum_{l=1}^m \beta_l'' = \sum_{l=1}^m (d - \beta_l) < d$. Using the ideas described at the end of the last section, it will be easy to modify the arguments of this section to cover Brownian motion in \mathbb{R}^2 .

Set $V_\varepsilon = (\mu_\lambda \times \mu_\lambda)(\Delta_\varepsilon)$ and

$$\kappa(\varepsilon) = \int_{|z| \leq 1} \prod_{l=1}^m u^{(l),1}(\varepsilon z) \, dz. \tag{3.1}$$

Using the Potter bound (2.6) and the dominated convergence theorem we can easily see that $\kappa(\varepsilon) \sim (\int_{|z| \leq 1} |z|^{-\zeta} \, dz) \prod_{l=1}^m u^{(l),1}(\varepsilon) \sim \varepsilon^{-\zeta} s(\varepsilon)$ as $\varepsilon \rightarrow 0$ where $s(\varepsilon)$ is some slowly varying function. We intend to show that for all $0 < \varepsilon' \leq \varepsilon \leq 1$

$$\left\| \frac{1}{\varepsilon^d \kappa(\varepsilon)} V_\varepsilon - \frac{1}{(\varepsilon')^d \kappa(\varepsilon')} V_{\varepsilon'} \right\|_2 \leq \varepsilon^\rho \tag{3.2}$$

for some $\rho > 0$ and our theorem will follow easily from this. It suffices to prove (3.2) for all $0 < \varepsilon/2 < \varepsilon' \leq \varepsilon \leq 1$, since the general case can be obtained from this by using a telescoping sum.

(3.2) involves the expectation of the square of a difference. By expanding this square as a sum of four terms, it suffices to show that for all $0 < \varepsilon/2 < \varepsilon' \leq \varepsilon \leq 1$

$$E \left(\frac{V_\varepsilon}{\varepsilon^d \kappa(\varepsilon)} \frac{V_{\varepsilon'}}{(\varepsilon')^d \kappa(\varepsilon')} \right) = \bar{A} + O(\varepsilon^\rho) \tag{3.3}$$

for some $\rho > 0$ and the same constant \bar{A} . (This constant is identified in (3.18).)

Let $h(x)$ denote the characteristic function of the unit ball, so that we can write

$$V_\varepsilon = \int \int h \left(\frac{x - y}{\varepsilon} \right) \, d\mu_\lambda(x) \, d\mu_\lambda(y).$$

For fixed $\varepsilon > 0$ it is not hard to show that

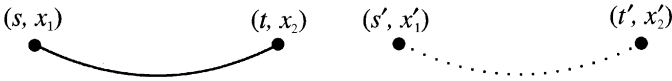
$$V_\varepsilon = \lim_{\delta \rightarrow 0} \int_{\times_{l=1}^m [0, \lambda^{(l)}]} \int_{\times_{l=1}^m [0, \lambda^{(l)}]} h \left(\frac{X_{s_l}^{(1)} - X_{t_l}^{(1)}}{\varepsilon} \right) \times \prod_{l=2}^m f_\delta(X_{s_l}^{(l)} - X_{s_l}^{(1)}) \prod_{l=2}^m f_\delta(X_{t_l}^{(l)} - X_{t_l}^{(1)}) \prod_{l=1}^m ds_l dt_l \tag{3.4}$$

in L^2 and consequently the expectation $E(V_\varepsilon V_{\varepsilon'})$ can be evaluated and then analyzed, see (3.14) and the discussion following. Before getting involved in the details, we wish to illustrate the main ideas by considering the case where $m = 1$, and even there highlighting only the critical concepts.

Thus, we consider

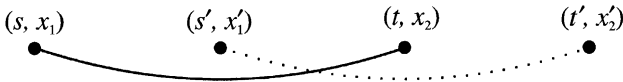
$$E(V_\varepsilon V_{\varepsilon'}) = E \left(\int_0^\lambda \int_0^\lambda h \left(\frac{X_s - X_t}{\varepsilon} \right) ds dt \int_0^\lambda \int_0^\lambda h \left(\frac{X_{s'} - X_{t'}}{\varepsilon'} \right) ds' dt' \right). \tag{3.5}$$

The expectation can be evaluated by breaking the integrand up into pieces depending on the relative positions of s, t, s', t' and there are three basic types of relative positions which are illustrated by the three figures below, in which the time coordinates increase as we go from left to right. Following each figure is the corresponding integral.



Case 1: $s < t < s' < t'$

$$J_1 = \int u^1(x_1) u^1(x_2 - x_1) u^1(x'_1 - x_2) u^1(x'_2 - x'_1) h \left(\frac{x_2 - x_1}{\varepsilon} \right) h \left(\frac{x'_2 - x'_1}{\varepsilon'} \right) dx \tag{3.6}$$



Case 2: $s < s' < t < t'$

$$J_2 = \int u^1(x_1) u^1(x'_1 - x_1) u^1(x_2 - x'_1) u^1(x'_2 - x_2) h \left(\frac{x_2 - x_1}{\varepsilon} \right) h \left(\frac{x'_2 - x'_1}{\varepsilon'} \right) dx \tag{3.7}$$



Case 3: $s < s' < t' < t$

$$J_3 = \int u^1(x_1) u^1(x'_1 - x_1) u^1(x'_2 - x'_1) u^1(x_2 - x'_2) h \left(\frac{x_2 - x_1}{\varepsilon} \right) h \left(\frac{x'_2 - x'_1}{\varepsilon'} \right) dx \tag{3.8}$$

Our goal now is to show that the contributions from the integrals corresponding to Cases 2 and 3 are of smaller order than the contribution from the integral corresponding to Case 1, and hence can be ignored in the limit as $\varepsilon, \varepsilon' \rightarrow 0$. After a change of variables we can write $J_i = (\varepsilon\varepsilon')^d \bar{J}_i$ where

$$\bar{J}_1 = \int u^1(x)u^1(\varepsilon z)u^1(x')u^1(\varepsilon'z')h(z)h(z') \, dx \, dz = \kappa(\varepsilon)\kappa(\varepsilon'), \tag{3.9}$$

$$\bar{J}_2 = \int u^1(x)u^1(x')u^1(-x' + \varepsilon z)u^1(x' + \varepsilon'z' - \varepsilon z)h(z)h(z') \, dx \, dz, \tag{3.10}$$

$$\bar{J}_3 = \int u^1(x)u^1(x')u^1(\varepsilon'z')u^1(-x' - \varepsilon'z' + \varepsilon z)h(z)h(z') \, dx \, dz, \tag{3.11}$$

where here $\kappa(\varepsilon) = \int_{|z| \leq 1} u^1(\varepsilon z) \, dz$, and we assume that $u^1(x)$ is regularly varying of index $\zeta < d$. The dx integral drops out and the basic idea now is that the dx' integration is ‘smoothing’, so that $\bar{J}_2, \bar{J}_3 = O(\kappa(\varepsilon)\kappa(\varepsilon')\varepsilon^\rho)$, which leads to (3.3) in the case where $m = 1$.

Here are the details. If $3\zeta \geq d$, we use the Potter bounds (2.6) with $\zeta < \zeta' < d$ and $3\zeta' > d$ and then scale in y to obtain the bound

$$\begin{aligned} \bar{J}_2 &= \int u^1(y)u^1(-y + \varepsilon z)u^1(y - \varepsilon z + \varepsilon'z')h(z)h(z') \, dz \, dz' \, dy \\ &\leq C \int \frac{1}{|y|^{\zeta'}} \frac{1}{|y - \varepsilon z|^{\zeta'}} \frac{1}{|y - \varepsilon z + \varepsilon'z'|^{\zeta'}} h(z)h(z') \, dz \, dz' \, dy \\ &\leq C\varepsilon^{d-3\zeta'} \int \frac{1}{|y|^{\zeta'}} \frac{1}{|y - z|^{\zeta'}} \frac{1}{|y - z + (\varepsilon'/\varepsilon)z'|^{\zeta'}} h(z)h(z') \, dz \, dz' \, dy \\ &\leq C\varepsilon^{d-3\zeta'} \int \frac{1}{|y|^{\zeta'}} \frac{1}{1 + |y|^{\zeta'}} \frac{1}{1 + |y|^{\zeta'}} \, dy \leq C\varepsilon^{d-3\zeta'} = O(\kappa^2(\varepsilon)\varepsilon^\rho), \end{aligned} \tag{3.12}$$

for some $\rho > 0$ by taking ζ' sufficiently close to ζ , while if $3\zeta < d$ we use the integrability of $(u^1(x))^3$ and the multiple Holder’s inequality to show that $\bar{J}_2 = O(1)$.

Turning to \bar{J}_3 , if $2\zeta \geq d$, we use the Potter bounds (2.6) with $\zeta < \zeta' < d$ and $2\zeta' > d$ and then scale in y to obtain the bound

$$\begin{aligned} \bar{J}_3 &= \int u^1(y)u^1(-y + \varepsilon z - \varepsilon'z')u^1(\varepsilon'z')h(z)h(z') \, dz \, dz' \, dy \\ &\leq C\varepsilon^{-\zeta'} \int \frac{1}{|y|^{\zeta'}} \frac{1}{|y - \varepsilon z + \varepsilon z'|^{\zeta'}} \frac{1}{|z'|^{\zeta'}} h(z)h(z') \, dz \, dz' \, dy \\ &\leq C\varepsilon^{d-3\zeta'} \int \frac{1}{|y|^{\zeta'}} \frac{1}{|y - z + (\varepsilon'/\varepsilon)z'|^{\zeta'}} \frac{1}{|z'|^{\zeta'}} h(z)h(z') \, dz \, dz' \, dy \\ &\leq C\varepsilon^{d-3\zeta'} \int \frac{1}{|y|^{\zeta'}} \frac{1}{1 + |y|^{\zeta'}} \frac{1}{|z'|^{\zeta'}} h(z') \, dz' \, dy \\ &\leq C\varepsilon^{d-3\zeta'} = O(\kappa^2(\varepsilon)\varepsilon^\rho) \end{aligned} \tag{3.13}$$

for some $\rho > 0$ by taking ζ' sufficiently close to ζ , while if $2\zeta < d$ we use the integrability of $(u^1(x))^2$ and the Cauchy–Schwarz inequality to show that $\bar{J}_3 = O(\kappa(\varepsilon))$.

We now return to write out the details of the general case with m arbitrary. We have

$$\begin{aligned}
 & E(V_\varepsilon V_{\varepsilon'}) \\
 &= \lim_{\delta \rightarrow 0} \int \left(\sum_{i=1}^3 \sum_{j,k=0}^1 (J_{i,j,k}^{(1)}(x_1, x_2, x'_1, x'_2) + J_{i,j,k}^{(1)}(x'_1, x'_2, x_1, x_2)) \right) \\
 &\quad \times \prod_{l=2}^m \left\{ \left(\sum_{i=1}^3 \sum_{j,k=0}^1 (J_{i,j,k}^{(l)}(u_{1,l}, u_{2,l}, u'_{1,l}, u'_{2,l}) + J_{i,j,k}^{(l)}(u'_{1,l}, u'_{2,l}, u_{1,l}, u_{2,l})) \right) \right. \\
 &\quad \times h\left(\frac{x_1 - x_2}{\varepsilon}\right) h\left(\frac{x'_1 - x'_2}{\varepsilon'}\right) f_\delta(x_1 - u_{1,l}) f_\delta(x_2 - u_{2,l}) \\
 &\quad \left. \times f_\delta(x'_1 - u'_{1,l}) f_\delta(x'_2 - u'_{2,l}) du_{1,l} du_{2,l} du'_{1,l} du'_{2,l} \right\} dx_1 dx_2 dx'_1 dx'_2 \\
 &= \int \prod_{l=1}^m \left\{ \sum_{i=1}^3 \sum_{j,k=0}^1 (J_{i,j,k}^{(l)}(x_1, x_2, x'_1, x'_2) + J_{i,j,k}^{(l)}(x'_1, x'_2, x_1, x_2)) \right\} \\
 &\quad \times h\left(\frac{x_1 - x_2}{\varepsilon}\right) h\left(\frac{x'_1 - x'_2}{\varepsilon'}\right) dx_1 dx_2 dx'_1 dx'_2 \tag{3.14}
 \end{aligned}$$

where

$$\begin{aligned}
 J_{i,0,0}^{(l)}(x_1, x_2, x'_1, x'_2) &= J_i^{(l)}(x_1, x_2, x'_1, x'_2), \\
 J_{i,0,1}^{(l)}(x_1, x_2, x'_1, x'_2) &= J_i^{(l)}(x_1, x_2, x'_2, x'_1), \\
 J_{i,1,0}^{(l)}(x_1, x_2, x'_1, x'_2) &= J_i^{(l)}(x_2, x_1, x'_1, x'_2), \\
 J_{i,1,1}^{(l)}(x_1, x_2, x'_1, x'_2) &= J_i^{(l)}(x_2, x_1, x'_2, x'_1),
 \end{aligned}$$

and

$$\begin{aligned}
 J_1^{(l)}(x, y, x', y') &= u^{(l),1}(x)u^{(l),1}(y - x)u^{(l),1}(x' - y)u^{(l),1}(y' - x'), \\
 J_2^{(l)}(x, y, x', y') &= u^{(l),1}(x)u^{(l),1}(x' - x)u^{(l),1}(y - x')u^{(l),1}(y' - y)
 \end{aligned}$$

and

$$J_3^{(l)}(x, y, x', y') = u^{(l),1}(x)u^{(l),1}(x' - x)u^{(l),1}(y' - x')u^{(l),1}(y - y').$$

Changing variables we see that

$$\begin{aligned}
 & E(V_\varepsilon V_{\varepsilon'}) \\
 &= (\varepsilon\varepsilon')^d \int \prod_{l=1}^m \left\{ \sum_{i=1}^3 \sum_{j,k=0}^1 (J_{i,j,k}^{(l)}(x, x + \varepsilon z, x', x' + \varepsilon' z') \right. \\
 &\quad \left. + J_{i,j,k}^{(l)}(x', x' + \varepsilon' z', x, x + \varepsilon z)) \right\} h(z)h(z') dx dx' dz dz'
 \end{aligned}$$

$$\begin{aligned}
 &= (\varepsilon\varepsilon')^d \int \prod_{l=1}^m \left\{ \sum_{i=1}^3 \sum_{j,k=0}^1 (K_{i,j,k,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') + K_{i,j,k,\varepsilon,\varepsilon'}^{(l)}(x',x; z',z)) \right\} \\
 &\quad \times h(z)h(z') \, dx \, dx' \, dz \, dz' \tag{3.15}
 \end{aligned}$$

where

$$\begin{aligned}
 K_{i,j,k,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') &= J_{i,j,k}^{(l)}(x,x + \varepsilon z, x', x' + \varepsilon' z') \\
 &= J_i^{(l)}(x + j\varepsilon z, x + (1-j)\varepsilon z, x' + k\varepsilon' z', x' + (1-k)\varepsilon' z') \\
 &= K_{i,\varepsilon,\varepsilon'}^{(l)}(x + j\varepsilon z, x' + k\varepsilon' z'; (1-2j)z, (1-2k)z')
 \end{aligned}$$

where

$$\begin{aligned}
 K_{1,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') &= u^{(l),1}(x)u^{(l),1}(\varepsilon z)u^{(l),1}(x' - x - \varepsilon z)u^{(l),1}(\varepsilon' z'), \\
 K_{2,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') &= u^{(l),1}(x)u^{(l),1}(x' - x)u^{(l),1}(x - x' + \varepsilon z)u^{(l),1}(x' - x + \varepsilon' z' - \varepsilon z)
 \end{aligned}$$

and

$$K_{3,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') = u^{(l),1}(x)u^{(l),1}(x' - x)u^{(l),1}(\varepsilon' z')u^{(l),1}(x - x' + \varepsilon z - \varepsilon' z').$$

We thus see that $E(V_\varepsilon V_{\varepsilon'})$ can be written as the sum of many integrals, each involving a product of m of the K 's.

Using the multiple Holder inequality in the following form:

$$\int \prod_{l=1}^m h_l(\varepsilon, \varepsilon', y) \, dy \leq \prod_{l=1}^m \left(\int (h_l(\varepsilon, \varepsilon', y))^{\zeta/\beta_l'} \, dy \right)^{\beta_l'/\zeta} \tag{3.16}$$

and using the estimates obtained when $m = 1$ we see that $E(V_\varepsilon V_{\varepsilon'})$ differs from

$$\begin{aligned}
 &(\varepsilon\varepsilon')^d \int \prod_{l=1}^m \left\{ \sum_{j,k=0}^1 (K_{1,j,k,\varepsilon,\varepsilon'}^{(l)}(x,x';z,z') + K_{1,j,k,\varepsilon,\varepsilon'}^{(l)}(x',x; z',z)) \right\} \\
 &\quad \times h(z)h(z') \, dx \, dx' \, dz \, dz' \tag{3.17}
 \end{aligned}$$

by terms which can be bounded by $C\varepsilon^{-2(d-\zeta)+\rho}$.

Using (1.4) now shows that

$$\begin{aligned}
 &E(V_\varepsilon V_{\varepsilon'}) \\
 &= 4^m (\varepsilon\varepsilon')^d \int \prod_{l=1}^m \{ u^{(l),1}(x)u^{(l),1}(x' - x) + u^{(l),1}(x')u^{(l),1}(x - x') \} \, dx \, dx' \\
 &\quad \times \int \prod_{l=1}^m u^{(l),1}(\varepsilon z)u^{(l),1}(\varepsilon' z')h(z)h(z') \, dz \, dz' + O(\varepsilon^{-2\zeta+(d-\zeta)}) \\
 &= 4^m (\varepsilon\varepsilon')^d \kappa(\varepsilon)\kappa(\varepsilon') \int \prod_{l=1}^m \{ u^{(l),1}(x)u^{(l),1}(x' - x) + u^{(l),1}(x')u^{(l),1}(x - x') \} \, dx \, dx' \\
 &\quad + O(\varepsilon^{-2(d-\zeta)+\rho}) \tag{3.18}
 \end{aligned}$$

which completes the proof of our theorem.

4. Proof of Theorems 1.1, 1.2 and 1.3

Proof of Theorems 1.1 and 1.2. Theorem 1.1 is a special case of Theorem 1.2. To prove the latter, for any $A \subseteq \mathbb{R}^d$, let $N_n(A)$ be the number of dyadic cubes in \mathbb{R}^d of edge length 2^{-n} which intersect A . As explained in Section 2 of Pemantle et al. (1996), see especially inequality (8) there, the following Corollary of Theorem 1.4 leads easily to the lower bound

$$v \in \mathcal{P}_1 \left(\bigcap_{l=1}^m X^{(l)}[0,1] \right) \quad \mathcal{E}_f(v) \geq C_\omega \int f(|x|) \prod_{l=1}^m u^{(l),1}(x) dx. \tag{4.1}$$

Corollary 4.1.

$$N_n \left(\bigcap_{l=1}^m X^{(l)}[0,1] \right) \leq C_\omega \frac{2^{nd}}{\prod_{l=1}^m u^{(l),1}(2^{-n})} \quad a.s. \tag{4.2}$$

Proof. By (1.1) we see that for almost every $(T_1, \dots, T_m) \in \mathbb{R}_+^m$ we have

$$\lim_{\varepsilon \rightarrow 0} \left(\prod_{l=1}^m u^{(l),1}(\varepsilon) \right) \left| \bigcap_{l=1}^m S_\varepsilon^{(l)}(T_l) \right| = \mathcal{S}(T_1, \dots, T_m) \quad a.s. \tag{4.3}$$

The result follows easily from this since we can choose (T_1, \dots, T_m) so that $1 \leq T_l \leq 2$ for all $1 \leq l \leq m$.

As explained in the proof of Theorem 1.4 in Section 3 of Pemantle et al. (1996), the following Corollary of Theorem 1.5 leads easily to the upper bound corresponding to (4.1)

$$v \in \mathcal{P}_1 \left(\bigcap_{l=1}^m X^{(l)}[0,1] \right) \quad \mathcal{E}_f(v) \leq C'_\omega \int f(|x|) \prod_{l=1}^m u^{(l),1}(x) dx \tag{4.4}$$

which will complete the proof of Theorem 1.2. \square

Corollary 4.2.

$$(\mu_{1, \dots, 1} \times \mu_{1, \dots, 1})(\Delta_{2^{-n}}) \leq C'_\omega 2^{-nd} \prod_{l=1}^m u^{(l),1}(2^{-n}) \quad a.s. \tag{4.5}$$

This corollary follows from Theorem 1.5 as in the proof of Corollary 1.

Proof of Theorem 1.3. This follows from the corollaries of this section as explained in the proof of Theorem 1.3 in Section 4 of Pemantle et al. (1996).

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References

- Aizenman, M., 1985. The intersections of Brownian paths as a case study of renormalization group methods for quantum field theory. *Comm. Math. Phys.* 97, 91–110.
- Albeverio, S., Zhou, X.Y., 1996. Intersections of random walls and Wiener sausages in four dimensions. *Acta Appl. Math.* 45, 195–237.
- Bingham, N., Goldie, C., Teugals, J., 1987. *Regular Variation*. Cambridge University Press, Cambridge.
- Bliedtner, J., Hanson, W., 1986. *Potential Theory*. Springer, New York.
- Blumenthal, R., Gettoor, R., 1968. *Dual Processes and Potential Theory*. Academic Press, New York.
- Blumenthal, R., Gettoor, R., 1970. Markov processes and potential theory. *Proceedings of the Twelfth Biennial Seminar of Canadian Mathematics Congress*, pp. 137–156.
- Chung, K.L., 1973. Probabilistic approach in potential theory to the equilibrium problem. *Ann. Inst. Fourier, Grenoble* 23, 313–322.
- Delmas, J.F., 1998. Some properties of the range of super-Brownian motion. *Probab. Theory Related Fields* 114, 505–547.
- Fitzsimmons, P.J., Salisbury, T., 1989. Capacity and energy for multiparameter Markov processes. *Ann. Inst. Henri Poincaré, Probab.* 25, 325–350.
- Hawkes, J., 1970. Polar sets, regular points and recurrent sets for symmetric and increasing stable processes. *Bull. London Math. Soc.* 2, 53–59.
- Lawler, G.F., 1982. The probability of intersection of independent random walks in four dimensions. *Commun. Math. Phys.* 86, 539–554.
- Pemantle, R., Peres, Y., Shapiro, J., 1996. The trace of spatial Brownian motion is capacity-equivalent to the unit square. *Probab. Theory Related Fields* 106, 379–399.
- Pemantle, R., Peres, Y., 1995. Galton-Watson trees with the same mean have the same polar sets. *Ann. Probab.* 23, 1102–1124.
- Peres, Y., 1996. Intersection-equivalence of Brownian paths and certain branching processes. *Commun. Math. Phys.* 177, 417–443.
- Rosen, J., 1986. Tanaka's formula for multiple intersections of planar Brownian motion. *Stochastic Process. Appl.* 23, 131–141.