Renormalized self-intersection local times and Wick power chaos processes

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ABSTRACT. Sufficient conditions are obtained for the continuity of renormalized self-intersection local times for the multiple intersections of a large class of strongly symmetric Lévy processes in R^m, $m = 1, 2$. In R^2 these include Brownian motion and stable processes of index greater than $3/2$, as well as many processes in their domains of attraction. In R^1 these include stable processes of index $3/4 < \beta \leq 1$ and many processes in their domains of attraction.

Let $(\Omega, \mathcal{F}(t), X(t), P^x)$ be one of these radially symmetric Lévy processes with 1-potential density $u^1(x,y)$. Let $\mathcal{G}_F^{2n}$ denote the class of positive finite measures μ on R^m for which

$$\int\int (u^1(x,y))^{2n}\, d\mu(x)\, d\mu(y) < \infty.$$

For $\mu \in \mathcal{G}_F^{2n}$, let

$$\alpha_{n,\epsilon}(\mu,\lambda) \stackrel{def}{=} \int\int_{\{0\leq t_1 \leq \cdots \leq t_n \leq \lambda\}} f_\epsilon(X(t_1)-x) \prod_{j=2}^{n} f_\epsilon(X(t_j)-X(t_{j-1}))\, dt_1 \cdots dt_n\, d\mu(x)$$

where f_ϵ is an approximate $\delta-$function at zero and λ is an random exponential time, with mean one, independent of X (with probability measure P_λ). The renormalized self-intersection local time of X with respect to the measure μ is defined as

$$\gamma_n(\mu) = \lim_{\epsilon\to 0} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (u_\epsilon^1(0))^k \alpha_{n-k,\epsilon}(\mu,\lambda)$$

where $u_\epsilon^1(x) \stackrel{def}{=} \int f_\epsilon(x-y)u^1(y)\, dy$, with $u^1(x) \stackrel{def}{=} u^1(x+z,z)$ for all $z \in R^m$. Conditions are obtained under which this limit exists in $L^2(\Omega\times R^+, P_\lambda^y)$ for all $y \in R^m$, where $P_\lambda^y \stackrel{def}{=} P^y \times P_\lambda$.

Let $\{\mu_x, x \in R^m\}$ denote the set of translates of the measure μ. The main result in this paper is a sufficient condition for the continuity of $\{\gamma_n(\mu_x),\, x \in R^m\}$, namely that this process is continuous P_λ^y almost surely for all $y \in R^m$, if the corresponding $2n$-th Wick power chaos process, $\{: G^{2n}\mu_x :,\, x \in R^m\}$ is continuous almost surely. This chaos process is obtained in the following way. A Gaussian process $G_{x,\delta}$ is defined which has covariance $u_\delta^1(x,y)$, where $\lim_{\delta\to 0} u_\delta^1(x,y) = u^1(x,y)$. Then

$$: G^{2n}\mu_x : \stackrel{def}{=} \lim_{\delta\to 0} \int : G_{y,\delta}^{2n} : \, d\mu_x(y)$$

where the limit is taken in L^2. ($: G_{y,\delta}^{2n} :$ is the $2n$-th Wick power of $G_{y,\delta}$, that is, a normalized Hermite polynomial of degree $2n$ in $G_{y,\delta}$). This process has a natural metric

$$\begin{aligned} d(x,y) &\stackrel{def}{=} \frac{1}{(2n)!} \left(E(: G^{2n}\mu_x : - : G^{2n}\mu_y :)^2 \right)^{1/2} \\ &= \left(\int\int \left(u^1(u,v)\right)^{2n} (d(\mu_x(u)-\mu_y(u)))\,(d(\mu_x(v)-\mu_y(v))) \right)^{1/2}. \end{aligned}$$

A well known metric entropy condition with respect to d gives a sufficient condition for the continuity of $\{: G^{2n}\mu_x :,\, x \in R^m\}$ and hence for $\{\gamma_n(\mu_x),\, x \in R^m\}$.

Contents

CHAPTER 1

Introduction

We study the continuity of renormalized self-intersection local times for the multiple intersections of a large class of strongly symmetric Lévy processes in R^m, $m = 1, 2$, including symmetric stable processes. We do this by comparing these processes to Wick power Gaussian chaos processes using an isomorphism theorem which generalizes an isomorphism theorem of Dynkin.

Intersection local times "measure" the amount of self-intersections of a stochastic process, say, $X(t) \in R^m$. To define the $n-$fold self-intersection local time, the natural approach is to set

$$\alpha_{n,\epsilon}(\mu, t) \stackrel{def}{=} \int \int_{\{0 \le t_1 \le \cdots \le t_n \le t\}} f_\epsilon(X(t_1) - x) \prod_{j=2}^{n} f_\epsilon(X(t_j) - X(t_{j-1}))\, dt_1 \cdots dt_n\, d\mu(x) \tag{1.1}$$

where f_ϵ is an approximate $\delta-$function at zero, and take the limit as $\epsilon \to 0$. Intuitively, this gives a measure of the set of times $(t_1, \ldots, t_n)$ such that

$$X(t_1) = \cdots = X(t_n) = x,$$

where the "n-multiple points" $x \in R^m$ are weighted by the measure μ. However, in general, this limit does not exist because of the effect of the integral in the neighborhood of the diagonal. The method used to compensate for this is called renormalization. One subtracts from $\alpha_{n,\epsilon}(\mu, t)$ terms involving lower order intersections $\alpha_{k,\epsilon}(\mu, t)$ for $k < n$, in such a way that a finite limit results. This was originally done by Varadhan [**25**] for double intersections of Brownian motion in the plane with μ taken to be Lebesgue measure. Varadhan's work stimulated a large body of research which is summarized by Dynkin in [**6**]. Renormalized intersection local times have turned out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener and stable sausage in the plane and fluctuations of the range of stable random walks. (See Le Gall [**10, 9**], Le Gall-Rosen [**12**] and Rosen [**23**]). For a clear account of progress concerning Brownian intersection local times up to 1990 see Le Gall's lecture notes [**11**]. For more recent results see Bass and Khoshnevisan [**2**] and Rosen [**22**].

For Brownian motion in the plane and with μ taken to be Lebesgue measure, Dynkin [**5**] introduced the idea of studying $\alpha_{n,\epsilon}(\mu, \lambda)$, where λ is an exponential random variable with mean one, independent of $X(t)$, and showed how this randomization of time leads to technical simplifications. Also, he introduced the following renormalization formula

$$\gamma_n(\mu) \stackrel{def}{=} \lim_{\epsilon \to 0} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (u^1_\epsilon(0))^k \alpha_{n-k,\epsilon}(\mu, \lambda) \tag{1.2}$$

where $u^1_\epsilon(x)=\int f_\epsilon(x-y)u^1(y)\,dy$ and $u^1(x)=\int_0^\infty e^{-t}p_t(x)\,dt$ is the 1-potential density.

Let

$$\gamma_{n,\epsilon}(\mu)=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(u^1_\epsilon(0))^k\alpha_{n-k,\epsilon}(\mu,\lambda). \tag{1.3}$$

Heuristically, one may think of $\gamma_{n,\epsilon}(\mu)$ as

$$\gamma_{n,\epsilon}(\mu)=\int\int_{\{0\le t_1\le\cdots\le t_n\}} f_\epsilon(X(t_1)-x)$$
$$\prod_{j=2}^{n}\left\{f_\epsilon(X(t_j)-X(t_{j-1}))-\delta(t_j-t_{j-1})u^1_\epsilon(0)\right\}\,dt_1\cdots dt_n\,d\mu(x). \tag{1.4}$$

This formulation compensates for the difficulties caused when various of the t_i are close to each other.

In this paper we consider renormalized self-intersection local times $\gamma_n(\mu)$ for a large class of radially symmetric Lévy processes in R^m, $m=1,2$ and positive finite measures μ on R^m. We define $\mu_x(\cdot)=\mu(x+\cdot)$ to be the measure μ translated by $x\in R^m$ and study the continuity of the stochastic process $\{\gamma_n(\mu_x),x\in R^m\}$.

Let $(\Omega,\mathcal{F}(t),X(t),P^x)$ be a radially symmetric Lévy processes in R^m, $m=1,2$ with 1-potential density $u^1(x,y)$. Since intersection local times are trivial for processes which have an actual local time we only consider Lévy processes for which $u^1(0)=\infty$. Clearly $u^1(x,y)=u^1(x-y,0)$ and since $X(t)$ is radially symmetric we sometimes write these terms as $u^1(x-y)$ or $u^1(|x-y|)$. The results obtained in this paper are valid for a large class of radially symmetric Lévy processes which we say are in Class A. This class contains the symmetric stable processes and many processes in their domains of attraction. Class A is defined later in this chapter, see (1.15) and (1.16), after which we give more details about the range of this class and the scope of our results.

We use $\mathcal{G}^{2n}$ to denote the class of positive measures μ for which

$$\int\int(u^1(x,y))^{2n}\,d\mu(x)\,d\mu(y)<\infty. \tag{1.5}$$

It should be understood that when we say $\mu\in\mathcal{G}^{2n}$, that this is with respect to the 1-potential of some given Lévy process.

As in our recent work [**17, 15**] in which an isomorphism theorem of Dynkin enables us to use ideas from the theory of Gaussian processes and Gaussian chaos processes to study the continuity of Markov local times and additive functionals, here too we develop an isomorphism theorem and use it to relate renormalized self-intersection local times to higher order Gaussian chaos processes. We call them $2n$-th Wick power chaos processes and denote them by $:G^{2n}\mu_x:$. These processes are described rigorously in Chapter 2. But, roughly, here is how one can think of them. Let us first note that the 1-potential, $u^1(x,y)$ of $X(t)$ is positive definite. This follows because

$$u^1(x,y)=\int_0^\infty e^{-t}p_t(x,y)\,dt$$

and a symmetric transition probability density, here denoted by $p_t(x,y)$, is easily seen to be positive definite for all $t>0$ by the Chapman-Kolmogorov equation. (See also Theorem 3.3, [**17**]). We would like to be able to consider a Gaussian process with covariance $u^1(x,y)$, however the 1-potentials that interest us are infinite at the origin. To deal with this we approximate $u^1(x,y)$ by a positive definite function $u^1_\delta(x,y)$ such that $u^1_\delta(0)<\infty$ and $\lim_{\delta\to 0} u^1_\delta(x,y)=u^1(x,y)$. We then consider the mean zero Gaussian process $\{G_\delta(y), y\in R^m\}$ with covariance $u^1_\delta(x,y)$ and take its $2n$-th Wick power $:G^{2n}_\delta(y):$. (This is the Hermite polynomial of degree $2n$ in G_δ with leading coefficient one). We define

$$:G^{2n}\mu_x := \lim_{\delta\to 0}\int :G^{2n}_\delta(y):\, d\mu_x(y) \tag{1.6}$$

where the limit is taken in L^2. This limit exists for $\mu\in\mathcal{G}^{2n}$. Thus, given a Lévy process with 1-potential u^1 and a measure $\mu\in\mathcal{G}^{2n}$ we consider the $2n$-th order Gaussian chaos process $\{:G^{2n}\mu_x:, x\in R^m\}$.

The main result of this paper is the following sufficient condition for the almost sure continuity of the renormalized self-intersection local time $\gamma_n(\mu_x)$ as a function of x. To be more precise we say that a random variable Y is a version of $\gamma_n(\mu)$ if

$$Y=\lim_{\epsilon\to 0}\gamma_{n,\epsilon}(\mu) \tag{1.7}$$

in $L^2(\Omega\times R_+, P^y_\lambda)$ for all $y\in R^m$, where P^y_λ is the product probability measure P^yP_λ, (and P_λ is the probability measure of λ). Often we simply say that a stochastic process has a property almost surely when we actually mean that the process has a version with this property almost surely.

THEOREM 1.1. *Let $X=\{X(t), t\in R^+\}$ be a Lévy process in Class A and μ be a finite positive measure in $\mathcal{G}^{2n}$. Let $\{:G^{2n}\mu_x:, x\in R^m\}$ be the $2n$-th Wick power chaos process associated with X and μ and let $\{\gamma_n(\mu_x), x\in R^m\}$ be the n-fold renormalized self-intersection local time process of X, with respect to μ. If $\{:G^{2n}\mu_x:, x\in R^m\}$ is continuous almost surely then $\{\gamma_n(\mu_x), x\in R^m\}$ is continuous almost surely.*

Theorem 1.1 requires the continuity of the $2n$-th Wick power Gaussian chaos process $\{:G^{2n}\mu_x:, x\in R^m\}$. Here is a well known sufficient condition for the continuity of this process. For $\mu\in\mathcal{G}^{2n}$ define a metric on R^m

$$\begin{aligned} d(x,y) &= \left(\int\!\!\int \left(u^1(u,v)\right)^{2n}\left(d(\mu_x(u)-\mu_y(u))\right)\left(d(\mu_x(v)-\mu_y(v))\right)\right)^{1/2} \\ &= \frac{1}{(2n)!}\left(E(:G^{2n}\mu_x:-:G^{2n}\mu_y:)^2\right)^{1/2}. \end{aligned} \tag{1.8}$$

(The last equality is explained in (3.14)). A sufficient condition for the almost sure continuity of $\{:G^{2n}\mu_x:, x\in R^m\}$ is that

$$\int_0^\infty (\log N_d(B,\epsilon))^n\, d\epsilon<\infty \tag{1.9}$$

where B is the unit ball in R^m and $N_d(B,\epsilon)$ is the minimum number of balls of radius ϵ, in the metric d, that covers B. ($\log N_d(B,\cdot)$ is called the metric entropy of B with respect to d). Thus we get the following corollary of Theorem 1.1:

COROLLARY 1.1. *Let $X=\{X(t), t\in R^+\}$ be a Lévy process in Class A and μ be a finite positive measure in $\mathcal{G}^{2n}$. Let $\{\gamma_n(\mu_x), x\in R^m\}$ be the n-fold renormalized self-intersection local time process of X, with respect to μ. If (1.9) holds then $\{\gamma_n(\mu_x), x\in R^m\}$ is continuous almost surely.*

Here is a concrete application of Corollary 1.1. Let $\tau(\xi)$ denote the Fourier transform of $(u^1(x))^{2n}$ so that

$$\int \tau(\xi)|\hat{\mu}(\xi)|^2\,d\xi = \int\int \left(u^1(x,y)\right)^{2n}\,d\mu(x)\,d\mu(y). \tag{1.10}$$

COROLLARY 1.2. *Let $X = \{X(t), t \in R^+\}$ be a Lévy process in R^m, $m = 1, 2$, in Class A and $\{\gamma_n(\mu_x),\, x \in R^m\}$ be the n-fold renormalized self-intersection local time process of X, with respect to a finite positive measure μ on R^m. If*

$$\int_1^\infty \frac{(\int_{|\xi|\geq x} \tau(\xi)|\hat{\mu}(\xi)|^2\,d\xi)^{1/2}}{x}(\log x)^{n-1}\,dx < \infty \tag{1.11}$$

then $\mu \in \mathcal{G}^{2n}$ and $\{\gamma_n(\mu_x),\, x \in R^m\}$ is continuous almost surely. In particular, for Brownian motion in R^2, this is the case when

$$|\hat{\mu}(\xi)| = O\left(\frac{1}{(\log|\xi|)^{2n+\epsilon}}\right) \qquad as \quad |\xi| \to \infty. \tag{1.12}$$

Furthermore for a Lévy process X in R^2 in Class A with Lévy exponent asymptotic to $\lambda^2/(\log|\lambda|)^a$, $a > 0$, as $\lambda \to \infty$, (see (1.19) below), the n-fold renormalized self-intersection local time process of X, with respect to a positive measure μ on R^2, is continuous almost surely if (1.12) holds with $2n$ replaced by $2n(1 + a/2)$.

We do not know whether (1.9) is a necessary condition for continuity of the $2n$-th Wick power chaos associated with u^1 for any measure μ. Based on Theorem 1.5, [**15**] and the results in [**19**] we suspect that at least for a class of smooth measures μ a necessary and sufficient condition for continuity of the $2n$-th Wick power chaos associated with $u^1(\cdot)$ is the one in (1.9) but with n replaced by $1/2$. We do not know how to prove this. The methods of [**19**], which prove a result of this nature for second order Wick power chaos processes do not extend to higher order Wick power chaos processes.

The isomorphism theorems we develop can be used to obtain other path properties of renormalized self-intersection local times besides continuity. In [**16**] and [**15**] we used Dynkin's isomorphism theorem to obtain moduli of continuity results for continuous additive functionals of Lévy processes. We can do the same here for $\{\gamma_n(\mu_x), x \in R^m\}$. However, since there is little new involved we will leave this to the interested reader. Instead, we demonstrate the power of the isomorphism theorem approach by obtaining a bound on the exponential moment of $\sup_{x\in[-1,1]^m} |\gamma_n(\mu_x)|^{1/n}$. The following theorem is proved in Chapter 9:

THEOREM 1.2. *Assume that all the hypotheses of Theorem 1.1 are satisfied. This implies, in particular, that $\{\gamma_n(\mu_x),\, x \in R^m\}$ is continuous almost surely. Assume further that the 1-potential of X, $u^1(x) = O(1/x^{m+\epsilon})$, as $x \to \infty$, for some $\epsilon > 0$. Then there exist constants $0 < c, C < \infty$ such that*

$$E^y_\lambda \exp\left(c \sup_{x\in[-1,1]^m} |\gamma_n(\mu_x)|^{1/n}\right) < C \tag{1.13}$$

for all $y \in R^m$.

The next chapter contains a brief survey of properties of Wick products. In Chapter 3 we define the Wick power chaos processes that enter into the various isomorphism theorems which are the heart of this paper. In Theorem 3.1 we obtain a critical estimate for the behavior of a Wick power chaos in the neighborhood of its diagonal.

The main isomorphism theorem of this paper, Theorem 4.1, relates renormalized self-intersection local times of Lévy processes in Class A to Wick power chaos processes. The result we obtain, (4.3) was obtained by Dynkin in [**5**], for Brownian motion in R^2 with μ taken to be Lebesgue measure. Dynkin's result inspired this paper but our proofs are quite different. Since the 1-potential of planar Brownian motion has a logarithmic singularity, all of its powers are integrable. This simplifies considerably the proofs of many of the estimates required. On the contrary, when dealing with processes whose 1-potential has a power type singularity, obtaining the necessary estimates is very delicate.

In proving the isomprphism theorem, Theorem 4.1, it is natural to work with a renormalized self-intersection local time $\mathcal{L}_n(\mu)$ that does not have the same appearance as $\gamma_n(\mu)$. Nevertheless, in Chapter 5 we show that the two formulations of renormalized self-intersection local time are stochastically equivalent and hence are interchangeable in these theorems. Another difficulty we must deal with is that in the context of the isomorphism theorem, Theorem 4.1, it is natural to define $\gamma_n(\mu)$, P^ρ_λ almost surely for some measure $\rho \in \mathcal{G}^1$, whereas it is more desireable to define it P^x_λ for all $x \in R^m$. We can do this but we have to add an additional hypothesis on the measures μ, namely

$$\sup_x \left| \int (u^1(x-y))^n \, d\mu(y) \right| < \infty. \tag{1.14}$$

The reader may note that the hypothesis (1.14) does not appear explicitly in Theorem 1.1. This is because (1.14) is implied by the condition that $\{: G^{2n}\mu_x :, \, x \in R^m\}$ is locally bounded. This is not easy to see. Indeed it requires two more isomorphism theorems, Theorems A.2 and A.3 which deal with the intersections of independent Lévy processes. We relegate all this material to Chapter A, an Appendix to this paper.

In a brief Chapter 6 we put all the results of Chapters 4 and 5 together and prove Theorem 1.1 and Corollaries 1.1 and 1.2. In Chapter 7 we describe a large class of measures which are contained in Class A. In a brief Chapter 8 we give examples of Lévy processes and corresponding measures μ for which the n-fold self-intersection local time process, $\{\gamma_n(\mu_x), \, x \in R^m\}$ is continuous. In Chapter 9 we prove our bound on the exponential moment of $\sup_{x\in[-1,1]^m} |\gamma_n(\mu_x)|^{1/n}$, Theorem 1.2.

* * *

We now describe the Lévy processes in Class A. Let $h : R^m \to R^1$ and $b \in R^m$, $m = 1, 2$. Define $\Delta_b h(s) = h(s+b) - h(s)$ and $\Delta^2_{b,c} h(s) = \Delta_b \Delta_c h(s)$. We say that a Lévy process belongs to Class A if it is radially symmetric and its 1-potential density $u^1(|s|)$ is regularly varying at the origin with index greater than minus two, u^1 is bounded away from the origin, and there exists an $s_0 > 0$ such that for $|s| \le s_0$

$$|\Delta_b u^1(s)| \le C|b| \frac{u^1(|s|)}{|s|} \qquad \text{for } |b| \le \frac{|s|}{4} \tag{1.15}$$

and

$$|\Delta^2_{b,c} u^1(s)| \le C(|bc|) \frac{u^1(|s|)}{|s|^2} \qquad \text{for } |b|, |c| \le \frac{|s|}{4}. \tag{1.16}$$

Also if $u^1(|s|)$ is slowly varying at the origin we require that it is asymptotic to a decreasing function at the origin. This condition is clearly satisfied when $u^1(|s|)$ is regularly varying at the origin with index less than zero. Finally, let $|s| = r$ and consider $(u^1(r))'$, the derivative of u^1 with respect to r. We require that for all $r_0 > 0$, $(u^1(r))' \vee (u^1(r))'' \leq C_{r_0}$ for all $r \geq r_0 > 0$, where C_{r_0} is a constant depending only on r_0.

Symmetric stable processes in R^2, including Brownian motion in R^2, are in Class A, as are symmetric stable processes in R^1 with index $\beta \leq 1$, but Class A is larger than this. Let

$$Ee^{i\lambda X(t)} = e^{-t\psi(|\lambda|)}. \tag{1.17}$$

We refer to ψ as the Lévy exponent of X. We show in Chapter 7 that "stable mixtures" are in Class A. These are Lévy processes, which we introduced in [**18**], which are defined in terms of their characteristic exponents

$$\psi(|\lambda|) = \int_a^\beta |\lambda|^s \, d\phi(s) \tag{1.18}$$

where ϕ is a probability measure such that support$(\phi) = [a, \beta]$, where $0 < a < \beta \leq 2$. It is easy to see that this is a Lévy exponent since it is the limit of a linear combination of the Lévy exponents of symmetric stable processes. The class of function given in (1.18) is fairly general. It follows from Lemma 3, [**18**] that for any function g which is regularly varying at infinity with positive exponent or which is an increasing slowly varying function at infinity we can find a Lévy exponent of the form of (1.18) for which

$$\psi(|\lambda|) \sim \frac{|\lambda|^\beta}{g(\log|\lambda|)} \qquad \text{as } |\lambda| \to \infty. \tag{1.19}$$

The condition that $\mu \in \mathcal{G}^4$, an hypothesis that is required by the approach we take to study double points in R^m, along with our interest only in those processes for which $u^1(0) = \infty$, restricts our consideration of stable mixtures to the cases $3/2 < \beta \leq 2$ when $m = 2$ and $3/4 < \beta \leq 1$ when $m = 1$.

If the integral appearing in the definition of $\mathcal{G}^{2n}$, (1.5), is finite for any positive measure on R^m, it is finite for Lebesgue measure on $[-1, 1]^m$. This shows us that for $n \geq 2$, $\mathcal{G}^{2n}$ is empty for positive measures on R^m, $m \geq 3$. Note that Lévy processes do not intersect in R^4 but in R^3 they can have (at most) double points. That is, one could consider intersections for Lévy processes in R^3 but Theorem 1.1 says nothing in this case. However, in R^1 and R^2, we get interesting results.

For symmetric β stable processes in R^2, $\beta < 2$, the requirement that $\mu \in \mathcal{G}^{2n}$ means that $\beta > 2 - (1/n)$. However for Brownian motion in R^2 and for certain Lévy processes in R^2, in the domain of attraction of Brownian motion, we can find positive measures $\mu \in \mathcal{G}^{2n}$ for all n. A similar situation exists in R^1. In this case $\mu \in \mathcal{G}^{2n}$ requires that $\beta > 1 - (1/2n)$. However for the symmetric Cauchy process in R^1 and for certain Lévy processes in R^1 in the domain of attraction of the symmetric Cauchy process, we can find positive measures $\mu \in \mathcal{G}^{2n}$ for all n.

It is well known, [**13**], [**8**] that

$$\int \left(u^1(x)\right)^n \, dx < \infty \tag{1.20}$$

is necessary and sufficient for n-fold self-intersections to exist almost surely. Let ν be Lebesgue measure on $[-1, 1]^m$. It is easy to see that for 1-potentials which are bounded away from the origin

$$\int_{[-1,1]^m} \left(u^1(x)\right)^n \, dx < \infty \tag{1.21}$$

if and only if $\nu \in \mathcal{G}^n$.

The criteria for the existence of $\gamma_n(\mu)$ are much stronger. We define $\gamma_n(\mu)$, P^x_λ almost surely, as a limit in L^2. Consequently we require that $E|\gamma_n(\mu)|^2 < \infty$. We show in Lemma 5.1 that for many Lévy processes in Class A and smooth measures μ, $E|\gamma_n(\mu)|^2 < \infty$ if and only if $\mu \in \mathcal{G}^{2n}$. Roughly speaking, the sufficient condition for the continuity of $\{\gamma_n(\mu_x), x \in R^m\}$ given in Corollary 1.2 is only slightly stronger than this.

We now discuss one way to interpret Theorem 1.1 which shows why it is interesting to have this result for a wide class of measures. First let us note that the 1-potential of a symmetric β-stable process in R^2 is asymptotic to $1/|x|^{2-\beta}$, at the origin, and is also the density of a measure on R^2. This shows that we can find measures μ on R^2, absolutely continuous with respect to Lebesgue measure, with density asymptotic to $1/|x|^{2-\epsilon}$ at the origin for any $\epsilon > 0$, for which (1.12) is satisfied for all n. Depending on the Lévy process considered, some, or all, of these measures are admissible in Theorem 1.1. Now, ideally in Theorem 1.1, one might want to take for the measure μ, the $\delta-$function at some point in the state space of a Lévy process. Then one could talk about renormalized self-intersection local times of the Lévy process at this point. But this is infinite. So, instead we ask, how strong a weight can we put near a point and still obtain a renormalized self-intersection local time for the process in the neighborhood of the point? We obtain some answers to this question and also show that the renormalized self-intersection local time process, which is obtained when this weight (measure) is translated through the state space, is continuous almost surely. Of course, we need not only consider measures which are absolutely continuous with respect to Lebesgue measure. Another interesting application of these results would be for measures supported on a subspace of R^2 with fractional Haussdorff dimension.

CHAPTER 2

Wick products

We begin by giving the definition of Wick products and develop several useful relations involving them. Let $\{G_t,\, t \in T\}$ denote a mean-zero Gaussian process indexed by some set T, and let

$$g(s,t) = E(G_s G_t)$$

denote its covariance. Recall the well known equation

$$E(\prod_{i=1}^{2m} G_{t_i}) = \sum_{\mathcal{P}} \prod_{k=1}^{m} g(t_{\mathcal{P}_{k,1}}, t_{\mathcal{P}_{k,2}}) \tag{2.1}$$

where the sum runs over all possible pairings $\mathcal{P}$, i.e. partitions of $\{1,\ldots,2m\}$, into two element subsets (pairs). We let $(\mathcal{P}_{k,1}, \mathcal{P}_{k,2})$ denote the $k-$th pair of $\mathcal{P}$. Note that the expectation of a product of an odd number of the terms G_{t_i} is zero.

The Wick product of $\{G_{t_i};\, i = 1,\ldots,n\}$ which we denote by $: \prod_{i=1}^{n} G_{t_i} :$ is defined by the equation

$$: \prod_{i=1}^{n} G_{t_i} := \sum_{j=0}^{[n/2]} (-1)^j \sum_{|\widetilde{\mathcal{P}}|=j} \prod_{k=1}^{j} g(t_{\widetilde{\mathcal{P}}_{k,1}}, t_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{i \in \widetilde{\mathcal{P}}^c} G_{t_i} \tag{2.2}$$

where, for fixed j, the second sum runs over all possible choices $\widetilde{\mathcal{P}}$ of j pairs of indices chosen from $\{1,\ldots,n\}$, and $(\widetilde{\mathcal{P}}_{k,1}, \widetilde{\mathcal{P}}_{k,2})$ denotes the $k-$th pair in $\widetilde{\mathcal{P}}$. Here $|\widetilde{\mathcal{P}}| = j$, the number of pairs in $\widetilde{\mathcal{P}}$.

Let $\mathcal{I}_n$ denote the closed subspace of L^2 generated by all products of the form $\prod_{i=1}^{m} G_{t_i}$ with $0 \leq m \leq n$. Let $\mathcal{J}_n = \mathcal{I}_n \ominus \mathcal{I}_{n-1}$ denote the orthogonal complement of $\mathcal{I}_{n-1}$ in $\mathcal{I}_n$. Let Q_n denote the orthogonal projection of L^2 onto $\mathcal{J}_n$. We claim that

$$Q_n(\prod_{i=1}^{n} G_{t_i}) =: \prod_{i=1}^{n} G_{t_i} : . \tag{2.3}$$

To verify (2.3) we must show that $: \prod_{i=1}^{n} G_{t_i} :$ satisfies

$$: \prod_{i=1}^{n} G_{t_i} : - \prod_{i=1}^{n} G_{t_i} \in \mathcal{I}_{n-1}$$

and that $: \prod_{i=1}^{n} G_{t_i} :$ is orthogonal to $\mathcal{I}_{n-1}$. The first requirement is obvious from the definition (2.2). To establish the second let us first note that by (2.1) we have

$$E(\prod_{i=1}^{j} G_{t_{1,i}} \prod_{i=1}^{m} G_{t_{2,i}}) \tag{2.4}$$

$$= \sum_{\mathcal{P}=\mathcal{P}^1\cup\mathcal{P}^2\cup\mathcal{P}^3} \prod_{k=1}^{|\mathcal{P}^1|} g(t_{\mathcal{P}^1_{k,1}}, t_{\mathcal{P}^1_{k,2}}) \prod_{k=1}^{|\mathcal{P}^2|} g(t_{\mathcal{P}^2_{k,1}}, t_{\mathcal{P}^2_{k,2}}) \prod_{k=1}^{|\mathcal{P}^3|} g(t_{\mathcal{P}^3_{k,1}}, t_{\mathcal{P}^3_{k,2}})$$

where the sum runs over all pairings $\mathcal{P}$ of $\{(1,1),\dots,(1,j),(2,1),\dots,(2,m)\}$, and $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3$. Here $\mathcal{P}^1$ denotes the subcollection of pairs in $\mathcal{P}$ whose first components are both 1, $\mathcal{P}^2$ denotes the subcollection of pairs in $\mathcal{P}$ in which the first component of one element is 1 and that of other element is 2, and $\mathcal{P}^3$ denotes the subcollection of pairs in $\mathcal{P}$ whose first components are both 2. By (2.2)

$$\begin{aligned} &E(:\prod_{i=1}^n G_{t_{1,i}} : \prod_{i=1}^m G_{t_{2,i}}) \qquad (2.5)\\ &= \sum_{j=0}^{[n/2]} (-1)^j \sum_{|\widetilde{\mathcal{P}}|=j} \prod_{k=1}^{j} g(t_{1,\widetilde{\mathcal{P}}_{k,1}}, t_{1,\widetilde{\mathcal{P}}_{k,2}}) E(\prod_{i\in\widetilde{\mathcal{P}}^c} G_{t_{1,i}} \prod_{i=1}^m G_{t_{2,i}}). \end{aligned}$$

Now let $\mathcal{P}$ be the pairings of $\{(1,1),\dots,(1,j),(2,1),\dots,(2,m)\}$ and define $\mathcal{P}^1$, $\mathcal{P}^2$ and $\mathcal{P}^3$ as above. We see that the left-hand side of (2.5)

$$\begin{aligned} &= \sum_{\mathcal{P}=\mathcal{P}^1\cup\mathcal{P}^2\cup\mathcal{P}^3} \left\{ \sum_{j=0}^{[n/2]} (-1)^j \binom{|\mathcal{P}^1|}{j} \right\} \prod_{k=1}^{|\mathcal{P}^1|} g(t_{\mathcal{P}^1_{k,1}}, t_{\mathcal{P}^1_{k,2}}) \qquad (2.6)\\ &\qquad \prod_{k=1}^{|\mathcal{P}^2|} g(t_{\mathcal{P}^2_{k,1}}, t_{\mathcal{P}^2_{k,2}}) \prod_{k=1}^{|\mathcal{P}^3|} g(t_{\mathcal{P}^3_{k,1}}, t_{\mathcal{P}^3_{k,2}}) \end{aligned}$$

since there are precisely $\binom{|\mathcal{P}^1|}{j}$ terms in the sum

$$\sum_{|\widetilde{\mathcal{P}}|=j} \prod_{k=1}^{j} g(t_{1,\widetilde{\mathcal{P}}_{k,1}}, t_{1,\widetilde{\mathcal{P}}_{k,2}}) \prod_{i\in\widetilde{\mathcal{P}}^c} G_{t_{1,i}}$$

which give rise to the partition $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3$. That is, there are precisely $\binom{|\mathcal{P}^1|}{j}$ ways to choose sets $\widetilde{\mathcal{P}}$ with j pairs from the $|\mathcal{P}^1|$ pairs of $\mathcal{P}^1$.

Since $|\mathcal{P}^1| \le [n/2]$, we have

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{|\mathcal{P}^1|}{j} = \sum_{j=0}^{|\mathcal{P}^1|} (-1)^j \binom{|\mathcal{P}^1|}{j} = 0 \qquad (2.7)$$

whenever $\mathcal{P}^1$ is not empty. Therefore we need only consider the terms in which $\mathcal{P}^1$ is empty. This gives us

$$E(:\prod_{i=1}^n G_{t_{1,i}} : \prod_{i=1}^m G_{t_{2,i}}) = \sum_{\mathcal{P}=\mathcal{P}^2\cup\mathcal{P}^3} \prod_{k=1}^{|\mathcal{P}^2|} g(t_{\mathcal{P}^2_{k,1}}, t_{\mathcal{P}^2_{k,2}}) \prod_{k=1}^{|\mathcal{P}^3|} g(t_{\mathcal{P}^3_{k,1}}, t_{\mathcal{P}^3_{k,2}}). \qquad (2.8)$$

If $m < n$ and $m + n$ is even, $\mathcal{P}^1$ can not be empty. Thus we see that

$$E(:\prod_{i=1}^n G_{t_{1,i}} : \prod_{i=1}^m G_{t_{2,i}}) = 0 \qquad (2.9)$$

if $m < n$, i.e. that $:\prod_{i=1}^n G_{t_i}:$ is orthogonal to $\mathcal{I}_{n-1}$. Furthermore (2.8) also shows that

$$E(:\prod_{i=1}^n G_{t_{1,i}} : \prod_{i=1}^n G_{t_{2,i}}) = \sum_{\mathcal{P}=\{\mathcal{P}^2\}} \prod_{k=1}^{|\mathcal{P}^2|} g(t_{\mathcal{P}^2_{k,1}}, t_{\mathcal{P}^2_{k,2}}). \tag{2.10}$$

This can be written more suggestively as

$$E(:\prod_{i=1}^n G_{t_i} :: \prod_{i=1}^n G_{s_i} :) = \sum_{\pi} \prod_{k=1}^n g(t_i, s_{\pi(i)}), \tag{2.11}$$

where the sum goes over all permutations π of $\{1, \dots, n\}$.

The following generalization of (2.11) will be useful.

$$E(\prod_{j=1}^k : \prod_{i=1}^{n_j} G_{t_{j,i}} :) = \sum_{\mathcal{P}} \prod_{m=1}^{N/2} g(t_{\mathcal{P}_{m,1}}, t_{\mathcal{P}_{m,2}}) \tag{2.12}$$

where the sum runs over all pairings $\mathcal{P}$ of the set $\{(j,i);\, 1 \le j \le k;\, 1 \le i \le n_j\}$ such that if $\mathcal{P} = \{(\mathcal{P}_{m,1}, \mathcal{P}_{m,2});\, 1 \le m \le N/2\}$ where $N = \sum_{j=1}^k n_j$ and for any m we have $(\mathcal{P}_{m,1}, \mathcal{P}_{m,2}) = ((j,i),(j',i'))$ then $j \neq j'$. In other words, we never pair two $G_{t_{(\cdot,\cdot)}}$ terms from the same Wick product. The equation in (2.12) can be proved similarly to the proof of (2.11). Consider

$$E(:\prod_{i=1}^{n_1} G_{t_{1,i}} : \prod_{j=2}^k \prod_{i=1}^{n_j} G_{t_{j,i}}). \tag{2.13}$$

We can use the same proof as in (2.5) and (2.6) to see that the terms in $\{G_{t_{1,i}}\}_{i=1}^{n_1}$ that are paired with themselves contribute nothing to (2.13). Next we consider

$$E(:\prod_{i=1}^{n_1} G_{t_{1,i}} :: \prod_{i=1}^{n_2} G_{t_{2,i}} : \prod_{j=3}^k \prod_{i=1}^{n_j} G_{t_{j,i}}). \tag{2.14}$$

and see that this is also true for the pairings of $\{G_{t_{2,i}}\}_{i=1}^{n_1}$ with themselves. Proceeding recursively we get (2.12).

Using (2.12) we can now establish the expansion formula

$$\prod_{j=1}^k : \prod_{i=1}^{n_j} G_{t_{j,i}} := \sum_{(A_1,\dots,A_k)} \sum_{\mathcal{P}=\mathcal{P}(\cup_{j=1}^k A_j)} \prod_{m=1}^{|\mathcal{P}|} g(t_{\mathcal{P}_{m,1}}, t_{\mathcal{P}_{m,2}}) : \prod_{j=1}^k \prod_{i \in A_j^c} G_{t_{j,i}} : \tag{2.15}$$

where the first sum runs over all k-tuples of subsets $(A_1, \dots, A_k)$ with $A_j \subseteq \{(j,i);\, 1 \le i \le n_j\}$, and the second sum runs over all pairings $\mathcal{P}$ of $\cup_{j=1}^k A_j$ such that if $\mathcal{P} = \{(\mathcal{P}_{m,1}, \mathcal{P}_{m,2});\, 1 \le m \le |\mathcal{P}|\}$ and for any m we have $(\mathcal{P}_{m,1}, \mathcal{P}_{m,2}) = ((j,i),(j',i'))$, then $j \neq j'$. In other words, we never pair two indices from the same A_j. To verify (2.15), it suffices to show that both sides of (2.15) have the same L^2 inner product with all Wick products, and that can be done using (2.12).

CHAPTER 3

Wick power chaos processes

In this chapter we define what we mean by a Wick power chaos and a Wick power chaos process. We also obtain a critical theorem on the behavior of a Wick product chaos in the neighborhood of the diagonal, that is, the rate at which a Wick product chaos approaches a Wick power chaos. On a first reading of this paper, we recommend studying the present chapter up to the statement of Theorem 3.1, and then going on to the following chapter.

For a given 1-potential u^1 and some positive integer p let $\mu, \nu \in \mathcal{G}^p$, i.e. (1.5) is satisfied with $2n$ replaced by p. Since u^1 is positive definite we can define the inner product

$$\langle \mu, \nu \rangle_{(p)} = \int\int (u^1(x-y))^p \, d\mu(x)\, d\nu(y). \tag{3.1}$$

Denote $\|\mu\|^2_{(p)} = \langle \mu, \mu \rangle_{(p)}$. Let us also note that, trivially, $\|\mu_a\|_{(p)} = \|\mu\|_{(p)}$ for all $a \in R^m$.

LEMMA 3.1. *Let $\mu \in \mathcal{G}^p$, then for all $a_1, \ldots, a_p \in R^m$*

$$\int\int \prod_{i=1}^{p} u^1(x-y+a_i)\, d\mu(x)\, d\mu(y) \tag{3.2}$$

is continuous in $(a_1, \ldots, a_p)$ and is bounded by $\|\mu\|^2_{(p)}$.

PROOF. Let $a = (a_1, \ldots, a_p)$ and denote the integral in (3.2) by $V(a)$. Taking Fourier transforms we see that

$$V(a) - V(b) = \int\int \left(\prod_{j=1}^{p} e^{ia_j\lambda_j} - \prod_{j=1}^{p} e^{ib_j\lambda_j} \right) \prod_{j=1}^{p} \frac{1}{1+\psi(\lambda_j)} \, |\hat{\mu}(\sum_{j=1}^{p} \lambda_j)|^2 \, d\lambda.$$

Note that $\mu \in \mathcal{G}^p$ is equivalent to

$$\int\int \prod_{j=1}^{p} \frac{1}{1+\psi(\lambda_j)} \, |\hat{\mu}(\sum_{j=1}^{p} \lambda_j)|^2 \, d\lambda < \infty. \tag{3.3}$$

Therefore the Lemma follows from the dominated convergence theorem. □

Let $f_\delta(y)$ be a continuous positive symmetric function on $(y, \delta) \in R^m \times (0, 1]$ with support in the ball of radius δ and such that $\int f_\delta(y)\, dy = 1$. That is, f_δ is a smooth approximate identity. We assume that $f_\delta(\cdot) \le C/\delta^m$ for some constant C. Set $f_{x,\delta}(y) = f_\delta(y-x)$. By $u^1(\delta)$, for $\delta > 0$, we mean the 1-potential density evaluated at any element in R^m with absolute value equal to δ. We note the following simple estimate:

LEMMA 3.2. *Let u^1 be the 1-potential of a Lévy process in Class A, and assume that it is in L^p with respect to Lebesgue measure on $[-1,1]^m$. Then for all $b \in R^m$, $m = 1, 2$*

$$\int (u^1(x-b))^p f_\delta(x)\, dx \le C(u^1(\delta))^p. \tag{3.4}$$

for $\delta \le \delta_0$, for some δ_0 sufficiently small.

PROOF. Since u^1 is radially symmetric and $u^1(|\cdot|)$ is regularly varying at the origin, and hence effectively decreasing in a neighborhood of the origin, it is easy to see that

$$\begin{aligned} \int (u^1(x-b))^p f_\delta(x)\, dx &\le \frac{C}{\delta^m} \int_{|x|\le\delta} (u^1(x-b))^p\, dx \\ &\le \frac{C}{\delta^m} \int_{\xi\le\delta} (u^1(|\xi|))^p \xi^{m-1}\, d\xi \le C(u^1(\delta))^p. \end{aligned} \tag{3.5}$$

□

We now define the $2n$-th Wick power chaos. Let u^1 be the 1-potential of a Lévy process in R^m. Let $\theta, \phi \in \mathcal{G}^1$, with respect to u^1. We define $\{G_\rho, \rho \in \mathcal{G}^1\}$ to be the mean zero Gaussian process with covariance

$$E(G_\theta G_\phi) = \int\int u^1(x,y)\, d\theta(x)\, d\phi(y). \tag{3.6}$$

Set $\rho_\delta(dx') = f_\delta(x')\, dx'$ and $\rho_{x,\delta}(dx') = f_{x,\delta}(x')\, dx'$. It is clear that $\rho_{x,\delta}(dy) \in \mathcal{G}^1$. Let $G_{x,\delta} \stackrel{def}{=} G_{\rho_{x,\delta}}$ and consider the mean zero Gaussian process $\{G_{x,\delta}, (x,\delta) \in R^m \times (0,1]\}$ with covariance

$$\begin{aligned} E(G_{x,\delta} G_{y,\delta'}) &= \int\int u^1(x',y')\rho_{x,\delta}(dx')\rho_{y,\delta'}(dy') \\ &= \int\int u^1(x+x', y+y')\rho_\delta(dx')\rho_{\delta'}(dy') \\ &\stackrel{def}{=} u^1_{\delta,\delta'}(x,y). \end{aligned} \tag{3.7}$$

Since $\rho_\delta(dy) \in \mathcal{G}^1$, $u^1_{\delta,\delta'}(0) < \infty$.

It follows from (2.2) that the $2n$-th Wick product formed from $G_{x,\delta}$ satisfies

$$:G^{2n}_{x,\delta}: = \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} \frac{(2j)!}{j!2^j} (u^1_{\delta,\delta}(0))^j G^{2(n-j)}_{x,\delta} \tag{3.8}$$

and, by (2.11), that

$$E(:G^{2n}_{x,\delta}::G^{2n}_{y,\delta'}:) = (2n)!(u^1_{\delta,\delta'}(x-y))^{2n}. \tag{3.9}$$

We note, for later use, that it also follows from (2.11) that

$$E(:\prod_{i=1}^{2n} G_{v_i,\delta} :: \prod_{i=1}^{2n} G_{w_i,\delta'}:) = \sum_\pi \prod_{k=1}^{2n} u^1_{\delta,\delta'}(v_{\pi_k} - w_k) \tag{3.10}$$

where the sum runs over all permutations π of $\{1,\dots,2n\}$.

Let $\mu \in \mathcal{G}^{2n}$. It follows from (3.9) and Fubini's theorem that

$$
\begin{aligned}
&E\left(\int\!\!\int : G^{2n}_{x,\delta} :: G^{2n}_{y,\delta'} : \, d\mu(x)\, d\mu(y)\right) \\
&\quad = (2n!) \int\!\!\int (u^1_{\delta,\delta'}(x,y))^{2n}\, d\mu(x)\, d\mu(y) \qquad (3.11)\\
&\quad = (2n!) \int \dots \int \prod_{j=1}^{2n} u^1(v_j, w_j) \prod_{j=1}^{2n} \rho_{x,\delta}(dv_j)\rho_{y,\delta'}(dw_j)\, d\mu(x)\, d\nu(y)\\
&\quad = (2n!) \int \dots \int \prod_{j=1}^{2n} u^1(x+v_j, y+w_j) \prod_{j=1}^{2n} \rho_{\delta}(dv_j)\rho_{\delta'}(dw_j)\, d\mu(x)\, d\nu(y)\\
&\quad = (2n!) \int \dots \int \left(\int\!\!\int \prod_{j=1}^{2n} u^1(x-y+v_j-w_j)\, d\mu(x)\, d\nu(y)\right)\\
&\qquad\qquad \prod_{j=1}^{2n} \rho_{\delta}(dv_j)\rho_{\delta'}(dw_j).
\end{aligned}
$$

By Lemma 3.1 the double integral in parentheses immediately above is continuous in $(v_1 - w_1, \dots, v_{2n} - w_{2n})$ and goes to

$$\int\!\!\int \prod_{j=1}^{2n} u^1(x-y)\, d\mu(x)\, d\nu(y) \tag{3.12}$$

as $\sup_{1 \le j \le 2n} |v_j - w_j| \to 0$. Hence for any $\mu \in \mathcal{G}^{2n}$, the $2n$-th Wick power chaos

$$: G^{2n}\mu : \stackrel{def}{=} \lim_{\delta \to 0} \int : G^{2n}_{x,\delta} : \, d\mu(x) \tag{3.13}$$

exists as a limit in L^2 and furthermore

$$E(: G^{2n}\mu :: G^{2n}\nu :) = (2n!) \int\!\!\int (u^1(x,y))^{2n}\, d\mu(x)\, d\nu(y) \tag{3.14}$$

for all $\mu, \nu \in \mathcal{G}^{2n}$. For later use we also define

$$: G^{2n}_{\delta}\mu := \int : G^{2n}_{x,\delta} : \, d\mu(x). \tag{3.15}$$

We define a $2n$-th Wick power chaos process to be the stochastic process $\{: G^{2n}\mu_x : , x \in R^m\}$. This process induces a natural metric d on R^m which is given in (1.8).

Our use of the term chaos to describe $: G^{2n}\mu :$ and $\{: G^{2n}\mu_x :, x \in R^m\}$ is consistent with classical usage. A good reference that describes processes of this sort is [**1**], in which they are called $\mathcal{H}$-chaos processes. This reference contains many interesting results about these processes, some of which are used in this paper. However, since our definition and the representation (2.4), [**1**] are not easily seen to be the same, we show how they are related.

An alternate way to define Wick powers, $: Y^n :$ of a Gaussian random variable Y with mean zero, is by the generating function equation

$$\exp\left(\lambda Y - \frac{\lambda^2 EY^2}{2}\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : Y^n : . \tag{3.16}$$

One can check that this is the same as (3.8). When g is a normal random variable with mean zero and variance 1, $(N(0,1))$, $(1/\sqrt{n!}) : g^n :$ is the Hermite polynomial of g of order n and $: g^n :$ is the Hermite polynomial of degree n normalized so that it has leading coefficient 1.

Let $\{G(x), x \in S\}$ be a real valued Gaussian process with Let $\{G(x), x \in R^m\}$ be a real valued Gaussian process with Karhunen-Loeve expansion

$$G(x) = \sum_i g_i \phi_i(x) \tag{3.17}$$

where $\{g_i\}_{i=0}^{\infty}$ is an independent identically distributed sequence of $N(0,1)$ random variables. By (3.16) we see that

$$\begin{aligned} \exp\left(\lambda G(x) - \frac{\lambda^2 EG^2(x)}{2}\right) &= \prod_i \exp\left(\lambda \phi_i(x) g_i - \frac{\lambda^2 \phi_i^2(x)}{2}\right) \qquad (3.18) \\ &= \prod_i \sum_{n=0}^{\infty} \frac{\lambda^n \phi_i^n(x)}{n!} : g_i^n : \end{aligned}$$

which implies that

$$:G^m(x):= \sum_{i_1,\ldots,i_m} \phi_{i_1}(x) \cdots \phi_{i_m}(x) \prod_{j \geq 1} H_{m_j(i_1,\ldots,i_m)}(g_j) \tag{3.19}$$

where H_m is the Hermite polynomial of degree m normalized so that it has leading coefficient 1 and $m_j(i_1,\ldots,i_m) = \sum_{r=1}^m I(i_r = j)$. Using this with an obvious change of notation we see that

$$\begin{aligned} &\int : G_{x,\delta}^{2n} : \, d\mu_y(x) \\ &\quad = \sum_{i_1,\ldots,i_{2n}} \int \phi_{i_1}(x,\delta) \cdots \phi_{i_{2n}}(x,\delta)\, d\mu_y(x) \prod_{j \geq 1} H_{m_j(i_1,\ldots,i_{2n})}(g_j) \qquad (3.20) \end{aligned}$$

For fixed $y \in R^m$ the $\mathcal{H}$-chaos random variables considered in [**1**] are the closure in L^2 of terms such as the right-hand side of (3.20).

In current parlance chaos generally has a different meaning than to describe processes such as (3.20). However, its usage in our context is well established. It dates back to Wiener's 1938 paper, *The homogeneous chaos*, [**26**], in which it is used to describe multiple stochastic integrals with respect to Brownian motion. (The expression on the right-hand side of (3.20) is a discrete version of such an integral). Wiener was motivated by problems in statistical mechanics. The development of his ideas in this direction is discussed Masani's biography of Wiener, [**20**], pages 149-151. Further references as to how his ideas were developed can be found in [**20**] and [**1**].

The next theorem is the principle result in this chapter.

THEOREM 3.1. *Let X be a Lévy process in Class A and let $G_{x,\delta}$ be a Gaussian process associated with X as defined in (3.7). Let $\mu \in \mathcal{G}^{2n}$ and $k \leq n$, then for all $\delta > 0$*

$$\begin{aligned} &\sup_{|x_i| \leq \epsilon} \| \int : \prod_{i=1}^{2k} G_{x+x_i,\delta} : \, d\mu(x) - : G_\delta^{2k} \mu : \|_2 \qquad (3.21) \\ &\quad = o((u^1(\epsilon))^{-(n-k)}) \qquad \text{as } \epsilon \to 0. \end{aligned}$$

The following lemmas are used in the proof of Theorem 3.1 although the full strength of Lemma 3.4 is not used until the proof of Theorem 5.1.

LEMMA 3.3. *Let μ be a positive measure on R^m and $f : R^m \to [0,\infty)$ be such that*

$$\int\int f(x-y))\,d\mu(x)\,d\mu(y) < \infty \tag{3.22}$$

Then for each $r > 0$ there exists a positive decreasing convex function g on $[0,\infty)$ such that $\lim_{x\downarrow 0} g(x) = \infty$, $x^r g(x)$ is increasing for $x \in [0,\infty)$ and

$$\int\int g(|x-y|)f(x-y))\,d\mu(x)\,d\mu(y) < \infty. \tag{3.23}$$

%noindent**Proof**

PROOF. (This is elementary without the condition that $|x|^r g(x)$ is increasing). Let $\{n_k\}_{k=1}^\infty$ be such that $2n_{k+1} \le n_k$ and

$$\int\int_{|x-y|\le n_k} f(x-y))\,d\mu(x)\,d\mu(y) \le 2^{-2^k}. \tag{3.24}$$

If $\lim_{k\to\infty} n_k > 0$ the assertion is trivial. Otherwise, define $g(n_k) = 2^{ka}$, $k = 1,2,\dots$ where $a < \log_2(r+2)-1$, and let $g(x)$ be its linear extension. (Let $g(x) = g(n_1)$ for $x > n_1$.) Since g is piecewise differentiable to show that $x^r g(x)$ is increasing it is enough to show that the derivative of $x^r g(x)$ is positive for $x > 0$. This is implied by the following inequality:

$$g(n_k) \ge \frac{(g(n_{k+1}) - g(n_k))n_k}{(n_k - n_{k+1})r} \qquad k = 1,2\dots \tag{3.25}$$

which follows from the definition of a. □

LEMMA 3.4. *Let X be a Lévy process in Class A with 1-potential u^1 and let $\mu \in \mathcal{G}^{2n}$. Then for all $a_1,\dots,a_k$ in R^m and $2 \le q \le 2n$*

$$\begin{aligned} I_{b,c} &\overset{def}{=} \sup_{a_i} \int\int |\Delta^2_{b,c} u^1(x-y+a_1)| \prod_{i=2}^q u^1(x-y+a_i)\,d\mu(x)\,d\mu(y) \\ &= o((u^1(|b|))^{-(n-q/2)}(u^1(|c|))^{-(n-q/2)}) \qquad \text{as } |b|,|c| \to 0 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &\sup_{a_i} \int\int |\Delta_b u^1(x-y+a_1)| \prod_{i=2}^q u^1(x-y+a_i)\,d\mu(x)\,d\mu(y) \\ &\qquad = o((u^1(|b|))^{-(2n-q)\wedge n}) \qquad \text{as } |b| \to 0. \end{aligned} \tag{3.27}$$

Furthermore, for $3 \le q \le 2n$

$$\begin{aligned} &\sup_{a_i} \int\int |\Delta_b u^1(x-y+a_1)||\Delta_c u^1(x-y+a_2)| \prod_{i=3}^q u^1(x-y+a_i) d\mu(x) d\mu(y) \\ &\quad = o((u^1(|b|))^{-(n-q/2)}(u^1(|c|))^{-(n-q/2)}) \quad \text{as } |b|,|c| \to 0. \end{aligned} \tag{3.28}$$

Proof. We prove (3.26) and (3.27). The proof of (3.28) is similar. We begin with (3.26). Without loss of generality we assume that $|b| \le |c|$. By the multiple Hölder inequality and Lemma 3.1

$$I_{b,c} \le C \sup_{a_1} \left(\int\int |\Delta^2_{b,c} u^1(x-y)|^p \, d\mu_{a_1}(x)\, d\mu(y) \right)^{1/p} \|\mu\|^{q-1}_{(2n)} \tag{3.29}$$

where $p = 2n/(2n-q+1)$.

By hypothesis $\mu \in \mathcal{G}^{2n}$. This implies, as we remarked after Theorem 1.1, that Lebesgue measure on $[-1,1]^m$ is also in $\mathcal{G}^{2n}$. Let $z \in R^m$. Since $u^1(|z|)$ is regularly varying at zero, we see that

$$|z|u^1(|z|))^{(n-(q/2))} = |z|^{\delta_1} L(|z|) \tag{3.30}$$

where $\delta_1 > 0$ and $L(|z|)$ is slowly varying at zero. The full range of δ_1 is $(0,1]$. $\delta_1 > 1$ is not possible because we assume $u^1(0) = \infty$. Let g be a function as determined in Lemma 3.3 for which (3.23) holds with $f(x-y) = (u^1(|x-y|))^{2n}$ and with $r = \delta_1 p$. Set $\phi(|z|) = (u^1(|z|))^{-(n-(q/2))} g(|z|)^{-1/2p}$. Since $g(|z|)$ is decreasing as $|z|$ increases and $u^1(|z|)$ is asymptotic to a decreasing function at the origin, for our purposes, we can assume that $\phi(|z|)$ is increasing on $[0, z_1]$ for some $z_1 > 0$.

Consider

$$|z|/\phi(|z|) = |z|^{\delta_1/2} L(|z|) \left(g(|z|)|z|^{\delta_1 p} \right)^{1/(2p)}. \tag{3.31}$$

Since $|z|^{\delta_1/2}L(|z|)$ is regularly varying at zero with a positive index it is asymptotic to an increasing function on $[0, z_2]$, for some $z_2 > 0$. Therefore, we can assume that $|z|^{\delta_1/2}L(|z|)$ is increasing on $[0, z_2]$. Also $g(|z|)|z|^{\delta_1 p}$ is increasing by Lemma 3.3. Thus $|z|/\phi(|z|)$ is increasing in $|z|$ for $|z| \le z_2$. Finally, we choose some $z_0 \le (z_1 \wedge z_2)$ so that $|z_0|/\phi(|z_0|) \le 1$. Clearly, to establish (3.26), we need only consider $|z|$ close to zero. Let s_0 be as given just before (1.15) and set $\tilde{c}_0 = (s_0 \wedge z_0)/6$. Since $|z|/\phi(|z|)$ is increasing in $|z|$ for $|z| \le \tilde{c}_0$, it follows from (1.16) that

$$|\Delta^2_{b,c} u^1(s)| \le C\phi(|b|)\phi(|c|) \frac{u^1(|s|)}{\phi^2(|s|)} \qquad |b|, |c| \le \frac{|s|}{4} \le \tilde{c}_0. \tag{3.32}$$

Without loss of generality we take $c \le \tilde{c}_0$.

Let $D = \{(x,y);\, |x-y| \ge 4\tilde{c}_0\}$. By hypothesis $(u^1(r))' \vee (u^1(r))''$ is bounded on $\{(x,y);\, |x-y| \ge 2\tilde{c}_0\}$. Therefore

$$\sup_{a_1} \int\int_D |\Delta^2_{b,c} u^1(x-y)|^p \, d\mu_{a_1}(x)\, d\mu(y) \le C|b|^p|c|^p. \tag{3.33}$$

It follows from (3.30) that $|b||c| = o((u^1(|b|))^{-(n-q/2)}(u^1(|c|))^{-(n-q/2)})$as $|b|, |c|$ $\to 0$. Thus we see that (3.26) holds if the range of integration is restricted to D.

Consider (3.29) integrated over D^c. Furthermore, without loss of generality we take $|c| \le \tilde{c}_0$. We decompose D^c into $A_1 \cup A_2$ where $A_1 = \{(x,y);\, 4|c| \le |x-y| \le 4\tilde{c}_0\}$ and $A_2 = \{(x,y);\, 4|c| \ge |x-y|\}$, and obtain (3.26) separately for the integrals over each of these two regions.

Using (3.32), with s replaced by $x-y$, we have that

$$\begin{aligned} &\sup_{a_1} \int\int_{A_1} |\Delta^2_{b,c} u^1(x-y)|^p \, d\mu_{a_1}(x)\, d\mu(y) \\ &\quad \le C\phi^p(|b|)\phi^p(|c|) \sup_{a_1} \int\int \left| \frac{u^1(|x-y|)}{\phi^2(|x-y|)} \right|^p d\mu_{a_1}(x)\, d\mu(y) \\ &\quad \le C\phi^p(|b|)\phi^p(|c|) \end{aligned} \tag{3.34}$$

$$\begin{aligned}&\sup_{a_1}\iint_{A_1} g(|x-y|)(u^1(|x-y|))^{2n}\,d\mu_{a_1}(x)\,d\mu(y)\\&=o((u^1(|b|)u^1(|c|))^{-(n-q/2)p})\\&\quad\sup_{a_1}\iint g(|x-y|)(u^1(|x-y|))^{2n}\,d\mu_{a_1}(x)\,d\mu(y).\end{aligned}$$

By construction $g(|x|)$ is convex and hence is a positive definite function on R^m. This can be shown similarly to the result in R^1 starting with the fact that for $a, \xi \in R^m$, $1-(|\xi|/|a|)$ is the characteristic function of a measure on R^m with density $C(1-\cos(a\cdot x))/(|a||x|^{m+1})$, for the appropriate constant C. This is easily seen using polar coordinates and the corresponding result in R^1. (See e.g. [**7**], pg.478). Furthermore, since the product of positive definite functions is positive definite $g(|x|)(u^1(|x|))^{2n}$ is a positive definite function on R^m. It now follows from Lemmas 3.1 and 3.3 that

$$\begin{aligned}&\iint g(|x-y|)(u^1(|x-y|))^{2n}\,d\mu_{a_1}(x)\,d\mu(y)\\&\qquad\le\iint g(|x-y|)(u^1(|x-y|))^{2n}\,d\mu(x)\,d\mu(y)<\infty.\end{aligned}\tag{3.35}$$

Therefore, we see from (3.34) and (3.35) that

$$\begin{aligned}&\sup_{a_1}\left(\iint_{A_1}|\Delta^2_{b,c}u^1(x-y)|^p\,d\mu_{a_1}(x)\,d\mu(y)\right)^{1/p}\\&\qquad=o((u^1(|b|)u^1(|c|))^{-(n-q/2)}).\end{aligned}\tag{3.36}$$

We next consider the integral over A_2. Since

$$|\Delta^2_{b,c}u^1(x-y)|\le|\Delta_b u^1(x-y+c)|+|\Delta_b u^1(x-y)|\tag{3.37}$$

it suffices to consider separately

$$\sup_{a_1}\iint_{A_2}|\Delta_b u^1(x-y+c)|^p\,d\mu_{a_1}(x)\,d\mu(y)\tag{3.38}$$

and

$$\sup_{a_1}\iint_{A_2}|\Delta_b u^1(x-y)|^p\,d\mu_{a_1}(x)\,d\mu(y).\tag{3.39}$$

To bound (3.39) we write $A_2=B_1\cup B_2$ where $B_1=\{(x,y);\,4|c|\ge|x-y|\ge4|b|\}$ and $B_2=\{(x,y);\,4|b|\ge|x-y|\}$, (recall that $|b|\le|c|$), and bound the integral separately on each region.

To handle the integral on B_1 we note that since $|b|/\phi(|b|)$ is increasing for $|b|\le\tilde c_0$ and (1.15) holds on B_1 we have, as above, that

$$|\Delta_b u^1(x-y)|\le C\phi(|b|)\frac{u^1(|x-y|)}{\phi(|x-y|)}\tag{3.40}$$

on B_1. On the other hand, since $\phi(4r)$ is increasing on $0\le r\le\tilde c_0$, we also have that

$$4|c|\ge|x-y|\Longrightarrow\phi(4|c|)\ge\phi(|x-y|)\tag{3.41}$$

on B_1. Combining (3.40) and (3.41) we see that

$$\sup_{a_1}\iint_{B_1}\left|\Delta_b u^1(x-y)\right|^p\,d\mu_{a_1}(x)\,d\mu(y)\tag{3.42}$$

$$\leq C\phi^p(|b|)\sup_{a_1}\iint_{B_1}\left|\frac{u^1(x-y)}{\phi(|x-y|)}\right|^p d\mu_{a_1}(x)\,d\mu(y)$$
$$\leq C\phi^p(|b|)\phi^p(4|c|)\sup_{a_1}\iint\left|\frac{u^1(x-y)}{\phi^2(|x-y|)}\right|^p d\mu_{a_1}(x)\,d\mu(y).$$

Proceeding as in (3.34) we get (3.36) but with the range of integration B_1.

To handle the integral on B_2 we first note that since

$$|\Delta_b u^1(x-y)| \leq |u^1(x-y+b)| + |u^1(x-y)| \tag{3.43}$$

it suffices to obtain a bound less than or equal to the last line of (3.42) for both

$$\sup_{a_1}\iint_{B_2}|u^1(x-y+b)|^p\,d\mu_{a_1}(x)\,d\mu(y) \tag{3.44}$$

and

$$\sup_{a_1}\iint_{B_2}|u^1(x-y)|^p\,d\mu_{a_1}(x)\,d\mu(y). \tag{3.45}$$

The last integral, (3.45) is easily bounded as above using (3.41) for both $|b|$ and $|c|$. To bound (3.44) we first note that

$$\{(x,y);\,4|b|\geq|x-y+b|\}\subset\{(x,y);\,5|b|\geq|x-y|\}. \tag{3.46}$$

Consequently, translating by $-b$ in the x variable in (3.44) we obtain

$$\begin{aligned}
&\sup_{a_1}\iint_{B_2}|u^1(x-y+b)|^p\,d\mu_{a_1}(x)\,d\mu(y)\\
&\qquad\leq\sup_{a_1}\iint_{(B_2)_b}|u^1(x-y)|^p\,d\mu_{a_1+b}(x)\,d\mu(y)\\
&\qquad\leq\sup_{a_1}\iint_{5|b|\geq|x-y|}|u^1(x-y)|^p\,d\mu_{a_1}(x)\,d\mu(y)\\
&\qquad\leq C\phi^{2p}(5|b|)\sup_{a_1}\iint\left|\frac{u^1(x-y)}{\phi^2(|x-y|)}\right|^p d\mu_{a_1}(x)\,d\mu(y).
\end{aligned} \tag{3.47}$$

where, at the last step we use (3.41) with $4|c|$ replaced by $5|b|$.

Finally to bound (3.38) we translate by $-c$ in the x variable and obtain

$$\begin{aligned}
&\sup_{a_1}\iint_{A_2}|\Delta_b u^1(x-y+c)|^p\,d\mu_{a_1}(x)\,d\mu(y)\\
&\qquad\leq\sup_{a_1}\iint_{(A_2)_c}|\Delta_b u^1(x-y)|^p\,d\mu_{a_1+c}(x)\,d\mu(y)\\
&\qquad\leq\sup_{a_1}\iint_{\{5|c|\geq|x-y|\}}|\Delta_b u^1(x-y)|^p\,d\mu_{a_1}(x)\,d\mu(y)
\end{aligned} \tag{3.48}$$

since

$$(A_2)_c=\{(x,y);\,4|c|\geq|x-y+c|\}\subset\{(x,y);\,5|c|\geq|x-y|\}. \tag{3.49}$$

Note that (3.48) is the same as (3.39), except that in the region of integration $4|c|$ has been replaced by $5|c|$. Going over the previous arguments we easily see that the methods used to bound (3.39) also work for (3.48). This completes the proof of (3.26).

The proof of (3.27) is similar to the proof of (3.26) except that we take $\phi(|b|)=(u^1(|b|))^{-(2n-q)\wedge n}g(|b|)^{-1/p}$. Note that the requirement that $|b|/\phi(|b|)$ is increasing necessitates the condition that the exponent of $u^1(|b|)$, in the definition of $\phi(|b|)$ not be smaller than $-n$. □

Proof of Theorem 3.1. Note that the left-hand side of (3.21)

$$\leq \sum_{j=1}^{2k} \sup_{|x_i|\leq\epsilon} \| \int \left(:\prod_{i=1}^{j} G_{x+x_i,\delta} \prod_{i=j+1}^{2k} G_{x,\delta} : - :\prod_{i=1}^{j-1} G_{x+x_i,\delta} \prod_{i=j}^{2k} G_{x,\delta} : \right) d\mu(x) \|_2. \tag{3.50}$$

The square of each L^2 norm in (3.50) is of the form

$$\begin{aligned} &\| \int :\prod_{i=1}^{2k-1} G_{x+c_i,\delta} \left(G_{x+c_{2k},\delta} - G_{x,\delta}\right): \, d\mu(x)\|_2^2 \qquad (3.51)\\ &= \int\int E\left(:\prod_{i=1}^{2k-1} G_{x+c_i,\delta} \left(G_{x+c_{2k},\delta} - G_{x,\delta}\right): \right.\\ &\qquad \left. :\prod_{i=1}^{2k-1} G_{y+c_i,\delta} \left(G_{y+c_{2k},\delta} - G_{y,\delta}\right): \right) d\mu(x)\, d\mu(y) \end{aligned}$$

where $|c_i| \leq \epsilon$, $i = 1, \ldots, 2k$ and we use the fact that Q_{2k} is a linear operator, (see (2.3)). In light of (3.10) the expectation of the Wick products in (3.51) is a sum of two types of terms. One type is of the form

$$\prod_{i=1}^{2k-1} u^1_{\delta,\delta}(x-y+a_j)\tilde{\Delta}^2_{c_{2k}} u^1_{\delta,\delta}(x-y) \tag{3.52}$$

which occurs when the Gaussian process $(G_{x+c_{2k},\delta} - G_{x,\delta})$ is paired with $(G_{y+c_{2k},\delta} - G_{y,\delta})$, where $\tilde{\Delta}^2_b u^1_{\delta,\delta}(x-y) \stackrel{def}{=} 2u^1_{\delta,\delta}(x-y) - u^1_{\delta,\delta}(x-y+b) - u^1_{\delta,\delta}(x-y-b)$. The other type is of the form

$$\prod_{i=1}^{2k-2} u^1_{\delta,\delta}(x-y+a_j)\Delta_{c_{2k-1}} u^1_{\delta,\delta}(x-y+d_1)\Delta_{c_{2k}} u^1_{\delta,\delta}(x-y+d_2) \tag{3.53}$$

which occurs when $(G_{x+c_{2k},\delta} - G_{x,\delta})$ is not paired with $(G_{y+c_{2k},\delta} - G_{y,\delta})$. Here $|a_i|$, $i = 1, \ldots, 2k-2$, $|d_1|$ and $|d_2|$ are all less than or equal to 2ϵ. Note that

$$|\tilde{\Delta}^2_b u^1_{\delta,\delta}(x-y)| = |\Delta^2_{b,b} u^1_{\delta,\delta}(x-y+b)|. \tag{3.54}$$

Using (3.52) and (3.53) we can estimate the terms in (3.51). In both (3.52) and (3.53) we replace $u^1_{\delta,\delta}$ by the second line in (3.7) and interchange the order of integration, integrating first with respect to $d\mu(x)\, d\mu(y)$. We can then use Lemmas 3.1 and 3.4 along with (3.54) to obtain Theorem 3.1. (Actually we get 2ϵ on the right-hand side of (3.21) but we can replace this by ϵ since u^1 is assumed to be regularly varying at zero). □

We also need an extended version of Theorem 3.1.

Theorem 3.2. *Let X be a Lévy process in Class A and let $G_{x,\delta}$ be a Gaussian process associated with X as defined in (3.7). Let $\{G_{(j),x,\delta}\}_{j=1}^m$ be independent copies of $G_{x,\delta}$. Let $\{D_k(x,\epsilon), x \in R^m\}$ be a stochastic process independent* *of* $\{G_{(j),x,\delta}\}_{j=1}^m$ *for which* $\|D_k(x,\epsilon)\|_2 = O((u^1(\epsilon))^k)$, $k \geq 1$, *and* $D_0 \equiv 1$. *Let* $n = \sum_{j=1}^m n_j$ *and* $\mu \in \mathcal{G}^{2\sigma}$ *where* $n \geq 1$ *and* $n+k \leq \sigma$, *then for all* $\delta > 0$

$$\sup_{|x_i|\leq\epsilon} \| \int \left(\prod_{j=1}^m : \prod_{i=1}^{2n_j} G_{(j),x+x_i,\delta} : - \prod_{j=1}^m : G^{2n_j}_{(j),x,\delta} : \right) D_k(x,\epsilon)\, d\mu(x) \|_2 \tag{3.55}$$
$$= o((u^1(\epsilon))^{-(\sigma-(n+k))}) \qquad \text{as } \epsilon \to 0.$$

PROOF. The left-hand side of (3.55)

$$\leq \sum_{p=1}^m \sup_{|x_i|\leq\epsilon} \| \int \prod_{j=1}^{p-1} : \prod_{i=1}^{2n_j} G_{(j),x+x_i,\delta} : \prod_{j=p+1}^m : G^{2n_j}_{(j),x,\delta} : \tag{3.56}$$
$$\left(: \prod_{i=1}^{2n_p} G_{(p),x+x_i,\delta} : - : G^{2n_p}_{(p),x,\delta} : \right) D_k(x,\epsilon)\, d\mu(x) \|_2.$$

Proceeding to expand the difference term as in (3.50) we see that the square of each L^2 norm in (3.56) is bounded above by the sum of terms of the form

$$\int\int \prod_{j=1}^{p-1}\prod_{i=1}^{2n_j} u^1_{\delta,\delta}(x-y+a_{i_j}) \prod_{j=p+1}^m (u^1_{\delta,\delta}(x-y))^{2n_j}$$
$$E : \prod_{i=1}^{2n_p-1} G_{x+c_i,\delta} \left(G_{x+c_{2n_p},\delta} - G_{x,\delta} \right) :: \prod_{i=1}^{2n_p-1} G_{y+c_i,\delta} \left(G_{y+c_{2n_p},\delta} - G_{y,\delta} \right) :$$
$$E(D_k(x,\epsilon)D_k(y,\epsilon))\, d\mu(x)\, d\mu(y)$$

where $|c_\cdot| \leq \epsilon$. Here we have already taken the expectation with respect to the probability spaces supporting $\{G_{(j),x,\delta}\}$, for $j = 1,\ldots,m$, excluding $j = p$.

Using the Schwarz inequality on the D_k term in the expression above and the argument preceding (3.52) on the expectation of the Wick products, we see that the left-hand side of (3.55) is bounded above by the sum of a finite number of terms of the form

$$C \int\int \prod_{i=1}^{2n-1} u^1_{\delta,\delta}(x-y+a_j)\tilde{\Delta}^2_{c_{2n}} u^1_{\delta,\delta}(x-y)(u^1(\epsilon))^{2k}\, d\mu(x)\, d\mu(y)$$

or

$$C \int\int \prod_{i=1}^{2n-2} u^1_{\delta,\delta}(x-y+a_j)\Delta_{c_{2n-1}} u^1_{\delta,\delta}(x-y+d_1)\Delta_{c_{2n}} u^1_{\delta,\delta}(x-y+d_2)$$
$$(u^1(\epsilon))^{2k}\, d\mu(x)\, d\mu(y)$$

where $|c_\cdot| \leq \epsilon$, $|a_\cdot| \leq 2\epsilon$ and $|d_\cdot| \leq 2\epsilon$. These are precisely terms of the form we dealt with in the proof of Theorem 3.1. The same argument used there completes the proof of this theorem. □

In Chapter 5 we will use the following variation of Lemma 3.4:

LEMMA 3.5. *Let* X *be a Lévy process in Class A with 1-potential* u^1 *and let* $\mu \in \mathcal{G}^{2n}$. *Assume further that*

$$\sup_x | \int (u^1(x-y))^n\, d\mu(y)| < \infty. \tag{3.57}$$

Then for all $a_0, a_1, \dots, a_k$ *in* R^m *and* $3 \le q \le 2n$

$$\begin{aligned}\tilde I_{b,c} &\stackrel{def}{=} \sup_{a_i} \int\!\!\int u^1(x-a_0)|\Delta^2_{b,c} u^1(x-y+a_1)| \prod_{i=3}^{q} u^1(x-y+a_i)\, d\mu(x)\, d\mu(y) \\ &= o((u^1(|b|))^{-(n-q/2)}(u^1(|c|))^{-(n-q/2)}) \qquad as\ |b|,|c|\to 0 \end{aligned} \tag{3.58}$$

and

$$\begin{aligned}&\sup_{a_i} \int\!\!\int u^1(x-a_0)|\Delta_b u^1(x-y+a_1)| \prod_{i=3}^{q} u^1(x-y+a_i)\, d\mu(x)\, d\mu(y) \\ &= o((u^1(|b|))^{-(2n-q)\wedge n}) \qquad as\ |b|\to 0. \end{aligned} \tag{3.59}$$

Furthermore, for $4 \le q \le 2n$

$$\begin{aligned}&\sup_{a_i} \int\!\!\int u^1(x-a_0)|\Delta_b u^1(x-y+a_1)||\Delta_c u^1(x-y+a_2)| \\ &\qquad \prod_{i=4}^{q} u^1(x-y+a_i) d\mu(x) d\mu(y) \\ &= o((u^1(|b|))^{-(n-q/2)}(u^1(|c|))^{-(n-q/2)}) \quad as\ |b|,|c|\to 0. \end{aligned} \tag{3.60}$$

Proof. We prove (3.58). The proofs of (3.59) and (3.60) are similar. By the multiple Hölder inequality and Lemma 3.1

$$\begin{aligned}\tilde I_{b,c} &\le C \sup_{a_1,a_2} \left(\int\!\!\int \left(u^1(x-a_1)u^1(x-y+a_2)\right)^n\, d\mu(x)\, d\mu(y)\right)^{1/n} \\ &\qquad \left(\int\!\!\int |\Delta^2_{b,c} u^1(x-y)|^p\, d\mu_{a_1}(x)\, d\mu(y)\right)^{1/p} \|\mu\|^{q-3}_{(2n)} \end{aligned} \tag{3.61}$$

where $p = 2n/(2n-q+1)$. Integrating first with respect to y we see that

$$\begin{aligned}&\int\!\!\int \left(u^1(x-a_1)u^1(x-y+a_2)\right)^n\, d\mu(x)\, d\mu(y) \\ &\qquad \le \sup_x \left(\int (u^1(x-y))^n\, d\mu(y)\right)^2 \end{aligned} \tag{3.62}$$

which is finite by hypothesis. Thus we get

$$\tilde I_{b,c} \le C \sup_{a_1} \left(\int\!\!\int |\Delta^2_{b,c} u^1(x-y)|^p\, d\mu_{a_1}(x)\, d\mu(y)\right)^{1/p} \|\mu\|^{q-3}_{(2n)}. \tag{3.63}$$

This is precisely the term in (3.29). Thus we get (3.58) just as we obtained (3.26). □

CHAPTER 4

Isomorphism theorem

Let $\rho \in \mathcal{G}^1$ be a compactly supported probability measure. As usual we set

$$P^\rho(\cdot) = \int P^x(\cdot)\, d\rho(x). \tag{4.1}$$

Recall that λ is a mean-1 exponential random variable, which is independent of X. We define the product measure

$$P^\rho_\lambda(\cdot) = P^\rho P_\lambda(\cdot) \tag{4.2}$$

and use E^ρ_λ to denote expectation with respect to this measure. P^ρ_λ is the probability measure of the Lévy process X, with initial distribution given by ρ which is killed at the exponential time λ. We denote expectation with respect to the chaos processes by E_G.

Let f denote a bounded, strictly positive, uniformly continuous integrable function on R^m and let $f \cdot dx$ denote the measure on R^m with density function f.

We now state an isomorphism theorem which relates the Wick power chaos $: G^{2n}\mu :$ and the renormalized intersection local time $\gamma_n(\mu)$. This isomorphism theorem is the main technical result of this paper. Its proof occupies the rest of this chapter. Immediately after stating this theorem we will define the terms $(: G^{2(n-k)} : \times \mathcal{L}_k)(\mu.)$ which appear in it. They will be defined in terms of the Lévy process X and the associated Gaussian process G defined in the previous chapter. We eventually show that $\mathcal{L}_n\mu = n!\gamma_n(\mu)$, P^ρ_λ *a.s.*

Theorem 4.1. *Let X be a Lévy process in Class A and let $\{\mu_i\}_{i=1}^\infty$ be sequence of finite positive measures in $\mathcal{G}^{2n}$. Then, for any compactly supported measure $\rho \in \mathcal{G}^1$ and $\mathcal{C}$ measurable non-negative function F on R^∞*

$$\begin{aligned} E_G E^\rho_\lambda &\left(F\left(\sum_{k=0}^n \binom{n}{k} \frac{1}{2^{n-k}} (: G^{2(n-k)} : \times \mathcal{L}_k)(\mu.) \right) f(X_\lambda) \right) \\ &= E_G \left(F\left(\frac{1}{2^n} : G^{2n}\mu. : \right) G_\rho G_{f\cdot dx} \right) \end{aligned} \tag{4.3}$$

where $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞.

In order to define the terms $(: G^{2(n-k)} : \times \mathcal{L}_k)(\mu.)$ we first define the 'chain factors'

$$ch^\epsilon_j \overset{def}{=} \int \prod_{k=2}^j u^1(x_k - x_{k-1}) \prod_{i=1}^j f_\epsilon(x_i)\, dx_i \tag{4.4}$$

for $j \geq 2$, and set $ch_1^\epsilon = 1$. (The reason for the name 'chain factor' will become clear in the course of proving Theorem 4.1). Let

$$B_{n,k}^\epsilon = \frac{n!}{k!} \sum_{\substack{j_1,\ldots,j_k \\ \sum_{b=1}^k j_b = n}} \prod_{b=1}^k ch_{j_b}^\epsilon. \tag{4.5}$$

By convention, we set $B_{0,0}^\epsilon = 1$ and $B_{n,0}^\epsilon = B_{0,n}^\epsilon = 0$, $n = 1, \ldots$. We note that $\{B_{j,k}^\epsilon\}_{j,k=0}^n$ is a lower triangular matrix with $B_{j,j}^\epsilon = 1$ for all $0 \leq j \leq n$. Let

$$L^{x,\epsilon} = \int_0^\lambda f_\epsilon(X(t) - x)\, dt. \tag{4.6}$$

We recursively define $\mathcal{L}_n^{x,\epsilon}$, $n = 0, 1, 2, \ldots$ by

$$(L^{x,\epsilon})^n = \sum_{k=0}^n B_{n,k}^\epsilon \mathcal{L}_k^{x,\epsilon}. \tag{4.7}$$

It is easy to see that $\mathcal{L}_0^{x,\epsilon} = 1$ and $\mathcal{L}_n^{x,\epsilon}$ is a polynomial of degree n in $L^{x,\epsilon}$ with leading term $(L^{x,\epsilon})^n$. Furthermore for each $\epsilon > 0$, $\mathcal{L}_k^{x,\epsilon}$ is continuous almost surely.

We now define the terms $(: G^{2(n-k)} : \times \mathcal{L}_k)(\mu.)$ which appear in the isomorhism theorem as

$$(: G^{2(n-k)} : \times \mathcal{L}_k)(\mu) \stackrel{def}{=} \lim_{\epsilon \to 0} \int : G_{x,\epsilon}^{2(n-k)} : \mathcal{L}_k^{x,\epsilon}\, d\mu(x). \tag{4.8}$$

In Lemma 4.2 below we will show that the limit in (4.8) exists in $L^2(f(X_\lambda)\, dP_\lambda^\rho\, dP_G)$ for each $k = 0, 1, \ldots, n$. Here we use the convention that $: G_{x,\epsilon}^0 := 1$. In particular

$$\mathcal{L}_n \mu \stackrel{def}{=} \lim_{\epsilon \to 0} \int \mathcal{L}_n^{x,\epsilon}\, d\mu(x) \tag{4.9}$$

exists in $L^2(f(X_\lambda)\, dP_\lambda^\rho)$. $\mathcal{L}_n \mu$ is a renormalized n-fold self-intersection local time which we will show in the following chapter can be taken to be $n! \gamma_n(\mu)$.

Our isomorphism therem, Theorem 4.1, will be derived as a consequence of the much simpler isomorphism theorem, Theorem 2.2, [**15**] which relates continuous additive functionals of strongly symmetric Markov processes to second order Gaussian chaos processes. For the precise hypotheses of this theorem we refer the reader to [**15**]. It does apply to the Lévy processes X in Class A that we are considering in this paper. We use L_t^μ to denote the continuous additive functional of X with Revuz measure μ and $Rev(X)$ to denote the class of Revuz measures of X.

Let $\mu \in \mathcal{G}^2$. To simplify the notation and in keeping with the notation of [**15**] we will often denote the second order Wick power chaos $: G^2 \mu :$, defined in (3.13), by $H(\mu)$. $H(\mu)$ is the second order Gaussian chaos associated with L_λ^μ in [**15**]. Let $\mathcal{G}_F$ denote the set of finite measures in $\mathcal{G}$. It follows by Hölder's inequality, that, $\mathcal{G}_F^j \subseteq \mathcal{G}_F^k$ for $j \leq k$.

The following is Theorem 2.2, [**15**] adapted to the needs of this paper:

THEOREM 4.2. *Let $\{\mu_i\}_{i=1}^\infty$ be a sequence of finite measures in $\mathcal{G}^2 \cap Rev(X)$. Set $L^{\mu.} = (L_\lambda^{\mu_1}, L_\lambda^{\mu_2}, \ldots)$ and $H(\mu.) = (H(\mu_1), H(\mu_2), \ldots)$. Then, for any compactly supported $\rho \in \mathcal{G}^1$ and $\mathcal{C}$ measurable non-negative function F on R^∞*

$$E_G E_\lambda^\rho \left(F\left(L^{\mu.} + \frac{1}{2} H(\mu.) \right) f(X_\lambda) \right) = E_G \left(F\left(\frac{1}{2} H(\mu.) \right) G_\rho G_{f \cdot dx} \right) \tag{4.10}$$

where $\mathcal{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

Let $\mathcal{M}(\mathcal{H})$ denote the set of functions measurable with respect to $\mathcal{H} \stackrel{def}{=} \sigma(H(\mu);\, \mu \in \mathcal{G}^2 \cap Rev(X))$. We define the ring homomorphism

$$\Phi : \mathcal{M}(\mathcal{H}) \mapsto \mathcal{M}(\mathcal{H} \times \mathcal{F}) \tag{4.11}$$

as the measurable extension of the mapping Φ such that $\Phi(1) = 1$ and

$$\Phi(\prod_{i=1}^n H(\mu_i)) = \prod_{i=1}^n (H(\mu_i) + 2L_\lambda^{\mu_i}), \qquad n = 1, \ldots, \tag{4.12}$$

where $\mathcal{F}$ is the σ-algebra generated by X. With this notation Theorem 4.2 can be reformulated as follows: Let $(h_1, h_2, \ldots)$ be a sequence of $\mathcal{H}$ measurable functions. Then for any $\mathcal{C}$ measurable non-negative function F on R^∞

$$E_G E_\lambda^\rho \left(F(\Phi(h_1), \Phi(h_2), \ldots) f(X_\lambda)\right) = E_G \left(F(h_1, h_2, \ldots) G_\rho G_{f\cdot dx}\right). \tag{4.13}$$

This will be explained in greater detail in the proof of Theorem 4.1.

Motivated by [**5**] we will obtain our isomorphism therem, Theorem 4.1, from (4.13) by taking $: G^{2n}\mu_i :$ for h_i and then finding $\Phi(: G^{2n}\mu_i :)$. This is accomplished in a series of lemmas. As a first step, we show that $: G^{2n}\mu_i :$ is $\mathcal{H}$ measurable, a point that is not at all obvious. We begin by defining the 'cycle factors'

$$cy_j^\epsilon \stackrel{def}{=} \frac{1}{2j} \int u^1(x_1 - x_j) \prod_{k=2}^j u^1(x_k - x_{k-1}) \prod_{i=1}^j f_\epsilon(x_i)\, dx_i \tag{4.14}$$

for $j \geq 2$, with the convention that $cy_1^\epsilon = 0$. (The reason for the name 'cycle factor' should also become clear in the course of proving Theorem 4.1). We next define

$$A_{n,k}^\epsilon \stackrel{def}{=} \frac{n!}{k!} \sum_{r=0}^\infty \frac{1}{r!} \sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_k \\ \sum_{a=1}^r i_a + \sum_{b=1}^k j_b = n}} \prod_{a=1}^r cy_{i_a}^\epsilon \prod_{b=1}^k ch_{j_b}^\epsilon. \tag{4.15}$$

By convention, we set $A_{0,0}^\epsilon = 1$ and $A_{n,0}^\epsilon = A_{0,n}^\epsilon = 0,\ n = 1, \ldots$. Note that for each $n \geq 0$, $\{A_{j,k}^\epsilon\}_{j,k=0}^n$ is a lower triangular matrix with $A_{j,j}^\epsilon = 1$ for all $0 \leq j \leq n$ and hence is invertible. Let $H(x,\epsilon) \stackrel{def}{=} H(f_{x,\epsilon} \cdot dx')$. We now inductively define $\Psi_k(x,\epsilon)$, $k = 0, 1, \ldots, n$ by the formula

$$\frac{H^n(x,\epsilon)}{2^n} = \sum_{k=0}^n A_{n,k}^\epsilon \Psi_k(x,\epsilon). \tag{4.16}$$

Let $\{\Lambda_{j,k}^\epsilon\}_{j,k=0}^n$ be the inverse of $\{A_{j,k}^\epsilon\}_{j,k=0}^n$, then

$$\Psi_n(x,\epsilon) = \sum_{k=0}^n \Lambda_{n,k}^\epsilon \frac{H^k(x,\epsilon)}{2^k}. \tag{4.17}$$

This shows us that $\Psi_0(x,\epsilon) = 1$ and for each n, $\Psi_n(x,\epsilon)$ is an $n-$th degree polynomial in $H(x,\epsilon)$. In particular $\Psi_n(x,\epsilon) \in \mathcal{M}(\mathcal{H})$.

Define

$$\Psi_{k,\epsilon}\mu = \int \Psi_k(x,\epsilon)\, d\mu(x). \tag{4.18}$$

LEMMA 4.1. *If $\mu \in \mathcal{G}^{2\sigma}$, then*

$$\frac{:G^{2k}\mu:}{2^k} = \lim_{\epsilon \to 0} \Psi_{k,\epsilon}\mu \tag{4.19}$$

in $L^2(dP_G)$, for all $1 \leq k \leq \sigma$.

This shows that for $\mu \in \mathcal{G}^{2n}$ we do indeed have that the $2n$-th Wick power chaos $:G^{2n}\mu: \in \mathcal{M}(\mathcal{H})$. Furthermore, by Theorem 4.2

$$\Phi(\frac{:G^{2n}\mu:}{2^n}) = \lim_{\epsilon \to 0} \Phi(\Psi_{n,\epsilon}\mu). \tag{4.20}$$

in L^2. The proof of Lemma 4.1 will be given later in this chapter.

The next lemma shows that the limit in (4.8) exists in $L^2(f(X_\lambda)\, dP^\rho_\lambda\, dP_G)$ for each $k = 1, \ldots, n$ and identifies $\Phi(\Psi_{n,\epsilon}\mu)$.

LEMMA 4.2. *If $\mu \in \mathcal{G}^{2n}$, then*

$$(:G^{2(n-k)}: \times \mathcal{L}_k)(\mu) \stackrel{def}{=} \lim_{\epsilon \to 0} \int :G^{2(n-k)}_{x,\epsilon}: \mathcal{L}^{x,\epsilon}_k \, d\mu(x) \tag{4.21}$$

exists in $L^2(f(X_\lambda)\, dP^\rho_\lambda\, dP_G)$ for each $k = 0, 1, \ldots, n$, and

$$\Phi(\frac{1}{2^n} :G^{2n}\mu:) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{(n-k)}} (:G^{2(n-k)}: \times \mathcal{L}_k)(\mu). \tag{4.22}$$

The proof of Lemma 4.2 is also given later in this chapter. We will now use these two lemmas to complete the proof of our isomorphism theorem, Theorem 4.1.

PROOF OF THEOREM 4.1. Let x_i; $i = 1, 2, \ldots$ be a countable dense set in R^m and let $H_1, H_2, \ldots$ be an enumeration of $H(x_i, j^{-1})$; $i, j = 1, 2, \ldots$. Let $L_1, L_2, \ldots$ be the associated continuous additive functionals. (Recall that $H(x_i, j^{-1})$ is an alternate notation for $H(f_{x_i,1/j}(x') \cdot dx')$). The continuous additive functional associated with $H(f_{x_i,1/j}(x') \cdot dx')$ in Theorem 4.2 is denoted by $L^{x_i,1/j}_\lambda$, (see (4.6))). Let $L_1, L_2, \ldots$ be the continuous additive functionals associated with $H_1, H_2, \ldots$. We note that for fixed $\epsilon > 0$, $H \stackrel{def}{=} \{H(x,\epsilon), x \in R^m\}$ can be taken to be continuous almost surely. This is easy to see since, similarly to (3.7), $EH(x,\epsilon)H(y,\epsilon) = u^1_{\epsilon,\epsilon}(x,y)$. We can choose a nice approximate identity f_ϵ, so that (1.11) holds when $n = 1$. This is a sufficient condition for the continuity of H. (See the proof of Theorem 1.6, [**15**]). Therefore $\mathcal{H} = \sigma(H_1, H_2, \ldots)$. Let $\{\mu_i,\, i = 1, 2, \ldots\}$ be a sequence of finite measures in $\mathcal{G}^{2n}$. By Lemma 4.1, $:G^{2n}\mu_i:$ is $\mathcal{H}$ measurable for each i, hence by a theorem of Doob, [**4**], page 12, we can write

$$\frac{1}{2^n} :G^{2n}\mu_i := D_i(H_1, H_2, \ldots) \tag{4.23}$$

for some $\mathcal{C}$ measurable random variable D_i where $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞. By our previous isomorphism theorem, Theorem 4.2, for any compactly supported measure $\rho \in \mathcal{G}^1$, and $\mathcal{C}$ measurable non-negative function F on R^∞

$$\begin{aligned} &E_G E^\rho_\lambda \left(F(D.(H_1 + 2L_1, H_2 + 2L_2, \ldots))f(X_\lambda)\right) \\ &\quad = E_G \left(F(D.(H_1, H_2, \ldots)G_\rho G_{f\cdot dx}\right). \end{aligned} \tag{4.24}$$

By the definition of Φ in (4.11) we have that (4.23) implies

$$\Phi(\frac{1}{2^n} : G^{2n}\mu_i :) = D_i(H_1 + 2L_1, H_2 + 2L_2, \ldots).$$

Thus (4.23), (4.24) and (4.22) immediately imply the isomorphism theorem, Theorem 4.1. □

It remains to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1:

Proof of Lemma 4.1. Recall that $H(\mu)$ is an alternate expression for $: G^2(\mu) :$. Therefore, consistent with (3.15)

$$H(x,\epsilon) \stackrel{def}{=} H(f_{x,\epsilon} \cdot dx') = \lim_{\delta\to 0} \int : G^2_{x',\delta} : f_{x,\epsilon}(x')\, dx' \tag{4.25}$$

where the limit is taken in L^2. Since $f_{x,\epsilon} \cdot dx' \in \mathcal{G}^1_F$, and hence $\mathcal{G}^2_F$, $H(x,\epsilon)$ is one of the basic random variables that generates $\mathcal{H}$. As explained above in the proof of Theorem 4.1, for fixed $\epsilon > 0$, $H \stackrel{def}{=} \{H(x,\epsilon), x \in R^m\}$ can be taken to be continuous almost surely. The same argument shows that the integral in (4.25) is continuous in x almost surely. Furthermore, the convergence in (4.25) is almost sure and in L^p, for all p, since Gaussian chaos processes have all moments.

In order to prove Lemma 4.1, we begin by deriving a formula for $\int H^n(x,\epsilon)\, d\mu(x)$. Clearly

$$H^n(x,\epsilon) = \lim_{\delta\to 0} \prod_{i=1}^n \int : G^2_{x_i,\delta} : f_{x,\epsilon}(x_i)\, dx_i. \tag{4.26}$$

We define

$$H^n_\epsilon \mu = \int H^n(x,\epsilon)\, d\mu(x). \tag{4.27}$$

Since the right-hand side of (4.26) converges in L^2 uniformly in x as, $\delta \to 0$, we see that

$$H^n_\epsilon \mu = \lim_{\delta\to 0} \int\!\!\int \prod_{i=1}^n : G^2_{x_i,\delta} : f_{x,\epsilon}(x_i)\, dx_i\, d\mu(x) \tag{4.28}$$

in L^2. (In fact by Lemma 3.3, [**1**], it also converges almost surely and in L^p for all $p \geq 0$).

Expand $\prod_{i=1}^n (: G^2_{x_i,\delta} : /2)$ as a sum of Wick products. Using (2.15) we can write

$$\prod_{i=1}^n \frac{: G^2_{x_i,\delta} :}{2} \tag{4.29}$$

$$= \sum_{R,S,T} \frac{1}{2^{|R|}} \sum_{\substack{pairings\ \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1} \neq \widetilde{\mathcal{P}}_{k,2}}} \prod_{k=1}^{|R|+|S|/2} u^1_{\delta,\delta}(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) : \prod_{i\in T} \frac{G^2_{x_i,\delta}}{2} \prod_{j\in S} G_{x_j,\delta} :$$

where the first sum runs over all partitions $R \cup S \cup T = \{1,2,\ldots,n\}$, with $|S|$ even, and the second sum runs over all pairings $\mathcal{P}$ of the set $(R \times \{1,2\}) \cup S$ such that $\widetilde{\mathcal{P}}_{k,1} \neq \widetilde{\mathcal{P}}_{k,2}$, where letting $(\mathcal{P}_{k,1}, \mathcal{P}_{k,2})$ denote the $k-$th pair of the pairing $\mathcal{P}$, we set $\widetilde{\mathcal{P}}_{k,1} = i$ if either $\mathcal{P}_{k,1} = i \times 1$ or $i \times 2$ for $i \in R$, or $\mathcal{P}_{k,1} = i$ for $i \in S$, and similarly for $\widetilde{\mathcal{P}}_{k,2}$.. Here we use the fact that for $i \in S$ one of the two $G_{x_i,\delta}$ terms is allocated to the $u^1_{\delta,\delta}$ terms and the other to the Wick product. Since there are two ways to do this the 1/2 is cancelled. The last formula, (4.29), will be abbreviated as

$$\prod_{i=1}^{n} \frac{:G^2_{x_i,\delta}:}{2} = \sum_{R,S,T} \mathcal{E}_\delta(x_1,\dots,x_n;R,S) : \prod_{i\in T} \frac{G^2_{x_i,\delta}}{2} \prod_{j\in S} G_{x_j,\delta} : \tag{4.30}$$

where

$$\mathcal{E}_\delta(x_1,\dots,x_n;R,S) \stackrel{def}{=} \frac{1}{2^{|R|}} \sum_{\substack{pairings\,\mathcal{P} \\ of\,(R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \prod_{k=1}^{|R|+|S|/2} u^1_{\delta,\delta}(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}). \tag{4.31}$$

Note that by an argument similar to the one used in (3.11) and (3.12) we see that for all $x \in R^m$

$$\lim_{\delta\to 0} \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \prod_{k=1}^{n} f_\epsilon(x_k)\,dx_k \tag{4.32}$$
$$= \frac{1}{2^{|R|}} \sum_{\substack{pairings\,\mathcal{P} \\ of\,(R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+|S|/2} u^1(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{k=1}^{n} f_\epsilon(x_k)\,dx_k$$

Using this, (4.28), Fubini's theorem and the definition of $f_{x,\epsilon}$ we have

$$\frac{H^n_\epsilon \mu}{2^n} = \sum_{R,S,T} \lim_{\delta\to 0} \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S)$$
$$\left(\int : \prod_{i\in T} \frac{G^2_{x+x_i,\delta}}{2} \prod_{j\in S} G_{x+x_j,\delta} : \, d\mu(x) \right) \prod_{k=1}^{n} f_\epsilon(x_k)\,dx_k. \tag{4.33}$$

Let us now consider

$$\int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \prod_{k=1}^{n} f_\epsilon(x_k)\,dx_k \tag{4.34}$$
$$= \frac{1}{2^{|R|}} \sum_{\substack{pairings\,\mathcal{P} \\ of\,(R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+|S|/2} u^1_{\delta,\delta}(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{k=1}^{n} f_\epsilon(x_k)\,dx_k.$$

Suppose $|R| + |S|/2 = p$. Then by the multiple Hölder inequality and (3.7), (4.34) is bounded above by

$$C \int\int |u^1_{\delta,\delta'}(|x-y|)|^p f_\epsilon(x) f_\epsilon(y)\,dx\,dy$$
$$\leq C \int\int |u^1(|x-y+(x'-y')|)|^p f_\epsilon(x) f_\epsilon(y)\,dx\,dy \rho_\delta(dx')\rho_\delta(dy') \tag{4.35}$$

It follows from this and Lemma 3.2 that (4.34) is $O((u^1(\epsilon))^{(|R|+|S|/2)})$ for all $\delta > 0$.

Using the last statement and Theorem 3.1 we see that for $\mu \in \mathcal{G}^{2\sigma}$

$$\| \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S)$$

$$\left\{ \int : \prod_{i\in T} \frac{G^2_{x+x_i,\delta}}{2} \prod_{j\in S} G_{x+x_j,\delta} : \, d\mu(x) - \frac{: G_\delta^{2|T|+|S|}\mu :}{2^{|T|}} \right\} \prod_{k=1}^n f_\epsilon(x_k)\, dx_k \|_2$$
$$\le \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \prod_{k=1}^n f_{x,\epsilon}(x_k)\, dx_k$$
$$\sup_{|x_i|\le\epsilon} \| \int : \prod_{i\in T} \frac{G^2_{x+x_i,\delta}}{2} \prod_{j\in S} G_{x+x_j,\delta} : \, d\mu(x) - \frac{: G_\delta^{2|T|+|S|}\mu :}{2^{|T|}} \|_2$$
$$\le o((u^1(\epsilon))^{-(\sigma-(|T|+|S|/2))} \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \prod_{k=1}^n f_\epsilon(x_k)\, dx_k$$
$$= o((u(\epsilon))^{-(\sigma-n)})$$

for all $\delta > 0$. Using this, (4.32) and (4.33) we see that for $\mu \in \mathcal{G}^{2\sigma}$

$$\begin{aligned} \frac{H^n_\epsilon \mu}{2^n} &= \sum_{R,S,T} \lim_{\delta\to 0} \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \\ &\qquad \prod_{i\in R\cup S} f_\epsilon(x_i)\, dx_i : \frac{G_\delta^{2|T|+|S|}}{2^{|T|}} \mu : + o((u(\epsilon))^{-(\sigma-n)}) \qquad (4.36) \\ &= \sum_{k=0}^n \sum_{\substack{R,S,T \\ 2|T|+|S|=2k}} \lim_{\delta\to 0} \int \mathcal{E}_\delta(x_1,\dots,x_n;R,S) \\ &\qquad \prod_{i\in R\cup S} f_\epsilon(x_i)\, dx_i \frac{1}{2^{|T|}} : G_\delta^{2k}\mu : + o((u(\epsilon))^{-(\sigma-n)}) \\ &= \sum_{k=0}^n \sum_{\substack{R,S,T \\ 2|T|+|S|=2k}} \frac{1}{2^{|R|}} \sum_{\substack{pairings\ \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+|S|/2} u^1(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \\ &\qquad \prod_{i\in R\cup S} f_\epsilon(x_i)\, dx_i \frac{1}{2^{|T|}} : G^{2k}\mu : + o((u(\epsilon))^{-(\sigma-n)}). \end{aligned}$$

in L^2, as $\epsilon \to 0$.

We now reorganize this in a form which is more useful. Fix some pairing $\mathcal{P}$ in the sum and pick any factor $u^1(x_i - x_j)$ in the product corresponding to $\mathcal{P}$. If both $i, j \in S$ we think of i, j as forming a two element chain. $u^1(x_i - x_j)$ is the factor associated with this chain. If say $j \in R$, there will be one other factor in the product corresponding to $\mathcal{P}$ which contains x_j, say $u^1(x_j - x_k)$. If both $i, k \in S$, we think of i, j, k as forming a three element chain. $u^1(x_i - x_j)u^1(x_j - x_k)$ is the factor associated with this chain. If either i or k or both are in R, we continue to find the other factors containing them, and continue in this manner until we can go no further. Two possibilities arise. Either we end up with a chain of elements $i_1, i_2, \dots, i_v$ with end points $i_1, i_v \in S$ and intermediate points $i_2, \dots, i_{v-1} \in R$ and associated factor

$$\prod_{j=2}^v u^1(x_{i_j} - x_{i_{j-1}}) \tag{4.37}$$

(such a chain is said to be of length v), or we have, what we call, a cycle $i_1, i_2, \ldots, i_v$ with all elements in R and associated factor

$$u^1(x_{i_1} - x_{i_v}) \prod_{j=2}^{v} u^1(x_{i_j} - x_{i_{j-1}}) \tag{4.38}$$

(such a cycle is also said to be of length v).

In this way the product

$$\prod_{k=1}^{|R|+|S|/2} u^1(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \tag{4.39}$$

in (4.36) associated with $\mathcal{P}$ breaks up into a product of factors associated with the chains and cycles of $\mathcal{P}$. Note that each $\mathcal{P}$ appearing in (4.36) will necessarily have precisely $|S|/2$ chains. For simplicity let $|S|/2$=p.

When $\mathcal{P}$ decomposes into m_l cycles of length l, $l = 2, \ldots$ and $\bar{m}_l$ chains of length l, $l = 2, \ldots$, we write $\mathcal{P} \to (m_2, \ldots; \bar{m}_2, \ldots)$. In this notation, recalling the definitions (4.4), (4.14) of the chain and cycle factors ch_j^ϵ and cy_j^ϵ, we have

$$\begin{aligned} &\frac{1}{2^{|R|}} \sum_{\substack{pairings\, \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq \widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+p} u^1(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{i\in R\cup S} f_\epsilon(x_i)\, dx_i \\ &= \frac{1}{2^{|R|}} \sum_{m_2,\ldots;\, \bar{m}_2,\ldots} \sum_{\substack{pairings\, \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \mathcal{P}\to(m_2,\ldots;\,\bar{m}_2,\ldots)}} \prod_{l=2}^{\infty} (2lcy_l^\epsilon)^{m_l} (ch_l^\epsilon)^{\bar{m}_l}. \end{aligned} \tag{4.40}$$

Note that when $\mathcal{P} \to (m_2, \ldots; \bar{m}_2, \ldots)$ and $\mathcal{P}$ is a pairing of $(R\times\{1,2\})\cup S$ we must have $\sum_{l=1}^{\infty} m_l l + \sum_{l=1}^{\infty} \bar{m}_l l = |R| + |S|$. We now further simplify (4.40) by observing that the number of pairings $\mathcal{P}$ of $(R\times\{1,2\})\cup S$ with $\mathcal{P} \to (m_2, \ldots; \bar{m}_2, \ldots)$ is

$$\frac{|R|!}{\prod_{l=2}^{\infty} (l!)^{m_l} (m_l!) ((l-2)!)^{\bar{m}_l} (\bar{m}_l!)} \prod_{l=2}^{\infty} (\frac{(l-1)!}{2})^{m_l} ((l-2)!)^{\bar{m}_l} \frac{|S|!}{2^{\sum_{l=2}^{\infty} \bar{m}_l}} 2^{|R|}. \tag{4.41}$$

Here, the first factor gives the number of ways to partition R into m_l cycles of length l, $l = 2, \ldots$ and $\bar{m}_l$ mid-chains (i.e. chains with end points deleted) of length $l-2$, $l = 2, \ldots$. To get the remainder of (4.41) we note that in each cycle of length l we can permute the points of the cycle in $(l-1)!$ distinct ways, except that we must divide by 2 to take into account the mirror image if $l > 2$, (we will explain shortly where the factor $1/2$ for cycles of length $l = 2$ comes from), while for each chain of length l we can permute the elements of the mid-chain in $(l-2)!$ ways, and the $|S|$ end points can be permuted among themselves in $|S|!$ ways, except that for any of the $\sum_{l=2}^{\infty} \bar{m}_l$ given chains we mustn't count an interchange of the end points of the same chain, since that has already been counted when we considered the permutations of the mid-chain. Finally, recall that the pairings are actually parings of $(R \times \{1,2\}) \cup S$, not of $R \cup S$, so that for any given pairing we can get analogous but distinct pairings by interchanging $i \times 1$ with $i \times 2$ for each $i \in R$. The only exception is that for any cycle of length $l = 2$ we get 2 rather than 4 distinct pairings. Altogether this gives rise to $2^{|R|}/2^{m_2}$ distinct pairings. (This explains where the factor $1/2$ for cycles of length $l = 2$ in (4.41) comes from). Therefore, combining (4.40) and (4.41) we see that

$$\frac{1}{2^{|R|}} \sum_{\substack{pairings\ \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+p} u^{\alpha}(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{i\in R\cup S} f_{\epsilon}(x_i)\, dx_i$$

$$= 2^{-\sum_{l=2}^{\infty}\bar{m}_l} \sum_{\substack{m_2,\dots;\,\bar{m}_2,\dots \\ \sum_{l=2}^{\infty} m_l l + \sum_{l=2}^{\infty}\bar{m}_l l = |R|+|S|}} \frac{|R|!}{\prod_{l=2}^{\infty}(l!)^{m_l}(m_l!)((l-2)!)^{\bar{m}_l}(\bar{m}_l!)} \tag{4.42}$$

$$\prod_{l=2}^{\infty}\left(\frac{(l-1)!}{2}\right)^{m_l}((l-2)!)^{\bar{m}_l}|S|! \prod_{l=2}^{\infty}(2lcy_l^{\epsilon})^{m_l}(ch_l^{\epsilon})^{\bar{m}_l}$$

$$= 2^{-\sum_{l=2}^{\infty}\bar{m}_l} \sum_{\substack{m_2,\dots;\,\bar{m}_2,\dots \\ \sum_{l=2}^{\infty} m_l l + \sum_{l=2}^{\infty}\bar{m}_l l = |R|+|S|}} \frac{|R|!|S|!}{\prod_{l=2}^{\infty}(m_l!)(\bar{m}_l!)} \prod_{l=2}^{\infty}(cy_l^{\epsilon})^{m_l}(ch_l^{\epsilon})^{\bar{m}_l}.$$

Instead of writing the last expression in terms of $m_2,\dots;\bar{m}_2,\dots$ it will be more convenient to write this expression in terms of ordered sequences of integers $i_1,\dots,i_r,\, j_1,\dots,j_p$ such that $m_l = |\{v;\, v = 1,\dots,r\,|\, i_v = l\}|$ and $\bar{m}_l = |\{v;\, v = 1,\dots,f\,|\, j_v = l\}|$. Since there are

$$\frac{r!}{\prod_{l=2}^{\infty}(m_l!)}\frac{p!}{\prod_{l=2}^{\infty}(\bar{m}_l!)}$$

such sequences which can be associated with a given $m_2,\dots;\bar{m}_2,\dots$, and noting that $r = \sum_{l=2}^{\infty} m_l$, $p = \sum_{l=2}^{\infty}\bar{m}_l$ and $\sum_{l=2}^{\infty} m_l l + \sum_{l=2}^{\infty}\bar{m}_l l = \sum_{a=1}^{r} i_a + \sum_{b=1}^{p} j_b$ we see from (4.42) that

$$\frac{1}{2^{|R|}} \sum_{\substack{pairings\ \mathcal{P} \\ of\ (R\times\{1,2\})\cup S \\ \widetilde{\mathcal{P}}_{k,1}\neq\widetilde{\mathcal{P}}_{k,2}}} \int \prod_{k=1}^{|R|+p} u^{\alpha}(x_{\widetilde{\mathcal{P}}_{k,1}} - x_{\widetilde{\mathcal{P}}_{k,2}}) \prod_{i\in R\cup S} f_{\epsilon}(x_i)\, dx_i \tag{4.43}$$

$$= 2^{-p} \sum_{\substack{r,i_1,\dots,i_r,\, j_1,\dots,j_p \\ j_b\geq 2,\,\forall b \\ \sum_{a=1}^{r} i_a + \sum_{b=1}^{p} j_b = |R|+|S|}} \frac{|R|!|S|!}{r!p!} \prod_{a=1}^{r} cy_{i_a}^{\epsilon} \prod_{b=1}^{p} ch_{j_b}^{\epsilon}.$$

Combining this with (4.36), and noting that for each $0 \leq p \leq k$ there are

$$\binom{n}{|S|,\ |T|,\ |R|} = \binom{n}{|S|,\ k-p,\ |R|} = \frac{n!}{|S|!(k-p)!|R|!}$$

ways to partition $\{1,\dots,n\} = S\cup T\cup R$ into three parts with $|S| = 2p$, $|T| = k-p$, $|R| = n-k-p$ we obtain

$$\frac{H_{\epsilon}^{n}\mu}{2^n} = \sum_{k=0}^{n} \sum_{\substack{R,S,T \\ 2|T|+|S|=2k}} \sum_{\substack{r,i_1,\dots,i_r,\, j_1,\dots,j_p \\ j_b\geq 2,\,\forall b \\ \sum_{a=1}^{r} i_a + \sum_{b=1}^{p} j_b = |R|+|S|}}$$

$$\frac{|R|!|S|!}{r!p!}\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{p} ch^{\epsilon}_{j_b} : \frac{G^{2k}}{2^k}\mu : +o((u^1(\epsilon))^{-(\sigma-n)}) \quad (4.44)$$

$$= \sum_{k=0}^{n}\sum_{r}\frac{1}{r!}\sum_{p=0}^{k}\frac{n!}{(k-p)!p!}\sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_p \\ j_b\geq 2,\,\forall b \\ \sum_{a=1}^{r} i_a+\sum_{b=1}^{p} j_b=n-(k-p)}}$$

$$\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{p} ch^{\epsilon}_{j_b} : \frac{G^{2k}}{2^k}\mu : +o((u^1(\epsilon))^{-(\sigma-n)})$$

$$= \sum_{k=0}^{n}\frac{n!}{k!}\sum_{r}\frac{1}{r!}\sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_{k-v} \\ j_b\geq 2,\,\forall b \\ \sum_{a=1}^{r} i_a+\sum_{b=1}^{k-v} j_b=n-v}}$$

$$\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{k-v} ch^{\epsilon}_{j_b} : \frac{G^{2k}}{2^k}\mu : +o((u^1(\epsilon))^{-(\sigma-n)})$$

in L^2, as $\epsilon \to 0$, where in the last equality we write $v = k - p$.

Recall that $ch^{\epsilon}_1 = 1$ and note that

$$\sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_k \\ j_b\geq 1,\,\forall b \\ \sum_{a=1}^{r} i_a+\sum_{b=1}^{k} j_b=n}}\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{k} ch^{\epsilon}_{j_b} = \sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_{k-v} \\ j_b\geq 2,\,\forall b \\ \sum_{a=1}^{r} i_a+\sum_{b=1}^{k-v} j_b=n-v}}\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{k-v} ch^{\epsilon}_{j_b} \quad (4.45)$$

since the summation on the left hand side allows us to include v factors of $ch^{\epsilon}_1 = 1$ among the k factors of $ch^{\epsilon}_{\cdot}$, $0 \leq v \leq k$. From now on all summation is over $j_b \geq 1$. Consequently, we can write

$$\frac{H^n_{\epsilon}\mu}{2^n} = \sum_{k=0}^{n}\frac{n!}{k!}\sum_{r}\frac{1}{r!}\sum_{\substack{i_1,\ldots,i_r,\, j_1,\ldots,j_k \\ \sum_{a=1}^{r} i_a+\sum_{b=1}^{k} j_b=n}}\prod_{a=1}^{r} cy^{\epsilon}_{i_a}\prod_{b=1}^{k} ch^{\epsilon}_{j_b} : \frac{G^{2k}}{2^k}\mu : +o((u^1(\epsilon))^{-(\sigma-n)}). \quad (4.46)$$

Recalling the definition (4.15) of $A^{\epsilon}_{n,k}$ we can write (4.46) as

$$\frac{H^n_{\epsilon}\mu}{2^n} = \sum_{k=0}^{n} A^{\epsilon}_{n,k} : \frac{G^{2k}}{2^k}\mu : +o((u^1(\epsilon))^{-(\sigma-n)}). \quad (4.47)$$

Comparing (4.47) and (4.16) we see that

$$\sum_{k=0}^{n} A^{\epsilon}_{n,k}\left(\frac{: G^{2k}\mu :}{2^k} - \Psi_{k,\epsilon}\mu\right) = o((u^1(\epsilon)^{-(\sigma-n)}) \quad (4.48)$$

in L^2.

It follows from the multiple Hölder inequality and Lemma 3.2 that

$$cy^{\epsilon}_i = O((u^1(\epsilon))^i) \quad (4.49)$$

and

$$ch_j^\epsilon = O((u^1(\epsilon))^{(j-1)}) \tag{4.50}$$

which implies that

$$A_{n,k}^\epsilon = O((u^1(\epsilon))^{(n-k)}). \tag{4.51}$$

Using (4.48) and arguing inductively for $k = 1, \dots, n$ we now see that

$$\Psi_{k,\epsilon}\mu = \frac{: G^{2k}\mu :}{2^k} + o((u^1(\epsilon)^{-(\sigma-k)}). \tag{4.52}$$

This completes the proof of Lemma 4.1. □

In preparation for the verification of (4.8) and (4.22) of Lemma 4.2, we first prove the following purely combinatorial lemma

LEMMA 4.3.

$$\Phi(\Psi_n(x,\epsilon)) = \sum_{k=0}^n \binom{n}{k} \Psi_{n-k}(x,\epsilon)\mathcal{L}_k^{x,\epsilon}. \tag{4.53}$$

PROOF OF LEMMA 4.3. Using the definition of Φ together with (**??**) and (4.7) we see that

$$\begin{aligned} \Phi(\frac{H^n(x,\epsilon)}{2^n}) &= (\frac{H(x,\epsilon)}{2} + L^{x,\epsilon})^n \\ &= \sum_{m=0}^n \binom{n}{m} \frac{H^m(x,\epsilon)}{2^m} (L^{x,\epsilon})^{n-m} \\ &= \sum_{m=0}^n \binom{n}{m} (A^\epsilon \Psi(x,\epsilon))_m (B^\epsilon \mathcal{L}^{x,\epsilon})_{n-m}, \end{aligned} \tag{4.54}$$

using the abbreviations $(A^\epsilon\Psi(x,\epsilon))_m = \sum_{k=0}^m A_{m,k}^\epsilon \Psi_k(x,\epsilon)$ and $(B^\epsilon\mathcal{L}^{x,\epsilon})_m = \sum_{k=0}^m B_{m,k}^\epsilon \mathcal{L}_k^{x,\epsilon}$. On the other hand, by the defining relation (4.16), we have

$$\Phi(\frac{H^n(x,\epsilon)}{2^n}) = \sum_{k=0}^n A_{n,k}^\epsilon \Phi(\Psi_k(x,\epsilon)). \tag{4.55}$$

Comparing (4.54) and (4.55) we see that we can verify (4.53) recursively for $n = 0, 1, \dots$ by showing that

$$\sum_{k=0}^n A_{n,k}^\epsilon \sum_{j=0}^k \binom{k}{j} \Psi_j(x,\epsilon)\mathcal{L}_{k-j}^{x,\epsilon} = \sum_{m=0}^n \binom{n}{m} (A^\epsilon\Psi(x,\epsilon))_m (B^\epsilon\mathcal{L}^{x,\epsilon})_{n-m} \tag{4.56}$$

for all $n \geq 0$.

To simplify notation, we drop the terms x and ϵ and rewrite (4.56) as

$$\sum_{k=0}^n A_{n,k} \sum_{j=0}^k \binom{k}{j} \Psi_j \mathcal{L}_{k-j} = \sum_{m=0}^n \binom{n}{m} (A\Psi)_m (B\mathcal{L})_{n-m}. \tag{4.57}$$

In the following set of equations we use the convention that the sum over an empty set of indices is one. We note that for any $0 \leq v \leq k$ by (4.15) and (4.5)

$A_{n,k}$

$$= \frac{n!}{k!}\sum_r \frac{1}{r!}\sum_{m=0}^n \left(\sum_{\substack{i_1,\ldots,i_r,j_1,\ldots,j_v \\ \sum_{a=1}^r i_a + \sum_{b=1}^v j_b = m}} \prod_{a=1}^r cy_{i_a} \prod_{b=1}^v ch_{j_b} \right) \left(\sum_{\substack{j_{v+1},\ldots,j_k \\ \sum_{b=v+1}^k j_b = n-m}} \prod_{b=v+1}^k ch_{j_b} \right)$$

$$= \binom{k}{v}^{-1} \sum_{m=0}^n \binom{n}{m} A_{m,v} B_{n-m,k-v}.$$

Using this we have

$$\begin{aligned} \sum_{k=0}^n A_{n,k} \sum_{v=0}^k \binom{k}{v} \Psi_v \mathcal{L}_{k-v} &= = \sum_{k=0}^n \sum_{v=0}^k \sum_{m=0}^n \binom{n}{m} A_{m,v} B_{n-m,k-v} \Psi_v \mathcal{L}_{k-v} \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{v=0}^n A_{m,v} \Psi_v \sum_{k=v}^n B_{n-m,k-v} \mathcal{L}_{k-v} \\ &= \sum_{m=0}^n \binom{n}{m} (A\Psi)_m (B\mathcal{L})_{n-m} \qquad (4.58) \end{aligned}$$

which is (4.57). This completes the proof of Lemma 4.3. □

Proof of Lemma 4.2. For the proof, we introduce $n+1$ independent copies of the Gaussian chaos processes considered in Theorem 4.1. Let $G_{(0),\rho}, G_{(1),\rho}, \ldots, G_{(n),\rho}$ denote independent Gaussian processes distributed like G_ρ, $\rho \in \mathcal{G}^1$. For each of these we define and construct all the processes that are defined and constructed in Theorem 4.1 from G_ρ, $\rho \in \mathcal{G}^1$. These different independent processes will be denoted by the subscript (j) $j = 0, \ldots, n$. We use the notation $\Phi_{(0)}$ to denote the ring homeomorphism defined in (4.11) applied (only) to the processes defined and constructed from $G_{(0),\rho}$, $\rho \in \mathcal{G}^1$. Using this notation it follows from (4.53) that

$$\Phi_{(0)}(\prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0}(x,\epsilon)) = \sum_{k=0}^{n_0} \binom{n_0}{k} \prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-k}(x,\epsilon)\mathcal{L}_k^{x,\epsilon}. \tag{4.59}$$

where $\Psi_{(j),n}(x,\epsilon)$ is the analog of $\Psi_n(x,\epsilon)$ built up from $G_{(j)}$.

We now verify that when $\mu \in \mathcal{G}^{2n}$, then for any m and $n_0, \ldots, n_m$ such that $n = \sum_{j=0}^m n_j$ and $k = 0, \ldots, n_0$

$$\lim_{\epsilon \to 0} \int \prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-k}(x,\epsilon)\mathcal{L}_k^{x,\epsilon}\, d\mu(x) \tag{4.60}$$

exists in L^2. Here the probability space is the product probability space generated by $\{G_{(j),\rho)}\}_{j=0}^m$, $\rho \in \mathcal{G}^1$ and $f(X_\lambda)\, dP_\lambda^\rho$. (Without loss of generality we can consider the last measure as a probability measure). To do this we need to generalize (4.47). We first remark that it is easy to check that

$$\|(L^{x,\epsilon})^k\|_2 = O((u^1(\epsilon))^k). \tag{4.61}$$

Also, since

$$B_{n,k}^\epsilon = O((u^1(\epsilon))^{(n-k)}) \tag{4.62}$$

by (4.50), we can use induction in (4.7) to see that

$$\|\mathcal{L}_n^{x,\epsilon}\|_2 = O((u^1(\epsilon))^n). \tag{4.63}$$

In (4.61) and (4.63) we take the norm to be that of $L^2(f(X_\lambda)\,dP^\rho_\lambda)$.

Analogous to (4.33) we expand

$$\prod_{j=1}^m \frac{H^{n_j}_{(j)}(x,\epsilon)}{2^{n_j}}$$

in terms of the corresponding Wick products and then integrate with respect to $\mathcal{L}^{x,\epsilon}_k\,d\mu(x)$. We then use Theorem 3.2 and proceed to follow all the steps up to (4.47), using also (4.63), to obtain

$$\begin{aligned}&\int \prod_{j=1}^m \frac{H^{n_j}_{(j)}(x,\epsilon)}{2^{n_j}}\mathcal{L}^{x,\epsilon}_k\,d\mu(x)\\ &\quad= \sum_{\substack{k_1,\ldots,k_j\\ 0\le k_j\le n_j}} \prod_{j=1}^m A^\epsilon_{n_j,k_j}\int\prod_{j=1}^m \frac{:G^{2k_j}_{(j),x,\epsilon}:}{2^{k_j}}\mathcal{L}^{x,\epsilon}_k\,d\mu(x) + o(u^1(\epsilon)^{-(\sigma-(n+k))})\end{aligned}$$

where the $A^\epsilon_{n_j,k_j}$ are given in (4.15) and we set $\mathcal{L}_0\equiv 1$. Recall that in this formula $n=\sum_{j=1}^m n_j$ and $\mu\in\mathcal{G}^{2\sigma}$ with $\sigma\ge n+k$. Define $\Psi_{(j),k_j}(x,\epsilon)$ in terms of $H^{k_j}_{(j)}(x,\epsilon)$ analogously to (4.16), as follows:

$$\int \prod_{j=1}^m \frac{H^{n_j}_{(j)}(x,\epsilon)}{2^{n_j}}\mathcal{L}^{x,\epsilon}_k\,d\mu(x) = \sum_{\substack{k_1,\ldots,k_j\\ 0\le k_j\le n_j}} \prod_{j=1}^m A^\epsilon_{n_j,k_j}\int\prod_{j=1}^m \Psi_{(j),k_j}(x,\epsilon)\mathcal{L}^{x,\epsilon}_k\,d\mu(x). \tag{4.64}$$

We now take $k=0$ and by the same arguments as those leading up to (4.19) we get

$$\lim_{\epsilon\to0}\int\prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\,d\mu(x) = \lim_{\epsilon\to0}\int\prod_{j=1}^m \frac{:G^{2n_j}_{(j),x,\epsilon}:}{2^{n_j}}\,d\mu(x). \tag{4.65}$$

By the same argument we used at (3.13) we see that the limit on the right-hand side of (4.65) exists in L^2. Hence the limit on the left-hand side of (4.65) also exists in L^2. Thus, with a slight change of notation, we obtain that when $\mu\in\mathcal{G}^{2n}$, (4.60) converges in L^2 in the case $k=0$ for all m as long as $n=\sum_{j=0}^m n_j$. We proceed to obtain this for all k by induction.

Assume that (4.60) has been established for all m and $n_0,\ldots,n_m$ such that $n=\sum_{j=0}^m n_j$, $k=0,\ldots,n_0$ and $n_0\le\nu-1$. We now show that it also holds when $n_0=\nu$. Let $\widetilde{G}_\rho$, $\rho\in\mathcal{G}^1$ denote another independent Gaussian process distributed like G_ρ, $\rho\in\mathcal{G}^1$, and let $\widetilde{\Psi}_\cdot(x,\epsilon)$ denote the analog of $\Psi_\cdot(x,\epsilon)$ built up from $\widetilde{G}_\rho$, $\rho\in\mathcal{G}^1$. By (4.59), with an obvious change of notation,

$$\begin{aligned}&\Phi_{\widetilde{G}}\left(\int\prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-v}(x,\epsilon)\widetilde{\Psi}_v(x,\epsilon)\,d\mu(x)\right)\\ &\quad=\sum_{k=0}^v\binom{v}{k}\int\prod_{j=1}^m \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-v}(x,\epsilon)\widetilde{\Psi}_{v-k}(x,\epsilon)\mathcal{L}^{x,\epsilon}_k\,d\mu(x)\end{aligned}\tag{4.66}$$

where $\Phi_{\widetilde{G}}$ denotes the ring homeomorphism defined in (4.11) applied to the processes defined and constructed from $\widetilde{G}_\rho$, $\rho\in\mathcal{G}^1$. By the induction hypothesis the integrals on the right-hand side of (4.66) for $k = 0, \ldots, v-1$ all converge in L^2. The $k = 0$ integral is the argument of $\Phi_{\widetilde{G}}$. Hence the right-hand side of (4.66) also converges in L^2 since $\Phi_{\widetilde{G}}$ is a L^2 isometry. Thus, we verify the assertion about (4.60).

We return to (4.64). Again, by the same arguments leading up to (3.13) we get, analogously to (4.65), that for $\mu \in \mathcal{G}^{2n}$

$$\lim_{\epsilon \to 0} \int \prod_{j=1}^{m} \frac{:G^{2n_j}_{(j),x,\epsilon}:}{2^{n_j}} \frac{:G^{2(n_0-k)}_{(0),x,\epsilon}:}{2^{n_0-k}} \mathcal{L}^{x,\epsilon}_k \, d\mu(x)$$
$$= \lim_{\epsilon \to 0} \int \prod_{j=1}^{m} \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-k}(x,\epsilon)\mathcal{L}^{x,\epsilon}_k \, d\mu(x). \tag{4.67}$$

Now we know that the limit on the right-hand side of (4.67) exists in L^2 and therefore so does the limit on the left-hand side. In particular, we have established (4.8).

Using (4.20), (4.53) and (4.67) we have

$$\begin{aligned} \Phi\left(\frac{:G^{2n}\mu:}{2^n}\right) &= \lim_{\epsilon \to 0} \Phi(\Psi_{n,\epsilon}\mu) \\ &= \lim_{\epsilon \to 0} \sum_{k=0}^{n} \binom{n}{k} \int \Psi_{n-k}(x,\epsilon)\mathcal{L}^{x,\epsilon}_k \, d\mu(x) \\ &= \lim_{\epsilon \to 0} \sum_{k=0}^{n} \binom{n}{k} \int \frac{:G^{2(n-k)}_{x,\epsilon}:}{2^{(n-k)}} \mathcal{L}^{x,\epsilon}_k \, d\mu(x) \end{aligned} \tag{4.68}$$

which gives us (4.22). This completes the proof of Lemma 4.2, and hence of Theorem 4.1. □

We have actually proved a more general isomorphism than Theorem 4.1 which will be used in Chapter 5. For any m and $n_0, \ldots, n_m$ such that $n = \sum_{j=0}^{m} n_j$ and $k = 0, \ldots, n_0$ define

$$(\times_{j=1}^{m} :G^{2n_j}_{(j)}: \times :G^{2(n_0-k)}_{(0)}: \times \mathcal{L}_k)(\mu) \tag{4.69}$$
$$= \lim_{\epsilon \to 0} \int \prod_{j=1}^{m} :G^{2n_j}_{(j),x,\epsilon}::G^{2(n_0-k)}_{(0),x,\epsilon}: \mathcal{L}^{x,\epsilon}_k \, d\mu(x).$$

We have just shown that when $\mu \in \mathcal{G}^{2n}$, the right-hand side of (4.69) converges in L^2. Therefore by (4.67) and the fact that $\Phi_{(0)}$ is an L^2 isometry we have that

$$\Phi_{(0)}((\times_{j=1}^{m} :G^{2n_j}_{(j)}: \times :G^{2n_0}_{(0)}:)(\mu)) \tag{4.70}$$
$$= \lim_{\epsilon \to 0} \Phi_{(0)}\left(\int 2^n \prod_{j=1}^{m} \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0}(x,\epsilon) \, d\mu(x)\right).$$

We next use (4.59), (4.67) and (4.69) to see that the left-hand side of (4.70) is equal to

$$= \lim_{\epsilon \to 0} \sum_{k=0}^{n_0} \binom{n_0}{k} 2^n \int \prod_{j=1}^{m} \Psi_{(j),n_j}(x,\epsilon)\Psi_{(0),n_0-k}(x,\epsilon)\mathcal{L}^{x,\epsilon}_k \, d\mu(x)$$

$$= \sum_{k=0}^{n_0} \binom{n_0}{k} 2^k (\times_{j=1}^m : G_{(j)}^{2n_j} : \times : G_{(0)}^{2(n_0-k)} : \times \mathcal{L}_k)(\mu) \tag{4.71}$$

Using (4.70) and (4.71) we get the following extension of Theorem 4.1:

THEOREM 4.3. *Let X be a Lévy process in Class A and let $\{\mu_i\}_{i=1}^\infty$ be sequence of finite positive measures in $\mathcal{G}^{2n}$. Then for any m and $n_0, \ldots, n_m$ such that $n = \sum_{j=0}^m n_j$, and any compactly supported measure $\rho \in \mathcal{G}^1$ and $\mathcal{C}$ measurable non-negative function F on R^∞*

$$E_{\mathbf{G}} E_\lambda^\rho (F(\sum_{k=0}^{n_0} \binom{n_0}{k} \frac{1}{2^{n_0-k}} (\times_{j=1}^m : G_{(j)}^{2n_j} : \times : G_{(0)}^{2(n_0-k)} : \times \mathcal{L}_k)(\mu.)) f(x_\lambda))$$

$$= E_{\mathbf{G}} (F(\frac{1}{2^{n_0}} (\times_{j=1}^m : G_{(j)}^{2n_j} : \times : G_{(0)}^{2n_0} :)(\mu.)) G_{(0),\rho} G_{(0),f\cdot dx}) \tag{4.72}$$

where $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞, and $\mathbf{G}$ denote the product probability space that generated by $G_{(0),\rho}, \ldots, G_{(m),\rho}$, $\rho \in \mathcal{G}^1$.

CHAPTER 5

Renormalized self-intersection local times

The renormalized self-intersection local times that appear in the isomorphism theorems, Theorems 4.1 and 4.3 are defined for each $n \geq 0$ in (**??**) and (4.9) as the following limits

$$\mathcal{L}_n(\mu) = \lim_{\epsilon \to 0} \mathcal{L}_{n,\epsilon}(\mu) = \lim_{\epsilon \to 0} \int \mathcal{L}_n^{x,\epsilon}\, d\mu(x) \tag{5.1}$$

where $\mathcal{L}_n^{x,\epsilon}$ is defined implicitly in (4.7). Let

$$L_{n,\epsilon}(\mu) \stackrel{def}{=} \int (L^{x,\epsilon})^n\, d\mu(x). \tag{5.2}$$

It follows from (4.7) that

$$L_{n,\epsilon}(\mu) = (B^\epsilon \mathcal{L}_\epsilon(\mu))_n = \sum_{k=0}^{n} B^\epsilon_{n,k} \mathcal{L}_{k,\epsilon}(\mu) \tag{5.3}$$

where $B^\epsilon_{n,k}$ is given in (4.5).

That we can not explicitly state what $\mathcal{L}_n(\mu)$ is, seems to diminish the significance of the isomorphism theorems. However, this is not a problem because we can show that $\mathcal{L}_n(\mu)$ can be taken to be $n!\gamma_n(\mu)$ where $\gamma_n(\mu)$ is given in (1.2).

THEOREM 5.1. *Let X be a Lévy process in Class A and let $\mu \in \mathcal{G}^{2n}$. Let $\gamma_{n,\epsilon}(\mu)$ be as defined in (1.3) and $\mathcal{L}_n(\mu)$ be the n-fold renormalized self-intersection local time of X as defined in (5.1). Then for any $\rho \in \mathcal{G}^1$*

$$\gamma_n(\mu) \stackrel{def}{=} \lim_{\epsilon \to 0} \gamma_{n,\epsilon}(\mu) \tag{5.4}$$

exists in $L^2(f(X_\lambda)dP^\rho_\lambda)$ and

$$\mathcal{L}_n(\mu) = n!\gamma_n(\mu) \tag{5.5}$$

in $L^2(f(X_\lambda)dP^\rho_\lambda)$.

PROOF. We show that $\mu \in \mathcal{G}^{2n}$ implies that

$$I_n(\mu,\rho,f) \stackrel{def}{=} 2\int U^1\rho(x)(u^1(x-y))^{2n-1}U^1 f(y)\, d\mu(x)\, d\mu(y) < \infty \tag{5.6}$$

and that

$$E^\rho_\lambda(\{\mathcal{L}_n(\mu) - n!\gamma_{n,\epsilon}(\mu)\}^2\, f(X_\lambda)) = o(1_\epsilon). \tag{5.7}$$

The limit in (5.7) is a consequence of the following three assertions:

$$E^\rho_\lambda(\{\mathcal{L}_n(\mu)\}^2\, f(X_\lambda)) = (n!)^2 I_n(\mu,\rho,f) \tag{5.8}$$

$$E^\rho_\lambda(\mathcal{L}_n(\mu)\gamma_{n,\epsilon}(\mu) f(X_\lambda)) = n! I_n(\mu,\rho,f) + o(1_\epsilon) \tag{5.9}$$

$$E^\rho_\lambda(\{\gamma_{n,\epsilon}(\mu)\}^2\, f(X_\lambda)) = I_n(\mu,\rho,f) + o(1_\epsilon). \tag{5.10}$$

We now prove (5.8). We first show by induction on $k = 0, 1, \dots, n$ that for any $m = 0, 1, \dots$ and $n_i \geq 0$, $i = 1, \dots, m$ with $\sum_{i=1}^m n_i = n - k$

$$E^\rho_\lambda E_G(\Big\{(\times_{i=1}^m : G^{2n_i}_{(i)} : \times \mathcal{L}_k)(\mu)\Big\}^2 f(X_\lambda)) = \prod_{i=1}^m (2n_i)!(k!)^2 I_n(\mu, \rho, f) \tag{5.11}$$

from which (5.8) follows when $m = 0$. To begin the induction first observe that when $\sum_{i=0}^m n_i = n$, that is, when $k = 0$, we have

$$\begin{aligned} &E^\rho_\lambda E_G(\Big\{(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2 f(X_\lambda)) \\ &= E_G(\Big\{(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2) E^\rho_\lambda(f(X_\lambda)) \\ &= E_G(\Big\{(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2) \int u^1(x-y)\, d\rho(x) f(y)\, dy \\ &= E_G(\Big\{(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2) E_G(G_\rho G_{f\cdot dx}). \end{aligned} \tag{5.12}$$

Using Theorem 4.3, one of the isomorphism theorems, with $F(x) = x^2$, for $x \in R^1$, and observing that Wick powers of different orders are orthogonal, we see that when $\sum_{i=0}^m n_i = n$

$$\begin{aligned} &E_G(\Big\{\frac{1}{2^{n_0}}(\times_{i=1}^m : G^{2n_i}_{(i)} : \times : G^{2n_0}_{(0)} :)(\mu)\Big\}^2 G_{(0),\rho} G_{(0),f\cdot dx}) \\ &= \sum_{k=0}^{n_0} \binom{n_0}{k}^2 E^\rho_\lambda E_G(\Big\{\frac{1}{2^{n_0-k}}(\times_{i=1}^m : G^{2n_i}_{(i)} : \times : G^{2(n_0-k)}_{(0)} : \times \mathcal{L}_k)(\mu)\Big\}^2 f(X_\lambda)). \end{aligned} \tag{5.13}$$

In evaluating the expectation on the left-hand side, note that either $G_{(0),\rho}$ is paired with $G_{(0),f\cdot dx}$, or necessarily $G_{(0),\rho}$ is paired with one of the Wick powers and $G_{(0),f\cdot dx}$ is paired with the other Wick power. Therefore, it follows from (2.12) that

$$\begin{aligned} &E_G(\Big\{\frac{1}{2^{n_0}}(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2 G_{(0),\rho} G_{(0),f\cdot dx}) \\ &= E_G(\Big\{\frac{1}{2^{n_0}}(\times_{i=0}^m : G^{2n_i}_{(i)} :)(\mu)\Big\}^2) E_G(G_\rho G_{f\cdot dx}) \\ &\quad + \frac{1}{2^{2n_0}} \prod_{i=1}^m (2n_i)!(2n_0)^2(2n_0-1)! I_n(\mu, \rho, f). \end{aligned} \tag{5.14}$$

Note that taking $m = 0$ shows that $I_n(\mu, \rho, f) < \infty$.

The first term on the right-hand side of (5.14) is, by (5.12), precisely the $k = 0$ term of the right-hand side of (5.13). Thus we see that

$$\begin{aligned} &\frac{1}{2^{2n_0}} \prod_{i=1}^m (2n_i)!(2n_0)! 2n_0 I_n(\mu, \rho, f) \\ &= \sum_{k=1}^{n_0} \binom{n_0}{k}^2 E^\rho_\lambda E_G(\Big\{\frac{1}{2^{n_0-k}}(\times_{i=1}^m : G^{2n_i}_{(i)} : \times : G^{2(n_0-k)}_{(0)} : \times \mathcal{L}_k)(\mu)\Big\}^2 f(X_\lambda)). \end{aligned} \tag{5.15}$$

Taking $n_0 = 1$ we obtain (5.11) when $k = 1$.

Assume now that for some $n_0 \leq n$ we have proved (5.11) for all $k < n_0$. Then (5.15) can be written as

$$\frac{1}{2^{2n_0}} \prod_{i=0}^{m} (2n_i)! 2n_0 I_n(\mu, \rho, f) \tag{5.16}$$
$$= \sum_{k=1}^{n_0-1} \binom{n_0}{k}^2 \frac{1}{2^{2(n_0-k)}} \prod_{i=1}^{m} (2n_i)!(2(n_0-k))!(k!)^2 I_n(\mu, \rho, f)$$
$$+ E_\lambda^\rho E_G(\left\{(\times_{i=1}^m : G_{(i)}^{2n_i} \times \mathcal{L}_{n_0})(\mu)\right\}^2 f(X_\lambda)).$$

Thus, to establish (5.11) for $k = n_0$ we need only show that

$$\frac{1}{2^{2n_0}} \prod_{i=0}^{m} (2n_i)! 2n_0 = \sum_{k=1}^{n_0} \binom{n_0}{k}^2 \frac{1}{2^{2(n_0-k)}} \prod_{i=1}^{m} (2n_i)!(2(n_0-k))!(k!)^2 \tag{5.17}$$

or equivalently, that

$$\frac{1}{2^{2n_0}} (2n_0)! 2n_0 = \sum_{k=1}^{n_0} \binom{n_0}{k}^2 \frac{1}{2^{2(n_0-k)}} (2(n_0-k))!(k!)^2. \tag{5.18}$$

Writing

$$\binom{n_0}{k}^2 (k!)^2 = \frac{(n_0!)^2}{((n_0-k)!)^2}$$

we see that (5.18) is equivalent to

$$\frac{1}{2^{2n_0}} 2n_0 \binom{2n_0}{n_0} = \sum_{k=1}^{n_0} \frac{1}{2^{2(n_0-k)}} \binom{2(n_0-k)}{n_0-k} \tag{5.19}$$

which we rewrite as

$$\frac{1}{2^{2n_0}} 2n_0 \binom{2n_0}{n_0} = \sum_{j=0}^{n_0-1} \frac{1}{2^{2j}} \binom{2j}{j}. \tag{5.20}$$

This, in turn, is equivalent to the combinatorial identity

$$\frac{1}{2^{2n_0}} (2n_0+1) \binom{2n_0}{n_0} = \sum_{j=0}^{n_0} \frac{1}{2^{2j}} \binom{2j}{j}. \tag{5.21}$$

This identity is well known, see e.g. [**21**], page 131. This completes the proof of (5.8).

We next prove (5.10). Note that

$$E_\lambda^\rho(\alpha_{n_1,\epsilon}(\mu,\lambda)\alpha_{n_2,\epsilon}(\mu,\lambda) f(X_\lambda))$$
$$= E_\lambda^\rho(\int \int_{\substack{\{0 \leq s_1 \leq \cdots \leq s_{n_1} \leq \lambda\} \\ \{0 \leq t_1 \leq \cdots \leq t_{n_2} \leq \lambda\}}} f_{\epsilon,x_1}(X_{s_1}) \prod_{j=2}^{n_1} f_\epsilon(X_{s_j} - X_{s_{j-1}}) \tag{5.22}$$
$$f_{\epsilon,x_2}(X_{t_1}) \prod_{j=2}^{n_2} f_\epsilon(X_{t_j} - X_{t_{j-1}})$$
$$ds_1 \cdots ds_{n_1}\, dt_1 \cdots dt_{n_2}\, d\mu(x_1)\, d\mu(x_2) f(X_\lambda)).$$

Considering the different ways in which we can arrange the n_1+n_2 terms $s_1,\dots,s_{n_1}$ and $t_1,\dots,t_{n_2}$ in increasing order and taking the expectation we see that the above equation

$$= \sum_{v\in\mathcal{V}} \int U^1\rho(z_{v(1)}) \prod_{p=2}^{n_1+n_2} u^1(z_{v(p)} - z_{v(p-1)}) U^1 f(z_{v(n_1+n_2)})$$
$$\prod_{i=1}^{2} f_{\epsilon,x_i}(z_{i,1}) \prod_{j=2}^{n_i} f_\epsilon(z_{i,j} - z_{i,j-1})\, dz_{i,j}\, d\mu(x_i) \tag{5.23}$$

where $\mathcal{V}$ is the set of bijections

$$v : \{1,\dots,n_1+n_2\} \mapsto \{(i,j);\ i=1,2;\ 1\le j\le n_i\}$$

such that when $v(p)=(i,j)$ and $v(\widetilde{p}) = (i,\widetilde{j})$ then $p<\widetilde{p}$ if and only if $j<\widetilde{j}$. We make the change of variables $y_{i,1} = z_{i,1} - x_i$ and $y_{i,j} = z_{i,j} - z_{i,j-1}$, $i=1,2$, $j\ge 2$ which leads to

$$E^\rho_\lambda(\alpha_{n_1,\epsilon}(\mu)\alpha_{n_2,\epsilon}(\mu) f(X_\lambda)) = \sum_{s\in\mathcal{S}} \int U^1\rho(x_{s(1)} + y_{s(1),1})$$
$$\prod_{p=2}^{n_1+n_2} u^1(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}) \tag{5.24}$$
$$U^1 f(x_{s(n_1+n_2)} + \sum_{j=1}^{c(n_1+n_2)} y_{s(n_1+n_2),j}) \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_\epsilon(y_{i,j})\, dy_{i,j}\, d\mu(x_i)$$

where $\mathcal{S}$ is the set of mappings

$$s : \{1,\dots,n_1+n_2\} \mapsto \{1,2\}$$

such that $|s^{-1}(i)| = n_i$, $i=1,2$ and $c(p) = |\{m\le p|\, s(m)=s(p)\}|$, $1\le p\le n_1+n_2$.

For a fixed $s\in\mathcal{S}$ we say that p is 'good' if $s(p)\neq s(p-1)$, while p is 'bad' if $s(p)=s(p-1)$. Here $p\ge 2$, and s(1) is always 'good'. Note that when p is bad, the corresponding u^1 term in (5.24) is

$$u^1(y_{s(p),c(p)}) \tag{5.25}$$

and when p is 'good' $x_{s(p)} \neq x_{s(p-1)}$. For each $s\in\mathcal{S}$ we set $B_s = \{p|\, s(p) = s(p-1)\}$, the set of 'bad' points p.

Let us analyze the changes which occur in (5.24) when we replace one of the factors $\alpha_{n_i,\epsilon}(\mu)$ by $\binom{n_i-1}{k_i}(u^1_\epsilon(0))^{k_i}\alpha_{n_i-k_i,\epsilon}(\mu)$. We claim that

$$E^\rho_\lambda(\binom{n_1-1}{k_1}(u^1_\epsilon(0))^{k_1}\alpha_{n_1-k_1,\epsilon}(\mu,\lambda)\alpha_{n_2,\epsilon}(\mu,\lambda) f(X_\lambda)) \tag{5.26}$$
$$= \sum_{\substack{D_1\subseteq\{2,\dots,n_1\}\\ |D_1|=k_1}} \sum_{s\in\mathcal{S}_{D_1}} \int U^1\rho(x_{s(1)} + y_{s(1),1}) \prod_{l\in D_1} u^1(y_{1,l}) \prod_{\substack{p=2\\ (s(p),c(p))\notin(1,D_1)}}^{n_1+n_2}$$
$$u^1(x_{s(p)} + \sum_{\substack{j=1\\ (s(p),j)\notin(1,D_1)}}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{\substack{j=1\\ (s(p-1),j)\notin(1,D_1)}}^{c(p-1)} y_{s(p-1),j})$$

$$U^1 f(x_{s(n_1+n_2)}+\sum_{\substack{j=1\\(s(p),j)\notin(1,D_1)}}^{c(n_1+n_2)} y_{s(n_1+n_2),j})\prod_{i=1}^{2}\prod_{j=1}^{n_i} f_\epsilon(y_{i,j})\,dy_{i,j}\,d\mu(x_i).$$

where $\mathcal{S}_{D_1}$ is the subset of $\mathcal{S}$ such that $\{p\,|\,(s(p),c(p))\in(1,D_1)\}\in B_s$.

Here is how we obtain (5.26). The analogue of (5.24) with n_1 replaced by n_1-k_1 will be a sum over the set of mappings $\widetilde{s}:\{1,\ldots,n_1-k_1+n_2\}\mapsto\{1,2\}$ such that $|\widetilde{s}^{-1}(1)|=n_1-k_1$ and $|\widetilde{s}^{-1}(2)|=n_2$. For each of the $\binom{n_1-1}{k_1}$ subsets $D_1\subseteq\{2,\ldots,n_1\}$ with $|D_1|=k_1$ there is a unique way to 'extend' $\widetilde{s}$ to a map $s:\{1,\ldots,n_1+n_2\}\mapsto\{1,2\}$ with $|s^{-1}(1)|=n_1$ and $|s^{-1}(2)|=n_2$, in such a way that $s\in\mathcal{S}_{D_1}$. (Think of $\widetilde{s}$ as defining a coloring of $n_1-k_1+n_2$ balls lined up in a row. We color the p'th ball red if $\widetilde{s}(p)=1$, and color it white otherwise. Now label the red balls successively with the numbers in $D_1^c=(j_1,\ldots,j_{n_1-k_1})$. Then, succesively place a new red ball with label $i\in D_1=(i_1,\ldots,i_{k_1})$ immediately after the red ball with label $i-1$. We obtain n_1 red and n_2 white balls lined up in a row. Finally, set $s(p)=1$ if the p'th ball is red, and $s(p)=2$ if the p'th ball is white). The u^1 terms corresponding to the set of phantom 'bad' points $\{p\,|\,(s(p),c(p))\in(1,D_1)\}$ are $\prod_{l\in D_1}u^1(y_{1,l})$. Note that the integral in (5.26) with respect to these bad points gives

$$\prod_{l\in D_1}\int f_\epsilon(y_{1,l})u^1(y_{1,l})\,dy_{1,l}=(u^1_\epsilon(0))^{k_1}. \tag{5.27}$$

Thus we get (5.26).

There may be other 'bad' points in the various $s\in\mathcal{S}_{D_1}$ in (5.26) besides the phantom 'bad' points. Including these 'bad' points in with the phantom 'bad' points we can write (5.26) as

$$\begin{aligned}&E^\rho_\lambda(\binom{n_1-1}{k_1}(u^1_\epsilon(0))^{k_1}\alpha_{n_1-k_1,\epsilon}(\mu,\lambda)\alpha_{n_2,\epsilon}(\mu,\lambda)f(X_\lambda))\\&=\sum_{\substack{D_1\subseteq\{2,\ldots,n_1\}\\|D_1|=k_1}}\sum_{s\in\mathcal{S}_{D_1}}\int U^1\rho(x_{s(1)}+y_{s(1),1})\prod_{p\in B_s}u^1(y_{s(p),c(p)})\\&\prod_{p\in B_s^c}u^1(x_{s(p)}+\sum_{\substack{j=1\\(s(p),j)\notin(1,D_1)}}^{c(p)}y_{s(p),j}-x_{s(p-1)}-\sum_{\substack{j=1\\(s(p-1),j)\notin(1,D_1)}}^{c(p-1)}y_{s(p-1),j})\\&U^1f(x_{s(n_1+n_2)}+\sum_{\substack{j=1\\(s(p),j)\notin(1,D_1)}}^{c(n_1+n_2)}y_{s(n_1+n_2),j})\prod_{i=1}^{2}\prod_{j=1}^{n_i}f_\epsilon(y_{i,j})\,dy_{i,j}\,d\mu(x_i).\end{aligned} \tag{5.28}$$

Generalizing (5.28) we can we can write out

$$E^\rho_\lambda(\binom{n_1-1}{k_1}\binom{n_2-1}{k_2}(u^1_\epsilon(0))^{k_1+k_2}\alpha_{n_1-k_1,\epsilon}(\mu,\lambda)\alpha_{n_2-k_2,\epsilon}(\mu,\lambda)f(X_\lambda)) \tag{5.29}$$

for which (5.28) is modified by summing also over the sets $D_2\subseteq\{2,\ldots,n_2\}$ of cardinality k_2 and $s\in\mathcal{S}_{D_1,D_2}$ where $\mathcal{S}_{D_1,D_2}$ is the subset of $\mathcal{S}$ such that $(1,D_1)\cup(2,D_2)\subseteq\{(s(p),c(p)):p\in B_s\}$. We also introduce an additional k_2 phantom 'bad' points and now, after the symbol $\prod_{p\in B_s^c}$, sum only over j such that $(s(p),j)\notin (1,D_1)\cup(2,D_2)$.

Recall the definition of $\gamma_{n,\epsilon}(\mu)$ in (1.3). Let $h(x)$ be a function of the variable x. We use the notation

$$\mathcal{D}_x h \equiv h(x)-h(0).$$

We claim that

$$\begin{aligned}
&E^\rho_\lambda(\gamma_{n,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_\lambda))\\
&\quad=\sum_{s\in\mathcal{S}}\int U^1\rho(x_{s(1)}+y_{s(1),1})\prod_{p\in B_s}u^1(y_{s(p),c(p)})\prod_{p\in B_s}\mathcal{D}_{y_{s(p),c(p)}} \qquad (5.30)\\
&\quad\left\{\prod_{p\in(B_s\cup\{1\})^c}u^1(x_{s(p)}+\sum_{j=1}^{c(p)}y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)}y_{s(p-1),j})\right\}\\
&\quad U^1f(x_{s(2n)}+\sum_{j=1}^{c(2n)}y_{s(2n),j})\prod_{i=1}^2\prod_{j=1}^n f_\epsilon(y_{i,j})\,dy_{i,j}\,d\mu(x_i).
\end{aligned}$$

Here is how we obtain (5.30). Abbreviate $y_p\equiv y_{s(p),c(p)}$ and set

$$\begin{aligned}
&F(y_1,\dots,y_{n_1+n_2})=\\
&\quad\left\{\prod_{p\in(B_s\cup\{1\})^c}u^1(x_{s(p)}+\sum_{j=1}^{c(p)}y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)}y_{s(p-1),j})\right\} \qquad (5.31)\\
&\quad U^1f(x_{s(2n)}+\sum_{j=1}^{c(2n)}y_{s(2n),j}).
\end{aligned}$$

Write the difference operator $\mathcal{D}_{y_p}=I_{y_p}-R_{y_p}$ where $\mathcal{D}_{y_p}h=0$ if the function h does not contain the variable y_p and otherwise, that is if h contains the variable y_p, $I_{y_p}h$ leaves h unchanged and $R_{y_p}h$ sets the variable y_p to 0. Thus

$$\prod_{p\in B_s}\mathcal{D}_{y_{s(p),c(p)}}=\sum_{D\subseteq B_s}\prod_{p\in B_s-D}I_{y_p}\prod_{p\in D}(-R_{y_p}). \qquad (5.32)$$

It follows that

$$\begin{aligned}
&\prod_{p\in B_s}\mathcal{D}_{y_{s(p),c(p)}}F(y_1,\dots,y_{n_1+n_2})\\
&\qquad=\sum_{D\subseteq B_s}(-1)^{|D|}F(y_1,\dots,y_{n_1+n_2})|_{y_p=0\text{ if }p\in D}. \qquad (5.33)
\end{aligned}$$

Let $D=D_1\cup D_2$ such that $s|_{D_1}=1$ and $s|_{D_2}=2$ and let $k_1=|D_1|$ and $k_2=|D_2|$. Assume, initially, that $k_2=0$. Fix $s\in\mathcal{S}_{D_1}$

$$\begin{aligned}
&\int U^1\rho(x_{s(1)}+y_{s(1),1})\prod_{p\in B_s}u^1(y_{s(p),c(p)})\\
&\qquad F(y_1,\dots,y_{n_1+n_2})|_{y_p=0\text{ if }p\in D_1}\prod_{i=1}^2\prod_{j=1}^{n_i}f_\epsilon(y_{i,j})\,dy_{i,j}\,d\mu(x_i) \qquad (5.34)
\end{aligned}$$

is precisely the term in (5.28) corresponding to this $s \in \mathcal{S}_{D_1}$. Considering the expansion of (5.29) the same argument gives the corresponding term when $|D_1| = k_1$ and $|D_2| = k_2$. Thus, substituting (5.33) into (5.30) we verify (5.30).

We now show that except in the case $B_s = \emptyset$ the terms in (5.30) are all $o(1_\epsilon)$. If $|B_s| = j$, $j \geq 2$, we use two of the difference operators in (5.30) and replace all the other differences by sums. This gives us an upper bound for the left-hand side of (5.30) since the u^1 terms are all positive. By (3.26) and (3.28) the $\prod_{p \notin B_s} dy_{s(p),c(p)}\, d\mu(x_1)\, d\mu(x_2)$ integral is bounded by

$$o((u^1(|\epsilon|))^{-j}) \int \prod_{p \in B_s} u^1(y_{s(p),c(p)}) f_\epsilon(y_{s(p),c(p)})\, dy_{s(p),c(p)},$$

while

$$\int \prod_{p \in B_s} u^1(y_{s(p),c(p)}) f_\epsilon(y_{s(p),c(p)})\, dy_{s(p),c(p)} \leq C(u^1(|\epsilon|))^j \tag{5.35}$$

by Lemma 3.2. Thus the terms in (5.30) for which $|B_s| \geq 2$ are $o(1_\epsilon)$. We get the same conclusion when $|B_s| = 1$ if we use (3.27). Therefore, up to an error which is $o(1_\epsilon)$, we can restrict the sum in (5.30) to $s \in \mathcal{S}_0$ where

$$\mathcal{S}_0 \equiv \{s \in \mathcal{S} \,|\, B_s = \emptyset\} = \{s \in \mathcal{S} \,|\, s(p) \neq s(p-1),\ \forall p\}. \tag{5.36}$$

Using (3.27) once again we now see that

$$\begin{aligned} E^\rho_\lambda(\gamma_{n,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu) f(X_\lambda)) &= \sum_{s \in \mathcal{S}_0} \int U^1\rho(x_{s(1)}) \prod_{p=2}^{2n} u^1(x_{s(p)} - x_{s(p-1)}) \\ &\qquad U^1 f(x_{s(2n)})\, d\mu(x_1)\, d\mu(x_2) + o(1_\epsilon). \end{aligned} \tag{5.37}$$

There are only two members in $\mathcal{S}_0$, s_1 and s_2, where $s_1(2j) = 1$, $s_1(2j+1) = 2$ for all j and $s_2(2j) = 2$, $s_2(2j+1) = 1$ for all j. Consequently, (5.37) implies (5.10).

Lastly, we turn to the proof of (5.9). Note that

$$L_{n,\epsilon}(\mu) = n! \int_{\{0 \leq t_1 \leq \cdots \leq t_n\}} \prod_{j=1}^n f_{\epsilon,x}(X_{t_j})\, dt_1 \cdots dt_n\, d\mu(x). \tag{5.38}$$

As in the transition from (5.22) to (5.23) we see that

$$\begin{aligned} &E^\rho_\lambda(L_{n_1,\epsilon}(\mu)\alpha_{n_2,\epsilon}(\mu,\lambda) f(X_\lambda)) \\ &= n_1! E^\rho_\lambda\Big(\int_{\substack{\{0 \leq s_1 \leq \cdots \leq s_{n_1}\} \\ \{0 \leq t_1 \leq \cdots \leq t_{n_2}\}}} \prod_{j=1}^{n_1} f_{\epsilon,x_1}(X_{s_j}) f_{\epsilon,x_2}(X_{t_1}) \prod_{j=2}^{n_2} f_\epsilon(X_{t_j} - X_{t_{j-1}}) \\ &\quad ds_1 \cdots ds_{n_1}\, dt_1 \cdots dt_{n_2}\, d\mu(x_1)\, d\mu(x_2) f(X_\lambda)\Big) \\ &= n_1! \sum_{v \in \mathcal{V}} \int U^1\rho(z_{v(1)}) \prod_{p=2}^{n_1+n_2} u^1(z_{v(p)} - z_{v(p-1)}) U^1 f(z_{v(n_1+n_2)}) \\ &\quad \prod_{j=1}^{n_1} f_{\epsilon,x_1}(z_{1,j}) f_{\epsilon,x_2}(z_{2,1}) \prod_{j=2}^{n_2} f_\epsilon(z_{2,j} - z_{2,j-1}) \prod_{i=1}^{2} \prod_{j=1}^{n_i} dz_{i,j}\, d\mu(x_i) \end{aligned} \tag{5.39}$$

where, as above, $\mathcal{V}$ is the set of bijections

$$v : \{1, \ldots, n_1 + n_2\} \mapsto \{(i,j);\ i = 1,2;\ 1 \leq j \leq n_i\}$$

such that when $v(p) = (i,j)$ and $v(\widetilde{p}) = (i,\widetilde{j})$ then $p < \widetilde{p}$ if and only if $j < \widetilde{j}$. We make the change of variables $y_{2,1} = z_{2,1} - x_2$, $y_{2,j} = z_{2,j} - z_{2,j-1}$, $j \geq 2$ and $y_{1,j} = z_{1,j} - x_1$, $j \geq 1$. This leads to

$$\begin{aligned} E^{\rho}_{\lambda}(L_{n_1,\epsilon}(\mu)\alpha_{n_2,\epsilon}(\mu)f(X_\lambda)) &= n_1! \sum_{s\in\mathcal{S}} \int U^1\rho(x_{s(1)} + y_{s(1),1}) \\ &\prod_{p=2}^{n_1+n_2} u^1(x_{s(p)} + \sum_{j\in D_p} y_{s(p),j} - x_{s(p-1)} - \sum_{j\in D_{p-1}} y_{s(p-1),j}) \qquad (5.40) \\ &U^1 f(x_{s(n_1+n_2)} + \sum_{j\in D_{n_1+n_2}} y_{s(n_1+n_2),j}) \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_\epsilon(y_{i,j})\, dy_{i,j}\, d\mu(x_i) \end{aligned}$$

where $\mathcal{S}$ is the set of mappings

$$s : \{1, \ldots, n_1 + n_2\} \mapsto \{1, 2\}$$

such that $|s^{-1}(i)| = n_i$, $i = 1, 2$. Furthermore, $D_p = \{c(p)\}$ if $s(p) = 1$ and $D_p = \{1, \ldots, c(p)\}$ if $s(p) = 2$. As above, $c(p) = |\{m \leq p|\, s(m) = s(p)\}|$.

Arguing as above, we see that for $n \geq n_1$

$$\begin{aligned} E^{\rho}_{\lambda}(L_{n_1,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_\lambda)) &= n_1! \sum_{s\in\mathcal{S}} \int U^1\rho(x_{s(1)} + y_{s(1),1}) \\ &\prod_{p\in B_{s,1}} u^1(y_{s(p),c(p)} - y_{s(p-1),c(p-1)}) \prod_{p\in B_{s,2}} u^1(y_{s(p),c(p)}) \qquad (5.41) \\ &\prod_{p\in B_{s,2}} \mathcal{D}_{y_{s(p),c(p)}} \prod_{p\in B_s^c} u^1(x_{s(p)} + \sum_{j\in D_p} y_{s(p),j} - x_{s(p-1)} - \sum_{j\in D_{p-1}} y_{s(p-1),j}) \\ &U^1 f(x_{s(n_1+n)} + \sum_{j\in D_{n_1+n}} y_{s(n_1+n),j}) \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_\epsilon(y_{i,j})\, dy_{i,j}\, d\mu(x_i) \end{aligned}$$

where $B_{s,i} \stackrel{def}{=} B_s \cap s^{-1}(i)$.

When $|B_{s,2}| \geq 2$ the term in (5.41) corresponding to s is $o((u(\epsilon))^{-(n-n_1)})$. This is obtained using Lemma 3.4, keeping in mind that $\mu \in \mathcal{G}^{2n}$. Note that $|B_{s,1}| \leq n_1 - 1$ so when $|B_{s,2}| = 1$, $|B_{s,1}| + |B_{s,2}| \leq n_1$. Thus it follows from (3.27) that when $|B_{s,2}| = 1$ the term in (5.41) corresponding to s is also $o((u(\epsilon))^{-(n-n_1)})$. Hence in (5.41), up to an error which is $o((u(\epsilon))^{-(n-n_1)})$, we can restrict the summation to $\mathcal{S}_2 \stackrel{def}{=} \{s \in \mathcal{S}|\, B_{s,2} = \emptyset\}$. Arguing as in (5.37), using (3.27), we see that

$$\begin{aligned} E^{\rho}_{\lambda}(L_{n_1,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_\lambda)) &= n_1! \sum_{s\in\mathcal{S}_2} \int U^1\rho(x_{s(1)}) \\ &\prod_{p\in B_{s,1}} u^1(y_{s(p),c(p)} - y_{s(p-1),c(p-1)}) \prod_{p\in B_s^c} u^1(x_{s(p)} - x_{s(p-1)}) \qquad (5.42) \\ &U^1 f(x_{s(n_1+n)}) \prod_{j=1}^{n_1} f_\epsilon(y_{1,j})\, dy_{1,j} \prod_{i=1}^{2} d\mu(x_i) + o((u(\epsilon))^{-(n-n_1)}). \end{aligned}$$

Note that $\mathcal{S}_2 = \emptyset$ when $n_1 \leq n - 2$. Therefore

$$E^{\rho}_{\lambda}(L_{n_1,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_\lambda)) = o((u(\epsilon))^{-(n-n_1)}) \qquad n_1 \leq n-2. \qquad (5.43)$$

When $n_1 = n-1$, note that $\mathcal{S}_2$ consists of the single element $\tilde{s}$ such that $\tilde{s}(2j-1) = 2$, $\tilde{s}(2j) = 1$, $j = i, \ldots, n-1$, and $\tilde{s}(2n-1) = 2$. In this case we get

$$E^{\rho}_{\lambda}(L_{n-1,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_{\lambda})) = (n-1)! J_n(\mu,\rho,f) + o((u(\epsilon))^{-1}) \tag{5.44}$$

where

$$J_n(\mu,\rho,f) \stackrel{def}{=} \int U^1\rho(x)(u^1(y-x))^{2n-2}U^1 f(y)\, d\mu(x)\, d\mu(y). \tag{5.45}$$

When $n_1 = n$, $\mathcal{S}_2 = \mathcal{S}_0 \cup \mathcal{S}_1$, where $\mathcal{S}_0$ is defined in (5.36) and $\mathcal{S}_1$ contains the $n-1$ elements with $|B_{s,1}| = 1$. The right-hand side of (5.42), with $n_1 = n$ and the sum restricted to $\mathcal{S}_0$, is equal to $n! I_n(\mu,\rho,f) + o(1_\epsilon)$, similarly to the calculation in (5.37). The right-hand side of (5.42), with $n_1 = n$ and the sum restricted to $\mathcal{S}_1$, is equal to $(n-1)n!ch^{\epsilon}_2 J_n(\mu,\rho,f) + o(1_\epsilon)$, ($ch^{\epsilon}_2$ is defined in (4.4)). Consequently we have

$$E^{\rho}_{\lambda}(L_{n,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_{\lambda})) = n! I_n(\mu,\rho,f) + (n-1)n!ch^{\epsilon}_2 J_n(\mu,\rho,f) + o(1_\epsilon). \tag{5.46}$$

It follows from (5.3) and (4.62), that

$$\mathcal{L}_{n,\epsilon}(\mu) = L_{n,\epsilon}(\mu) - B^{\epsilon}_{n,n-1}L_{n-1,\epsilon}(\mu) + \sum_{k=1}^{n-2} a_k L_{k,\epsilon}(\mu) \tag{5.47}$$

where

$$a_k = O((u^1(\epsilon))^{(n-k)}). \tag{5.48}$$

Using (5.43), (5.44), and (5.46) and the fact that $B^{\epsilon}_{n,n-1} = n(n-1)ch^{\epsilon}_2$, (see (4.5)), we now see that

$$E^{\rho}_{\lambda}(\mathcal{L}_{n,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu)f(X_{\lambda})) = n! I_n(\mu,\rho,f) + o(1_\epsilon). \tag{5.49}$$

By (5.1) this gives (5.9) which completes the proof of this theorem. □

Using (5.7) and the triangle inequality we see that

$$\lim_{\epsilon,\epsilon'\to 0} E^{\rho}_{\lambda}((\gamma_{n,\epsilon}(\mu) - \gamma_{n,\epsilon'}(\mu))^2 f(X_{\lambda})) = 0. \tag{5.50}$$

and, by techniques used in Chapter 9, that

$$\lim_{\epsilon,\epsilon'\to 0} E^{\rho}_{\lambda}(\gamma_{n,\epsilon}(\mu) - \gamma_{n,\epsilon'}(\mu))^2 = 0. \tag{5.51}$$

Thus we can find a limit for $\{\gamma_{n,\epsilon}\}$, P^{ρ}_{λ} almost surely and consequently P^{x}_{λ} almost surely for q.e. $x \in R^m$. However, we need more than this, specifically we want the limit to exist P^{x}_{λ} almost surely for all $x \in R^m$. We can do this if we impose an additional condition on the measures $\mu \in \mathcal{G}^{2n}$, namely that (3.57) holds. This condition does not restrict the scope of Theorem 1.1. We show in the Appendix that if $\{: G^{2n}\mu_x :, x \in R^m\}$ is locally bounded then (3.57) is satisfied.

THEOREM 5.2. *Let X be a Lévy process in Class A and let $\mu \in \mathcal{G}^{2n}$ also satisfy (3.57). Let $\gamma_{n,\epsilon}(\mu)$ be as defined in (1.3). Then for all $x \in R^m$*

$$\gamma_n(\mu) \stackrel{def}{=} \lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu) \tag{5.52}$$

exists in $L^2(\Omega \times R^+, P^x_\lambda)$. Furthermore

$$\lim_{\epsilon\to 0} E^{x}_{\lambda}(\gamma_{n,\epsilon}(\mu_y) - \gamma_n(\mu_y))^2 = 0 \tag{5.53}$$

uniformly in $y \in R^m$ *and*

$$\lim_{y \to y'} E^x_\lambda(\gamma_n(\mu_y) - \gamma_n(\mu_{y'}))^2 = 0 \tag{5.54}$$

for all $x \in R^m$.

Note that we can't simply prove Theorem 5.1 with ρ replaced by δ_x, the delta function at x, and $f \cdot dy$ replaced by dy, Lebesgue measure on R^m. In the proof of (5.8) we use the isomorphism theorem, Theorem 4.1. This requires that both measures ρ and $f \cdot dy$ are in $\mathcal{G}^1$. Neither δ_x, nor dy are in $\mathcal{G}^1$. However, in the proof of (5.10), we do not use Theorem 4.1. The proof of (5.10) can be adapted to prove Theorem 5.2.

Proof of Theorem 5.2. We show that the proof of (5.10) goes through with ρ replaced by δ_x and $f(X_\lambda)$ replaced by one. The proof of (5.10) begins with (5.22). In place of (5.22) and (5.23) we now get

$$E^x_\lambda(\alpha_{n_1,\epsilon}(\mu,\lambda)\alpha_{n_2,\epsilon}(\mu,\lambda)) = \sum_{v \in \mathcal{V}} \int u^1(z_{v(1)} - x) \prod_{p=2}^{n_1+n_2} u^1(z_{v(p)} - z_{v(p-1)})$$
$$\prod_{i=1}^{2} f_{\epsilon,x_i}(z_{i,1}) \prod_{j=2}^{n_i} f_\epsilon(z_{i,j} - z_{i,j-1})\, dz_{i,j}\, d\mu(x_i). \tag{5.55}$$

We continue to trace the proof of Theorem 5.1, step by step, and in place of (5.30) we get

$$\begin{aligned} &E^x_\lambda(\gamma_{n,\epsilon}(\mu)\gamma_{n,\epsilon}(\mu))) \\ &= \sum_{s \in \mathcal{S}} \int u^1(x_{s(1)} + y_{s(1),1} - x) \prod_{p \in B_s} u^1(y_{s(p),c(p)}) \prod_{p \in B_s} \mathcal{D}_{y_{s(p),c(p)}} \\ &\left\{ \prod_{p \in (B_s \cup \{1\})^c} u^1(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}) \right\} \\ &\prod_{i=1}^{2} \prod_{j=1}^{n} f_\epsilon(y_{i,j})\, dy_{i,j}\, d\mu(x_i). \end{aligned} \tag{5.56}$$

In the proof of Theorem 5.1 we applied Lemma 3.4 to (5.30) to get (5.10). In a similar fashion we use Lemma 3.5 to obtain

$$E^x_\lambda(\gamma_{n,\epsilon}(\mu)\gamma_{n,\epsilon'}(\mu)) = 2 \int u^1(w - x)(u^1(w - y))^{2n-1}\, d\mu(w)\, d\mu(y) + o(1_\epsilon) \tag{5.57}$$

where $\epsilon > \epsilon'$. (Actually, in the proof of Theorem 5.1 we took $\epsilon = \epsilon'$. It follows from Lemma 3.1 that we can also prove this more general result). Using the multiple Hölder inequality and (3.62) we see that

$$\begin{aligned} &\int u^1(x)(u^1(x - y))^{2n-1}\, d\mu(x)\, d\mu(y) \\ &\leq \left(\int\int (u^1(x)u^1(x - y))^n\, d\mu(x)\, d\mu(y) \right)^{1/n} \|\mu\|^{2n-2}_{(2n)} \end{aligned} \tag{5.58}$$

$$\leq \sup_x \left(\int (u^1(x-y))^n \, d\mu(y) \right)^{2/n} \|\mu\|_{(2n)}^{2n-2}.$$

Since this is finite by hypothesis we see that

$$\lim_{\epsilon,\epsilon' \to 0} E_\lambda^x (\gamma_{n,\epsilon}(\mu) - \gamma_{n,\epsilon'}(\mu))^2 = 0 \tag{5.59}$$

which establishes (5.53) and also (5.52).

The above analysis is clearly true with μ replaced by μ_y. Thus to obtain (5.54) all we need to show is that the $o(1_\epsilon)$ term can be taken to be independent of y. This error term is estimated in (3.34) and (3.35) . It is clear that (3.35) remains the same when μ is replaced by μ_y.

Using the facts that the $o(1_\epsilon)$ term can be taken to be independent of y and that $\gamma_n(\,\cdot\,)$ is linear we see from (5.57) and (5.58) that

$$E_\lambda^x (\gamma_n(\mu_y) - \gamma_n(\mu_{y'}))^2 \leq C \sup_x \left(\int (u^1(x-y))^n \, d\mu(y) \right)^{2/n} \|\mu_y - \mu_{y'}\|_{(2n)}^{2n-2}.$$

It is easy to see from the Fourier transform of $\|\mu_y - \mu_{y'}\|_{(2n)}^{2n}$ that $\lim_{y \to y'} \|\mu_y - \mu_{y'}\|_{(2n)} = 0$. Thus we obtain (5.54). □

It follows from (5.57) that when the hypotheses of Theorem 5.2 are satisfied

$$E_\lambda^x |\gamma_n(\mu)|^2 = 2 \int u^1(w-x)(u^1(w-y))^{2n-1} \, d\mu(w) \, d\mu(y) \qquad \forall x \in R^m. \tag{5.60}$$

Thus we see that the finiteness of this integral is a necessary condition for $\gamma_n(\mu)$ to exist P^x almost surely and to have a second moment. (It is not clear that one may be able to define a renormalized self-intersection local time without a second moment, under weaker conditions, or that one would want to). The next Lemma gives a smoothness condition on the Lévy exponent and measure μ that makes the condition $\mu \in \mathcal{G}^{2n}$ and the finiteness of the integral in (5.60) equivalent. This smoothness condition might seem rather strong and even arbitrary but it is satisfied by all the examples in Chapter 8. In general it is satisfied when the relevant functions are regularly varying at infinity.

Lemma 5.1. *Let $\hat{\mu} \geq 0$. Assume that*

$$\theta_{2n}(\tau) \approx \theta_{2n-1}(\tau) \int_0^{|\tau|/2} \frac{d\xi}{1+\psi(\xi)} \qquad \text{as } |\tau| \to \infty \tag{5.61}$$

and

$$\int \frac{\hat{\mu}(\xi)\, d\xi}{1+\psi(\tau-\xi)} \approx \hat{\mu}(\tau) \int_0^{|\tau|/2} \frac{d\xi}{1+\psi(\xi)} \qquad \text{as } |\tau| \to \infty. \tag{5.62}$$

Then $\mu \in \mathcal{G}^{2n}$ if and only if

$$\int u^1(x)(u^1(x-y))^{2n-1} \, d\mu(x) \, d\mu(y) < \infty. \tag{5.63}$$

(We write $f(x) \approx g(x)$, as $|x| \to \infty$, to indicate that there exists constants $0 < C \leq C' < \infty$ such that $Cf(x) \leq g(x) \leq C'f(x)$ for all $|x|$ sufficiently large).

PROOF. (Note that $\widehat{u^1}(\xi) = (1+\psi(\xi))^{-1}$ and $\theta_k(\tau) = ((1+\psi(\xi))^{-1})^{*(k)}$, the k-fold convolution of $(1+\psi(\xi))^{-1}$). The proof is immediate since the Fourier transform of the integral in (5.63) is

$$\int \theta_{2n-1}(\tau) \int \frac{\hat{\mu}(\xi)}{1+\psi(\tau-\xi)}\, d\xi \hat{\mu}(-\tau)\, d\tau \tag{5.64}$$

whereas the Fourier transform of (1.5), the defining condition that $\mu \in \mathcal{G}^{2n}$, is

$$\int \theta_{2n}(\tau) |\hat{\mu}(\tau)|^2 \, d\tau. \tag{5.65}$$

□

In light of the above Lemma, we may say that $\mu \in \mathcal{G}^{2n}$ is a necessary condition for $\gamma_n(\mu)$ to exist P^x almost surely. (As we have seen it is a sufficient condition to define $\gamma_n(\mu)$, P^ρ almost surely).

It is interesting to note that the proof of (5.10) alone gives (5.4), without having to consider $\mathcal{L}_n(\mu)$ or the isomorphism theorem, as long as we can show (5.6). The statement in (5.4) is an interesting result in its own right. The next lemma gives a simple proof of (5.6) and hence, together with the proof of (5.10), a simpler proof of (5.4).

LEMMA 5.2. *Let $\rho \in \mathcal{G}^1$ and $\mu \in \mathcal{G}^{2n}$. Let $U^1\rho(x) \cdot d\mu(x)$ denote the measure which has Radon Nykodym derivative $U^1\rho(x)$ with respect to $\mu(x)$. Then*

$$\|U^1\rho(x) \cdot d\mu(x)\|_{(2n-1)} \le \|\rho\|_{(1)} \|\mu\|_{(2n)} \tag{5.66}$$

where $\|\cdot\|_{(\,\cdot\,)}$ is defined right after (3.1).

Note that by the Schwarz inequality

$$\begin{aligned} I_n(\mu,\rho,f) &= \langle U^1\rho(x)\cdot d\mu(x), U^1 f(y)\cdot d\mu(y)\rangle_{(2n-1)} \\ &\le \|U^1\rho(x)\cdot d\mu(x)\|^{1/2}_{(2n-1)} \|U^1 f(y)\cdot d\mu(y)\|^{1/2}_{(2n-1)}. \end{aligned} \tag{5.67}$$

Since both ρ and $f \cdot dx$ are assumed to be in $\mathcal{G}^1$, Lemma 5.2 states that $\mu \in \mathcal{G}^{2n}$ implies that $I_n(\mu,\rho,f) < \infty$.

PROOF OF LEMMA 5.2. Let $\theta_k(\tau)$ denote the Fourier transform of $(u^1(x))^k$. Taking the Fourier transform we see that

$$\|U^1\rho(x)\,\cdot\, d\mu(x)\|^2_{(2n-1)} = \int \theta_{2n-1}(\tau) \frac{\hat{\rho}(\xi)}{1+\psi(\xi)} \frac{\hat{\rho}(\eta)}{1+\psi(\eta)} \hat{\mu}(\tau-\xi)\hat{\mu}(-\tau-\eta) d\xi d\eta d\tau \tag{5.68}$$

where $\widehat{u^1}(\xi) = (1+\psi(\xi))^{-1}$ and $\theta_{2n-1}(\tau) = ((1+\psi(\xi))^{-1})^{*(2n-1)}$, the $2n-1$-fold convolution of $(1+\psi(\xi))^{-1}$. By the Schwarz inequality

$$\int \frac{\hat{\rho}(\xi)\hat{\mu}(\tau-\xi)}{1+\psi(\xi)}\, d\xi \le \|\rho\|^{1/2}_{(1)} \left(\int \frac{|\hat{\mu}(\tau-\xi)|^2}{1+\psi(\xi)}\, d\xi \right)^{1/2} \tag{5.69}$$

Using (5.69), the Schwarz inequality and the fact that ψ is symmetric we see that

$$\begin{aligned} \|U^1\rho(x)\,\cdot\, d\mu(x)\|^2_{(2n-1)} &\le \|\rho\|_{(1)} \int \frac{\theta_{2n-1}(\tau)|\hat{\mu}(\tau-\xi)|^2}{1+\psi(\xi)}\, d\xi\, d\tau \\ &= \|\rho\|_{(1)} \int \frac{\theta_{2n-1}(\tau)|\hat{\mu}(\xi)|^2}{1+\psi(\xi-\tau)}\, d\xi\, d\tau \end{aligned} \tag{5.70}$$

$$= \quad \|\rho\|_{(1)} \int \theta_{2n}(\xi) |\hat{\mu}(\xi)|^2 \, d\xi$$

which is the right-hand side of (5.66). □

CHAPTER 6

Continuity

In this chapter we prove Theorem 1.1 and Corollaries 1.1 and 1.2.

PROOF OF THEOREM 1.1. It will be convenient to work with an explicit version of $\{\gamma_n(\mu_x);\, x \in R^m\}$. Recall that, in our usage, a random variable Y is called a version of $\gamma_n(\mu)$ if $Y = \lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu)$ in $L^2(\Omega \times R_+,\, P^y_\lambda)$ for all $y \in R^m$. Here Ω is the disjoint union

$$\Omega = \bigcup_{y\in R^m} \Omega_y$$

where Ω_y denotes the set of cadlag paths $\omega : R_+ \mapsto R^m$ with $\omega(0) = y$. The measure P^y_λ is concentrated on $\Omega_y \times R_+$ and we have

$$\int h(\omega, s)\, dP^{y+z}_\lambda(\omega, s) = \int h(z + \omega, s)\, dP^y_\lambda(\omega, s) \tag{6.1}$$

for all measurable functions h on $\Omega_{y+z} \times R_+$.

To begin, for each $x \in R^m$ and $\omega \in \Omega_0$ we choose $\widetilde{\gamma}_n(\mu_x)(\omega)$ to be some version of $\lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu_x)$ in $L^2(\Omega \times R_+,\, P^0_\lambda)$, which we know exists by Theorem 5.2. We then set

$$\widetilde{\gamma}_n(\mu_x)(\omega) = \widetilde{\gamma}_n(\mu_{x+y})(-y + \omega), \qquad \omega \in \Omega_y \tag{6.2}$$

which defines $\widetilde{\gamma}_n(\mu_x)$ for paths $\omega \in \Omega_y$. Note that since

$$\gamma_{n,\epsilon}(\mu_x)(\omega) = \gamma_{n,\epsilon}(\mu_{x+y})(-y + \omega), \qquad \omega \in \Omega_y \tag{6.3}$$

we see from (6.1) that

$$\widetilde{\gamma}_n(\mu_x) = \lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu_x) \tag{6.4}$$

in $L^2(\Omega \times R_+,\, P^y_\lambda)$ for all $y \in R^m$ so that $\{\widetilde{\gamma}_n(\mu_x);\, x \in R^m\}$ is indeed a version of $\{\gamma_n(\mu_x);\, x \in R^m\}$. We also observe by (6.2) that for any $x, y, z \in R^m$ we have

$$\widetilde{\gamma}_n(\mu_x)(\omega) = \widetilde{\gamma}_n(\mu_{x+z})(-z + \omega) = \widetilde{\gamma}_n(\mu_{x-y+z})(y - z + \omega), \qquad \omega \in \Omega_z. \tag{6.5}$$

By Theorem 5.2 convergence in (6.4) in $L^2(\Omega \times R_+,\, P^0_\lambda)$ is uniform in $x \in R^m$, and hence from the definition (6.2) we see that for fixed x the convergence in (6.4) in $L^2(\Omega \times R_+,\, P^y_\lambda)$ is uniform in $y \in R^m$. Therefore we have that (6.4) holds also in $L^2(\Omega \times R_+,\, P^\rho_\lambda)$ for any probability measure ρ, in particular for any probability measure $\rho \in \mathcal{G}^1$. Theorem 5.1 now shows that $\widetilde{\gamma}_n(\mu_x) = \mathcal{L}_n(\mu_x)$ as random variables in $L^2(\Omega \times R_+,\, P^\rho_\lambda)$ for any probability measure $\rho \in \mathcal{G}^1$. Hence the isomorphism theorem, Theorem 4.1 holds with $\{\mathcal{L}_n(\mu_x);\, x \in R^m\}$ replaced by $\{\widetilde{\gamma}_n(\mu_x);\, x \in R^m\}$.

Let $K \subset R^m$ be compact and let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of K. Consider Theorem 4.1, with $\{\widetilde{\gamma}_n(\mu_x);\, x \in R^m\}$ replaced by $\{\mathcal{L}_n(\mu_x);\, x \in R^m\}$, and take F to be a norm on $C(K)$, the space of continuous functions on K. Note that by the convexity of F and the fact that the Wick power chaos processes of order

$2k$, $k = 1, \dots, n$, all have mean zero and are independent of the $\{\widetilde{\gamma}_k \mu.\}_{k=1}^n$, (and that $\widetilde{\gamma}_0 =: G^0 := 1$ by definition) we have that

$$F(\widetilde{\gamma}_n \mu.) \leq E_G\left(F\left(\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^{n-k}}(: G^{2(n-k)} : \times \widetilde{\gamma}_k)(\mu.)\right)\right). \tag{6.6}$$

Therefore, by (4.3), and the Schwarz inequality

$$E_\lambda^\rho F(\widetilde{\gamma}_n \mu.) f(X_\lambda) \leq (E_G F(: G^{2n}\mu. :)^2)^{1/2}(E_G G_\rho^4 \, E_G G_{f\cdot dx}^4)^{1/4}. \tag{6.7}$$

By hypothesis, $\{: G^{2n}\mu_x :, x \in R^m\}$ is continuous almost surely. Since, in general, a Banach space valued Gaussian chaos has all moments, it follows that $E\sup_{x\in K} | : G^{2n}\mu_x : | < \infty$. Therefore, by the dominated convergence theorem

$$\lim_{\epsilon\to 0} E \sup_{\substack{x,y\leq\epsilon \\ x,y\in K}} | : G^{2n}\mu_x : - : G^{2n}\mu_y : | = 0. \tag{6.8}$$

Taking F to be this 'uniform modulus' norm and $f > 0$, it follows from (6.7) and (6.8) that $\{\widetilde{\gamma}_n \mu_{x_i}, \{x_i\} \in K\}$ is continuous almost surely with respect to P^ρ.

Let $D \subseteq R^m$ be a countable dense set. We have shown that $\{\widetilde{\gamma}_n(\mu_x);\, x \in D\}$ is P_λ^ρ almost surely locally uniformly continuous in $x \in D$ for any probability measure $\rho \in \mathcal{G}^1$. This implies that $\{\widetilde{\gamma}_n(\mu_x);\, x \in D\}$ is $P_\lambda^{y_0}$ almost surely locally uniformly continuous in $x \in D$ for some $y_0 \in R^m$. Let $\{\Gamma_n(\mu_x);\, x \in R^m\}$ denote the $P_\lambda^{y_0}$ almost surely continuous extension of $\{\widetilde{\gamma}_n(\mu_x);\, x \in D\}$. Thus

$$\Gamma_n(\mu_x)(\omega), \qquad \omega \in \Omega_{y_0}$$

is continuous in $x \in R^m$ for $P_\lambda^{y_0}$ almost every ω. Define $\{\Gamma_n(\mu_x);\, x \in R^m\}$ for paths $\omega \in \Omega_z$ by

$$\Gamma_n(\mu_x)(\omega) = \Gamma_n(\mu_{x-y_0+z})(y_0 - z + \omega), \qquad \omega \in \Omega_z. \tag{6.9}$$

With this definition we see that $\{\Gamma_n(\mu_x);\, x \in R^m\}$ is continuous P_λ^z almost surely for all $z \in R^m$. It remains to show that $\{\Gamma_n(\mu_x);\, x \in R^m\}$ is a version of $\{\gamma_n(\mu_x);\, x \in R^m\}$.

Note that from (6.5) and (6.9) we see that $\Gamma_n(\mu_x)$ is a P_λ^z almost surely continuous extension of $\{\widetilde{\gamma}_n(\mu_x);\, x \in D + y_0 - z\}$. If $x \in R^m$ is arbitrary, choose a sequence $x_j \in D + y_0 - z$ such that $x_j \to x$. By the last remark we have

$$\Gamma_n(\mu_x) = \lim_{j\to\infty} \widetilde{\gamma}_n(\mu_{x_j}) \tag{6.10}$$

P_λ^z almost surely. On the other hand, by Theorem 5.2 we see that

$$\widetilde{\gamma}_n(\mu_x) = \lim_{j\to\infty} \widetilde{\gamma}_n(\mu_{x_j}) \tag{6.11}$$

in $L^2(P_\lambda^z)$. These two facts complete the proof. □

Proof of Corollary 1.1. It follows from Lemmas 3.2 and 3.3, [**1**] and the remarks between them that there exists a $\lambda > 0$ such that

$$E\exp\left(\lambda\left(\frac{| : G^{2n}\mu_x : - : G^{2n}\mu_y : |}{d(x,y)}\right)^{1/n}\right) \leq 1. \tag{6.12}$$

(The fact that the Wick power chaos and the formulation in [**1**] are the same in pointed out in Chapter 3). The proof is an immediate consequence of (6.12), Theorem 11.6 [**14**], and the last paragraph on page 300 of [**14**]. □

Proof of Corollary 1.2. Corollary 1.2 holds because (1.11) implies (1.9). The proof of this fact is completely analogous to the proof of Theorem 1.6, [**15**] where this is proved in the case $n = 1$. □

CHAPTER 7

Stable mixtures

In this chapter we show that stable mixtures which are of interest to us in this paper are Lévy processes in Class A. (See Chapter 1, in particular the comments after (1.19).) We first note that the characteristic exponents of stable mixtures are quite regular.

LEMMA 7.1. *Let $|\lambda| = \rho$ in (1.18). Then $\psi(\rho)$ is regularly varying at infinity with index β and is regularly varying at zero with index a. Furthermore, for $1 < \beta \le 2$*

$$\begin{aligned} \rho\psi'(\rho) &\sim \beta\psi(\rho) \\ \rho^2\psi''(\rho) &\sim \beta(\beta-1)\psi(\rho) \end{aligned} \tag{7.1}$$

as $\rho \to \infty$, and for $0 < \beta \le 1$

$$\begin{aligned} \rho\psi'(\rho) &\sim \beta\psi(\rho) \\ |\rho^2\psi''(\rho)| &\le (1/4)\psi(\rho) \end{aligned} \tag{7.2}$$

as $\rho \to \infty$. In general, for all $k \ge 1$

$$|\psi^{(k)}(\rho)| \le C_k \frac{\psi(\rho)}{\rho^k} \qquad \text{as } \rho \to \infty \tag{7.3}$$

for some constant C_k depending on k.

PROOF. Let $a < \beta' < \beta$. Since $\phi(s)$ is supported on $[a, \beta]$ it puts positive mass on $[b, \beta]$ where $b = (\beta + \beta')/2$. Thus

$$\psi(\rho) \sim \int_b^\beta \rho^s \, d\phi(s) \qquad \text{as } \rho \to \infty. \tag{7.4}$$

Taking β' arbitrarily close to β shows that $\psi(2\rho) \sim 2^\beta \psi(\rho)$ as $\rho \to \infty$, which, by definition, means that $\psi(\rho)$ is regularly varying at infinity with index β. By similar reasoning we see that

$$\rho\psi'(\rho) \sim \int_b^\beta s\rho^s \, d\phi(s) \sim \beta\psi(\rho) \qquad \text{as } \rho \to \infty. \tag{7.5}$$

The rest of this lemma is proved similarly. □

In order to show that stable mixtures belong to Class A we first need estimates for the 1-potential of these processes near the origin.

THEOREM 7.1. *Let X be a stable mixture in R^2 with 1-potential density $u^1(|x|)$. When $3/2 < \beta < 2$*

$$u^1(|x|) = \int \frac{\cos(p \cdot x)}{1 + \psi(|p|)} \, d^2p \sim C_\beta \frac{1}{|x|^2 \psi(|1/x|)} \qquad \text{as } |x| \to 0 \tag{7.6}$$

where

$$C_\beta = \frac{4\pi\Gamma(1-(\beta/2))}{2^\beta\Gamma(\beta/2)}. \tag{7.7}$$

Let X be a stable mixture in R^1 with 1-potential density $u^1(|x|)$. When $3/4 < \beta < 1$

$$u^1(|x|) = \int \frac{\cos(px)}{1+\psi(|p|)}\,dp \sim C'_\beta \frac{1}{|x|\psi(|1/x|)} \qquad \text{as } |x| \to 0 \tag{7.8}$$

where

$$C'_\beta = \Gamma(1-\beta)\cos\left(\frac{(1-\beta)\pi}{2}\right). \tag{7.9}$$

The next lemma is used in the proof of Theorem 7.1.

LEMMA 7.2. *Let $h : R^+ \to R^+$ be of the form $h(y) = L(y)/y^\beta$ where L is a quasi-monotone slowly varying function at infinity and $1 < \beta < 2$. Then*

$$\int \cos(p\cdot x)h(|p|)\,d^2p \sim C_\beta \frac{L(1/|x|)}{|x|^{2-\beta}} \qquad \text{as } |x| \to 0. \tag{7.10}$$

PROOF. Changing to polar coordinates and performing one integral, we can write the left hand side of (7.10) as

$$\frac{2\pi}{|x|^{2-\beta}} \int_0^\infty \frac{J_0(s)}{s^{\beta-1}} L(s/|x|)\,ds \tag{7.11}$$

where J_0 is the zero-th order Bessel function. We now use Theorem 4.1.5, [**3**] to get (7.10). (Note that (7.4.1) of [**24**] implies that the two integrals at the bottom of page 200, [**3**] exist. This shows that the hypotheses of Theorem 4.1.5, [**3**] are satisfied). □

PROOF OF THEOREM 7.1. The asymptotic relationship in (7.6) follows immediately from Lemma 7.2 once we show that

$$L(|p|) = \frac{|p|^\beta}{1+\psi(|p|)} \tag{7.12}$$

is non-decreasing, as $|p| \to \infty$, since this is a sufficient condition for L to be quasi-monotone. (See [**3**] page 105). This is easy to see. Setting $|p| = \rho$ and treating $L(|p|)$ and $\psi(|p|)$ as functions of the positive real value ρ we have by (1.18) that

$$\rho\psi'(\rho) = \int_a^\beta s\rho^s\,d\mu(s) \le \beta\psi(\rho) \tag{7.13}$$

which shows that $L(\rho)$ is strictly increasing.

The asymptotic relationship in (7.8) follows similarly, and more simply, from Theorem 4.1.5, [**3**], than the one in (7.6). It is also stated explicitly in (4.3.7), [**3**]. □

Estimates of the 1-potential in the borderline cases of $\beta = 2$ in R^2 and $\beta = 1$ in R^1 are more complicated, as we see from the next theorem.

THEOREM 7.2. *Let X be a stable mixture in R^2 with 1-potential density $u^1(|x|)$. When $\beta = 2$ there exist constants $0 < C_1 \le C_2 < \infty$ such that for all $|x|$ sufficiently small*

$$C_1 \int_0^{1/|x|} \frac{1}{1+\psi(|p|)}\,d^2p \le u^1(|x|) \le C_2 \int_0^{1/|x|} \frac{1}{1+\psi(|p|)}\,d^2p. \tag{7.14}$$

Let X be a stable mixture in R^1 with 1-potential density $u^1(|x|)$. When $\beta = 1$ there exist constants $0 < C_1 \le C_2 < \infty$ such that for all $|x|$ sufficiently small

$$C_1 \int_0^{1/|x|} \frac{1}{1+\psi(p)}\, dp \le u^1(|x|) \le C_2 \int_0^{1/|x|} \frac{1}{1+\psi(p)}\, dp. \tag{7.15}$$

PROOF. Changing to polar coordinates and using the fact that ψ is radially symmetric, we can write

$$\begin{aligned} u^1(|x|) &= \int_0^1 \frac{1}{\sqrt{1-u^2}} \int_0^\infty \cos(u\rho|x|) \frac{\rho}{1+\psi(\rho)}\, d\rho\, du \\ &= -\int_0^1 \frac{1}{\sqrt{1-u^2}} \int_0^\infty \frac{\sin(u\rho|x|)}{u|x|} \frac{d}{d\rho} \frac{\rho}{1+\psi(\rho)}\, d\rho\, du. \end{aligned} \tag{7.16}$$

Note that $\int_{\pi/3}^{\pi} \frac{\sin v}{v}\, dv > -\int_{\pi}^{2\pi} \frac{\sin v}{v}\, dv$. This inequality is easy to verify. ¿Consider the intervals $(\pi/3, \pi/2)$, $(\pi/2, 3\pi/4)$, and the remaining intervals of length $\pi/4$. Bound $\sin v$ above and below as required and integrate $1/v$. Use a calculator to verify the inequality. Also, it is straightforward, using (7.1) with $\beta = 2$, to check that $(-\rho \frac{d}{d\rho} \frac{\rho}{1+\psi(\rho)})$ is decreasing for ρ sufficiently large. Considering these observations and the fact that $1/v$ is decreasing on $[0, \infty)$, we see that for all $|x|$ sufficiently small and $0 < u \le 1$

$$\int_{\pi/(3u|x|)}^\infty \frac{\sin(u\rho|x|)}{u|x|\rho} \left(-\rho \frac{d}{d\rho} \frac{\rho}{1+\psi(\rho)} \right) d\rho \ge 0. \tag{7.17}$$

Consequently

$$\begin{aligned} \int_0^\infty \frac{\sin(u\rho|x|)}{u|x|\rho} \left(-\rho \frac{d}{d\rho} \frac{\rho}{1+\psi(\rho)} \right) d\rho \ge \\ C \int_0^{\pi/(3u|x|)} \left(-\rho \frac{d}{d\rho} \frac{\rho}{1+\psi(\rho)} \right) d\rho. \end{aligned} \tag{7.18}$$

Using (7.1) to estimate the derivative in (7.18) and the fact that $0 \le u \le 1$, we obtain the left-hand side of (7.14).

To obtain the upper bound in (7.14) we write $(x \cdot p) = x_1 p_1 + x_2 p_2$. Without loss of generality we can assume that $|x_1| \ge |x_2|$. Integrating by parts with respect to p_1, we have

$$\begin{aligned} u^1(|x|) &= -\frac{1}{x_1} \int \sin(x \cdot p) \frac{d}{dp_1} \left(\frac{1}{1+\psi(|p|)} \right) d^2p \\ &\le \frac{C}{|x|} \int_{|p| \le 1/|x|} |xp| \frac{\psi'(|p|)}{(1+\psi(|p|))^2}\, d^2p \\ &\quad + \frac{C}{|x|} \int_{|p| \ge 1/|x|} \frac{\psi'(|p|)}{(1+\psi(|p|))^2}\, d^2p \\ &\le C \int_0^{1/|x|} \frac{\rho^2 \psi'(\rho)}{(1+\psi(\rho))^2}\, d\rho + \frac{C}{|x|} \int_{1/|x|}^\infty \frac{\rho \psi'(\rho)}{(1+\psi(\rho))^2}\, d\rho. \end{aligned} \tag{7.19}$$

Using the regular variation of $\psi(\rho)$ at infinity and (7.1) we see that for all $|x|$ sufficiently small

$$u^1(|x|) \le C \int_0^{1/|x|} \frac{\rho}{(1+\psi(\rho))}\, d\rho + \frac{C}{|x|^2 \psi(1/|x|)}. \tag{7.20}$$

Since $\psi(\rho)$ is effectively increasing for ρ sufficiently large we also see that

$$\frac{1}{|x|^2\psi(1/|x|)} \leq C\int_0^{1/|x|} \frac{\rho}{(1+\psi(\rho))}\,d\rho \tag{7.21}$$

for all $|x|$ sufficiently small. (Actually, in general, the left-hand side of (7.21) is little "o" of the right-hand side). This completes the proof of (7.14).

The proof of (7.15) is similar to the proof of (7.14) and simpler. We will not repeat the arguments but only note for future use that in this case, parallel to (7.21), we have

$$\frac{1}{|x|\psi(1/|x|)} \leq C\int_0^{1/|x|} \frac{1}{(1+\psi(\rho))}\,d\rho \tag{7.22}$$

for all $|x|$ sufficiently small. □

We can now show that the stable mixtures are in Class A.

THEOREM 7.3. *A stable mixture in R^2, which is regularly varying at infinity with index $3/2<\beta\leq 2$, belongs to Class A. In particular, symmetric stable processes in R^2 with index $3/2<\beta\leq 2$ are in Class A.*

PROOF. To avoid confusing notation in this proof we set $u^1\equiv u$. Let $r\geq 0$ and consider $u(r)$ as a function on R^1. Let $r=|x|$ and note that $|\nabla u(|x|)|=|u'(r)|$. In polar coordinates, for radially symmetric functions, $\Delta\equiv\partial^2/\partial\rho^2+(1/\rho)\partial/\partial\rho$. Since $\Delta(1/(1+\psi(|p|)))\in L^1(R^2)$ we can write

$$\begin{aligned} u(|x|) &= -\frac{1}{|x|^2}\int \cos(p\cdot x)\Delta\frac{1}{1+\psi(|p|)}\,d^2p \\ &= -\frac{C}{r^2}\int_0^\infty\int_0^1 \frac{\cos(u\rho r)}{\sqrt{1-u^2}}\Delta\frac{1}{1+\psi(|p|)}\rho\,du\,d\rho \end{aligned} \tag{7.23}$$

where, in the last line, we consider $\Delta(1/(1+\psi(|p|)))$ as a function of $\rho=|p|$. By (7.3) we see that $\rho^2\Delta(1/(1+\psi(|p|)))\in L^1(R^1)$. Thus we can differentiate $u(r)$ with respect to r and obtain

$$u'(r) = -\frac{Cu(r)}{r} - \frac{C}{r^2}\int_0^\infty\int_0^1 \frac{u\sin(u\rho r)}{\sqrt{1-u^2}}\rho^2\Delta\frac{1}{1+\psi(|p|)}\,du\,d\rho. \tag{7.24}$$

Therefore,

$$|u'(r)| \leq \frac{C|u(r)|}{r} + \left|\frac{C}{r^2}\int_0^\infty\int_0^1 \frac{u\sin(u\rho r)}{\sqrt{1-u^2}}\rho^2\Delta\frac{1}{1+\psi(|p|)}\,du\,d\rho\right|. \tag{7.25}$$

The last term in (7.25)

$$\begin{aligned} &\leq \frac{C}{r^2}\int_0^\infty\int_0^1 \frac{u}{\sqrt{1-u^2}}(\rho r\wedge 1)\rho^2|\Delta\frac{1}{1+\psi(|p|)}|\,du\,d\rho \\ &\leq \frac{C}{r^2}\left(\int_0^{1/r} r\rho^3|\Delta\frac{1}{1+\psi(|p|)}|\,d\rho + \int_{1/r}^\infty \rho^2|\Delta\frac{1}{1+\psi(|p|)}|\,d\rho\right). \end{aligned} \tag{7.26}$$

Since

$$\left|\Delta\frac{1}{1+\psi(|p|)}\right| \leq \frac{C}{\rho^2(1+\psi(\rho))} \tag{7.27}$$

and the latter function is regularly varying at infinity, we see that for $r \leq r_0$, for some $r_0 > 0$

$$\begin{aligned} |u'(r)| &\leq \frac{C|u(r)|}{r} + \frac{C}{r^3(1+\psi(1/r)} \\ &< \frac{C|u(r)|}{r} \end{aligned} \tag{7.28}$$

where, at the last step we use (7.6) when $3/2 < \beta \leq 2$ and (7.14) along with (7.21) when $\beta = 2$. This shows that $\Delta_b u$ satisfies condition (1.15). Also, by (7.23), (7.25), (7.26) and (7.27), $|u'(r)|$ is bounded for $r \geq r_0 > 0$ for all $r_0 > 0$.

When $r = |x|$

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} u(|x|) \right| \leq |u''(r)| + \frac{|u'(r)|}{r} \qquad i,j = 1,2. \tag{7.29}$$

Therefore, by (7.28), to show that $\Delta^2_{b,c} u$ satisfies (1.16) we need only consider $|u''(r)|$. We first integrate (7.24) by parts and then differentiate with respect to r and use (7.28) to get

$$\begin{aligned} |u''(r)| &\leq \frac{C|u(r)|}{r^2} + \\ &\qquad \left| \frac{C}{r^3} \int_0^\infty \int_0^1 \frac{u \sin(u\rho r)}{\sqrt{1-u^2}} (\rho \frac{d}{d\rho}(\rho^2 \Delta \frac{1}{1+\psi(|p|)}))\, du\, d\rho \right|. \end{aligned} \tag{7.30}$$

This is possible since both $\frac{d}{d\rho}(\rho^2 \Delta \frac{1}{1+\psi(|p|)})$ and $\rho \frac{d}{d\rho}(\rho^2 \Delta \frac{1}{1+\psi(|p|)})$ are in $L^1(R^1)$. Since

$$\rho \frac{d}{d\rho}(\rho^2 \Delta \frac{1}{1+\psi(|p|)}) \leq \frac{C}{(1+\psi(\rho)} \tag{7.31}$$

we can proceed as in (7.26) and what follows to see that for $r \leq r_0$, for some $r_0 > 0$

$$|u''(r)| \leq \frac{C}{r^4(1+\psi(1/r)} \leq \frac{C|u(r)|}{r^2}. \tag{7.32}$$

This shows that $\Delta^2_{b,c} u$ satisfies condition (1.15). Similarly, we can show that $|u''(r)|$ is bounded for $r \geq r_0 > 0$ for all $r_0 > 0$. □

THEOREM 7.4. *A stable mixture in R^1 which is regularly varying with index $3/4 < \beta \leq 1$ belongs to Class A. In particular, symmetric stable processes in R^1 with index $3/4 < \beta \leq 1$ are in Class A.*

PROOF. To avoid confusing notation we set $u^1(p) = u(p)$. The 1-potential of these processes is given by

$$u(x) = 2 \int_0^\infty \frac{\cos xp}{1+\psi(p)}\, dp \tag{7.33}$$

Without loss of generality we can assume $x > 0$. Writing (7.33) as the limit, as $N \to \infty$, of the integral, we can integrate by parts and then pass to the limit to obtain

$$u(x) = \frac{2}{x} \int_0^\infty \sin xp \frac{\psi'(p)}{(1+\psi(p))^2}\, dp. \tag{7.34}$$

Splitting the range of integration in (7.34) into two parts and then integrating the integral over $[\pi/(2x), \infty)$ by parts we obtain

$$u(x) = \frac{2}{x}\int_0^{\pi/(2x)} \sin xp \frac{\psi'(p)}{(1+\psi(p))^2}\,dp - \frac{2}{x^2}\int_{\pi/(2x)}^{\infty} \cos xp \frac{d}{dp}\frac{\psi'(p)}{(1+\psi(p))^2}\,dp. \tag{7.35}$$

It follows from (7.3) that the last integral is absolutely integrable as well as all the integrals that follow in this proof. Differentiating (7.35) we get

$$\begin{aligned} u'(x) \quad = \quad & -\frac{u(x)}{x} + \frac{2}{x^3}\int_{\pi/(2x)}^{\infty} \cos xp \frac{d}{dp}\frac{\psi'(p)}{(1+\psi(p))^2}\,dp \\ & + \frac{2}{x}\int_0^{\pi/(2x)} \cos xp \frac{p\psi'(p)}{(1+\psi(p))^2}\,dp \\ & + \frac{2}{x^2}\int_{\pi/(2x)}^{\infty} (\sin xp) p \frac{d}{dp}\frac{\psi'(p)}{(1+\psi(p))^2}\,dp \\ & - \frac{\pi}{x^3}\left(\frac{\psi'(\pi/(2x))}{(1+\psi(\pi/(2x)))^2}\right). \end{aligned} \tag{7.36}$$

Using Theorem 7.1 we see that

$$\begin{aligned} \left| -\frac{2}{x^3}\int_{\pi/(2x)}^{\infty} \cos xp \frac{d}{dp}\frac{\psi'(p)}{(1+\psi(p))^2}\,dp \right| \quad & \leq \quad \frac{C}{x^3}\int_{\pi/(2x)}^{\infty} \frac{\psi(p)}{p^2(1+\psi(p))^2}\,dp \\ & \leq \quad \frac{C}{x^2\psi(1/x)}. \end{aligned} \tag{7.37}$$

Similarly

$$\begin{aligned} \frac{2}{x}\int_0^{\pi/(2x)} \cos xp \frac{p\psi'(p)}{(1+\psi(p))^2}\,dp \quad & \leq \quad \frac{C}{x}\int_0^{\pi/(2x)} \frac{\psi(p)}{(1+\psi(p))^2}\,dp \\ & \leq \quad \frac{C}{x}\int_0^{\pi/(2x)} \frac{1}{(1+\psi(p))}\,dp \end{aligned} \tag{7.38}$$

$$\left| \frac{2}{x^2}\int_{\pi/(2x)}^{\infty} (\sin xp) p \frac{d}{dp}\frac{\psi'(p)}{(1+\psi(p))^2}\,dp \right| \leq \frac{C}{x^2\psi(1/x)} \tag{7.39}$$

and

$$\frac{\pi}{x^3}\left(\frac{\psi'(\pi/(2x))}{(1+\psi(\pi/(2x)))^2}\right) \leq \frac{C}{x^2\psi(1/x)} \tag{7.40}$$

for x sufficiently small. When $\beta < 1$ the last line in (7.38) is $\mathrm{O}(1/(x^2\psi(1/x)))$. Therefore, it follows from (7.8), (7.15) and (7.22) that $|u'(x)| \leq C|u(x)/x|$ for x sufficiently small and also that $u'(x)$ is bounded away from the origin.

Proceeding we first integrate the last integral in (7.36) by parts and then differentiate the resulting expression for $u'(x)$. There are many terms but they are all easy to estimate using the techniques of the proceeding paragraph. Doing this we see that $|u''(x)| \leq C|u(x)/x^2|$ for x sufficiently small and also that $u''(x)$ is bounded away from the origin. □

CHAPTER 8

Examples

Using Corollary 1.2 we give some examples of Lévy processes in Class A and measures $\mu \in \mathcal{G}^{2n}$, described by their Fourier transforms, for which $\{\gamma_n(\mu_x), x \in R^m\}$, $m = 1, 2$ is continuous almost surely. We use $\hat{u}^1(|\xi|) = 1/(1+\psi(|\xi|)$, the Fourier transform of $u^1(x)$, to estimate $\tau(\xi)$. Thus for Lévy processes in Class A in R^2, when

$$\psi(|\xi|) \sim \frac{|\xi|^2}{(\log|\xi|)^a} \qquad \text{as } |\xi| \to \infty \tag{8.1}$$

where $a \geq 0$, we have

$$\tau(|\xi|) = \left(\frac{1}{1+\psi(|\xi|)}\right)^{*2n} = O(\frac{(\log|\xi|)^{2n(1+a)-1}}{|\xi|^2}) \qquad \text{as } |\xi| \to \infty. \tag{8.2}$$

Substituting this in (1.11) gives (1.12), ($a = 0$ for Brownian motion), and the assertions in the paragraph following (1.12). Recall that in (1.19) we pointed out that stable mixtures include Lévy processes for which (8.1) is satisfied. The same is true for all the other Lévy exponents mentioned in this chapter.

Continuing to consider Lévy processes in R^2, let $2 - 1/n < \beta < 2$. If

$$\psi(|\xi|) \sim \frac{|\xi|^\beta}{(\log|\xi|)^a} \qquad \text{as } |\xi| \to \infty \tag{8.3}$$

for $a \geq 0$, then

$$\tau(|\xi|) = \left(\frac{1}{1+\psi(|\xi|)}\right)^{*2n} = O\left(\frac{(\log|\xi|)^{2na}}{|\xi|^{2n\beta-2(2n-1)}}\right) \qquad \text{as } |\xi| \to \infty. \tag{8.4}$$

Substituting this in (1.11) shows that for Lévy process with characteristic exponent given by (8.3), $\{\gamma_n(\mu_x), x \in R^2\}$ is continuous almost surely if

$$|\hat{\mu}(\xi)| = O\left(\frac{1}{|\xi|^{n(2-\beta)}(\log|\xi|)^{n(a+1)+1/2+\epsilon}}\right) \qquad \text{as } |\xi| \to \infty \tag{8.5}$$

for some $\epsilon > 0$.

For Lévy processes in Class A in R^1, when

$$\psi(|\xi|) \sim \frac{|\xi|}{(\log|\xi|)^a} \qquad \text{as } |\xi| \to \infty \tag{8.6}$$

where $a \geq 0$, we have

$$\tau(|\xi|) = O\left(\frac{(\log|\xi|)^{2n(1+a)-1}}{|\xi|^2}\right) \qquad \text{as } |\xi| \to \infty. \tag{8.7}$$

Substituting this in (1.11) shows that for Lévy process in Class A, with characteristic exponent ψ given by (8.1), $\{\gamma_n(\mu_x), x \in R^1\}$ is continuous almost surely when

$$|\hat{\mu}(\xi)| = O\left(\frac{1}{(\log|\xi|)^{2n(1+a/2)+\epsilon}}\right) \qquad \text{as } |\xi| \to \infty \tag{8.8}$$

for some $\epsilon > 0$.

Continuing to consider Lévy processes in R^1, let $1 - 1/(2n) < \beta < 1$. When

$$\psi(|\xi|) \sim \frac{|\xi|^\beta}{(\log|\xi|)^a} \qquad \text{as } |\xi| \to \infty \tag{8.9}$$

for $a > 0$, then

$$\tau(|\xi|) = O\left(\frac{(\log|\xi|)^{2na}}{|\xi|^{2n\beta-(2n-1)}}\right) \qquad \text{as } |\xi| \to \infty. \tag{8.10}$$

Consequently, $\{\gamma_n(\mu_x), x \in R^2\}$ is continuous almost surely if

$$|\hat{\mu}(\xi)| = O\left(\frac{1}{|\xi|^{n(1-\beta)}(\log|\xi|)^{n(a+1)+1/2+\epsilon}}\right) \qquad \text{as } |\xi| \to \infty \tag{8.11}$$

for some $\epsilon > 0$.

CHAPTER 9

Large Deviations

We first prove Theorem 1.2. This theorem applies to Lévy processes in Class A with an additional condition on their 1-potential at infinity. We next show that stable mixtures satisfy this additional condition.

Proof of Theorem 1.2. It follows from Corollary 4.4 [**1**] that whenever $\{: G^{2n}\mu_x :, x \in R^m\}$ is continuous almost surely, there exists a constant $d > 0$ such that

$$II \stackrel{def}{=} E\exp\left(d \sup_{x\in[-2,2]^m} |: G^{2n}\mu_x :|^{1/n}\right) < \infty. \tag{9.1}$$

We now show how to use (9.1) and an isomorphism theorem, Theorem 4.1 to obtain (1.13).

By the hypothesis, when X takes values in R^m, there exists a $\delta > m$ such that $u^1(x) = O(1/x^\delta)$. Let $q > 1$ be such that $\delta/q > m$. Set $c = d/(2p)$, where $1/p + 1/q = 1$. It is easy to see that there is a convex function Φ on $[0,\infty]$ such that $\exp(c(x)^{1/n}) \le \Phi(x) \le C\exp(c(x)^{1/n})$ for some constant C depending on c. Let $\|\cdot\|$ denote $\sup_{x\in[-2,2]^m} |\cdot|$. By the Hölder inequality

$$E^\rho_\lambda \Phi(\|\mathcal{L}_n\mu_\cdot\|) f(X_\lambda) \le \left(E^\rho_\lambda \Phi^p(\|\mathcal{L}_n\mu_\cdot\|) f(X_\lambda)\right)^{1/p} \left(E^\rho_\lambda f(X_\lambda)\right)^{1/q}. \tag{9.2}$$

By (6.7), which only depends on the convexity of F, we see that

$$E^\rho_\lambda \Phi^p(\|\mathcal{L}_n\mu_\cdot\|) f(X_\lambda) \le (E\Phi^{2p}(\|: G^{2n}\mu_\cdot :\|))^{1/2}(9EG^2_\rho\, EG^2_{f\cdot dx})^{1/4}. \tag{9.3}$$

Let $f_k(u) \stackrel{def}{=} I_{(k-1<u\le k]}$. In this case, for all $-\infty < k < \infty$

$$\begin{aligned} EG^2_{f_k\cdot dx} &= \int\int u^1(x-y) f_k(x) f_k(y)\,dx\,dy \\ &= \int_0^1\int_0^1 u^1(x-y)\,dx\,dy \stackrel{def}{=} III. \end{aligned} \tag{9.4}$$

Thus we get

$$E^\rho_\lambda \Phi^p(\|\mathcal{L}_n\mu_\cdot\|) f_k(X_\lambda) \le (9\cdot III\cdot EG^2_\rho)^{1/4}(II)^{1/2}. \tag{9.5}$$

Using (9.2) we see that

$$\begin{aligned} E^\rho_\lambda \Phi(\|\mathcal{L}_n\mu_\cdot\|) &= \sum_{k=-\infty}^{\infty} E^\rho_\lambda \Phi(\|\mathcal{L}_n\mu_\cdot\|) f_k(X_\lambda) \\ &\le \sum_{k=-\infty}^{\infty} (E^\rho_\lambda \Phi^p(\|\mathcal{L}_n\mu_\cdot\|) f_k(X_\lambda))^{1/p} P^{1/q}(k-1 < X_\lambda \le k) \end{aligned} \tag{9.6}$$

$$\leq \quad (9 \cdot III \cdot EG_\rho^2)^{1/(4p)}(II)^{1/(2p)} \sum_{k=-\infty}^{\infty} P^{1/q}(k-1 < X_\lambda \leq k).$$

We now note that the sum in the last line of (9.6) is finite. Since the probability density of X_λ is $u^1(x)$ this comes down to showing that

$$\sum_{k=N}^{\infty} k^{m-1}(u^1(k))^{1/q} < \infty \tag{9.7}$$

where $u^1(k) = u^1(|x|)$ for $|x| = k$, which is true by hypothesis. Thus we see that

$$E_\lambda^\rho \exp\left(c\|\gamma_n(\mu_x)\|^{1/n}\right) < \infty. \tag{9.8}$$

Furthermore, we can actually give an upper bound for (9.8), in terms of μ, ρ and u^1.

Let $y \in R^m$ and let ρ_y be Lebesgue measure on $A_y \stackrel{def}{=} y + [-1/2, 1/2]^m$. Then, since $EG^2_{\rho_y}$ is independent of y, we can write

$$E_\lambda^{\rho_y} \exp\left(c\|\gamma_n(\mu_x)\|^{1/n}\right) < C \tag{9.9}$$

for some constant C, which depends on μ, u^1 and ρ_0 but is independent of y. Recalling the definition of E^ρ in (4.1), we see that it follows from (9.9) that there exists a $z \in A_y$ such that

$$E_\lambda^z \exp\left(c\|\gamma_n(\mu_x)\|^{1/n}\right) < C. \tag{9.10}$$

To complete the proof we simply note that

$$\begin{aligned} E_\lambda^y \exp\left(c \sup_{[-1,1]^m} |\gamma_n(\mu_x)|^{1/n}\right) &= E_\lambda^z \exp\left(c \sup_{[-1,1]^m} |\gamma_n(\mu_{x+z-y})|^{1/n}\right) \\ &= E_\lambda^z \exp\left(c \sup_{[-1,1]^m+z-y} |\gamma_n(\mu_x)|^{1/n}\right) \end{aligned} \tag{9.11}$$

which is bounded by (9.10). Thus we obtain (1.13). □

The next lemma shows that the 1-potential of stable mixtures satisfy the hypotheses of Theorem 1.2.

LEMMA 9.1. *Let $u^1(|x|)$ be the 1-potential of a stable mixture in R^2 with $a > 1$. Then for $|x| \geq |x_0|$, for some $|x_0|$ sufficiently large*

$$u^1(|x|) \leq \frac{C}{|x|^3}. \tag{9.12}$$

Let $\tilde{u}^1(|x|)$ be the 1-potential of a stable mixture in R^1 with $0 < a < \beta \leq 1$. Then for $|x| \geq |x_0|$, for some $|x_0|$ sufficiently large

$$\tilde{u}^1(|x|) \leq C_1 \frac{1}{|x|^{1+\epsilon}} \tag{9.13}$$

for some $\epsilon > 0$.

PROOF. Set $x = (x_1, x_2)$ and $p = (p_1, p_2)$ in the first line of (7.23). Without loss of generality we may assume that $|x_1| \geq |x_2|$. Integrate by parts with respect to p_1 in (7.23) to obtain

$$u^1(|x|) = \frac{1}{|x_2||x_1|} \int \sin(p \cdot x) \frac{d}{dp_1} \left(\Delta \frac{1}{1 + \psi(|p|)} \right) d^2 p. \tag{9.14}$$

(Recall that in (7.23), $u(x)$ is an abbreviation for $u^1(x)$). Using (7.3) one can check that $\frac{d}{dp_1} \left(\Delta \frac{1}{1+\psi(|p|)} \right) \in L^1(R^2)$. Since $2|x_1| \geq |x|$ we get (9.12).

To get (9.13) we use (7.34). It suffices to take $x > 0$. We note that since $\psi'(p)/(1 + \psi(p))^2$ is decreasing and regularly varying at zero with index $a > 0$, we have

$$\begin{aligned} u(x) &\leq \beta \int_0^{(2\pi)/x} \frac{p\psi'(p)}{(1 + \psi(p))^2} \, dp \\ &\leq C \int_0^{(2\pi)/x} \psi(p) \, dp \sim C \frac{1}{x} \psi(\frac{1}{x}) \end{aligned} \tag{9.15}$$

as $x \to \infty$. Thus we obtain (9.13). □

APPENDIX A

Necessary conditions

In this Appendix we prove the following theorem which enables us to exclude the important condition (3.57) in the statement of Theorem 1.1.

THEOREM A.1. *Let $\mu \in \mathcal{G}_F^{2n}$.*

(i) *If $\{: G^{2n}\mu_x :, x \in R^m\}$ is locally bounded almost surely then*

$$\int (u^1(y-x))^n \, d\mu(x) \tag{A.1}$$

is bounded on R^m

(ii) *If $\{: G^{2n}\mu_x :, x \in R^m\}$ is continuous almost surely then (A.1) is continuous on R^m.*

This theorem is actually a corollary of a line of work which uses yet another version of Dynkin's isomorphism theorem. This version relates an intersection local time for n independent identically distributed Lévy processes to a $2n$-th order Gaussian chaos process on the space of measures $\mathcal{G}^{2n}$. This process is not the same as the one we have been studying. However, an applicaiton and generalization of a decoupling theorem of Arcones and Gine [**1**] shows that the two are closely related. In a future paper we plan to study intersections of independent Lévy processes in detail. Here we will concentrate on proving Theorem A.1.

For $\rho \in \mathcal{G}^1$ let $G_{(1),\rho}, \dots, G_{(n),\rho}$ denote n independent copies of G_ρ as defined on page 14. Set $H_j(x,\epsilon) = G^2_{(j),\rho_{x,\epsilon}} - EG^2_{(j),\rho_{x,\epsilon}}$, $j = 1, \dots, n$. For $\epsilon > 0$ and $\mu \in \mathcal{G}^{2n}$ define

$$(\times_{j=1}^n H_j)(\epsilon, \mu) = \int \prod_{j=1}^n H_j(x,\epsilon) \, d\mu(x). \tag{A.2}$$

Because of the independence it is easy to see that

$$\begin{aligned} &E_{G_{(1)},\dots,G_{(n)}}((\times_{j=1}^n H_j)(\epsilon,\mu)(\times_{j=1}^n H_j)(\epsilon',\nu)) \\ &\quad = 2^n \iint \prod_{j=1}^n (u^1(v_j,w_j))^2 \prod_{j=1}^n \rho_{x,\epsilon}(dv_j)\rho_{y,\epsilon'}(dw_j)\, d\mu(x)\, d\nu(y) \qquad (A.3) \\ &\quad = 2^n \iint \prod_{j=1}^n (u^1(x+v_j, y+w_j))^2 \prod_{j=1}^n \rho_{\epsilon}(dv_j)\rho_{\epsilon'}(dw_j)\, d\mu(x)\, d\nu(y). \end{aligned}$$

Here $E_{G_{(1)},\dots,G_{(n)}}$ denotes expectation with respect to the product probability space generated by $\times_{j=1}^n \{G_{(j),\rho}, \rho \in \mathcal{G}^1\}$ and we use ρ_ϵ to denote $\rho_{0,\epsilon}$, etc. Since $\mu, \nu \in \mathcal{G}^{2n}$, a slight generalization of Lemma 3.1 implies that

$$(v_1, w_1, \dots, v_n, w_n) \mapsto \iint \prod_{j=1}^n (u^1(x+v_j, y+w_j))^2 \, d\mu(x)\, d\nu(y) \tag{A.4}$$

is continuous. By an argument similar to the one in the paragraph containing (3.11) we see that

$$(\times_{j=1}^n H_j)(\mu) \stackrel{def}{=} \lim_{\epsilon\to 0}(\times_{j=1}^n H_j)(\epsilon,\mu) \tag{A.5}$$

exists as a limit in L^2 and satisfies

$$E_{G_{(1)},\dots,G_{(n)}}((\times_{j=1}^n H_j)(\mu)) = 0 \tag{A.6}$$

$$E_{G_{(1)},\dots,G_{(n)}}((\times_{j=1}^n H_j)(\mu)(\times_{j=1}^n H_j)(\nu)) = 2^n \iint (u^1(x,y))^{2n}\, d\mu(x)\, d\nu(y) \tag{A.7}$$

for all $\mu,\nu\in\mathcal{G}^{2n}$, and

$$\begin{aligned}&\left(E_{G_{(1)},\dots,G_{(n)}}((\times_{j=1}^n H_j)(\mu) - (\times_{j=1}^n H_j)(\nu))^2\right)^{1/2}\\ &\qquad = \left(2^n \iint (u^1(x,y))^{2n}\,(d(\mu(x)-\nu(x)))\,(d(\mu(y)-\nu(y)))\right)^{1/2}\end{aligned} \tag{A.8}$$

for all $\mu,\nu\in\mathcal{G}^{2n}$.

We now show how Theorem 4.2 can be extended to give an isomorphism theorem for the intersections of independent Lévy processes. Recall that in Theorem 4.2, given a Lévy process $(\Omega,\mathcal{F}(t),X(t),P^x,\lambda)$ and a measure $\mu\in\mathcal{G}^2$, we consider the additive functional L^μ_λ. Let $(\Omega_1,\mathcal{F}_1(t),X_1(t),P^x_1,\lambda_1),\dots,$ $(\Omega_n,\mathcal{F}_n(t),X_n(t),P^x_n,\lambda_n)$ denote n independent copies of $(\Omega,\mathcal{F}(t),X(t),P^x,\lambda)$, and $L^{\mu_1}_{1,\lambda_1},\dots,L^{\mu_n}_{n,\lambda_n}$ denote the corresponding additive functionals. It follows from the definition of L^μ_λ that

$$L_j^{x,\epsilon} \stackrel{def}{=} L_j^{\rho_{x,\epsilon}} = \int_0^{\lambda_j} f_\epsilon(X_j(t)-x)\,dt. \tag{A.9}$$

For $\rho_1,\dots,\rho_n\in\mathcal{G}^1$ we use $P^{\rho_1,\dots,\rho_n}_{\lambda_1,\dots,\lambda_n}$ to denote the product measure $P^{\rho_1}_{1,\lambda_1}\times\cdots\times P^{\rho_n}_{n,\lambda_n}$. Let Φ be as given in (4.11) and let Φ_j denote the copy of Φ associated with H_j and X_j. As noted in Chapter 4, we can take continuous versions of $H_1(x,\epsilon),\dots,H_n(x,\epsilon)$. From this it is easy to see that when $\mu\in\mathcal{G}^{2n}$, $(\times_{j=1}^n H_j)(\epsilon,\mu)\in\mathcal{M}(\bigotimes_{j=1}^n\mathcal{H}_j)$ and

$$\tilde\Phi((\times_{j=1}^n H_j)(\epsilon,\mu)) = \sum_A \int \prod_{i\in A^c} H_i(x,\epsilon)\prod_{j\in A} 2L_j^{x,\epsilon}\,d\mu(x) \tag{A.10}$$

where the sum runs over all subsets $A\subseteq\{1,\dots,n\}$ and $\tilde\Phi$ is the natural extension of $\times_{j=1}^n\Phi_j$ to $\mathcal{M}(\bigotimes_{j=1}^n\mathcal{H}_j)$. Let

$$(\times_{i\in A^c}H_i\times_{j\in A}2L_j)(\epsilon,\mu) \stackrel{def}{=} \int\prod_{i\in A^c}H_i(x,\epsilon)\prod_{j\in A}2L_j^{x,\epsilon}\,d\mu(x).$$

It follows from Theorem 4.2 that

$$\begin{aligned}&E_{G_{(1)},\dots,G_{(n)}}E^{\rho_1,\dots,\rho_n}_{\lambda_1,\dots,\lambda_n}\left(F\left(\sum_A(\times_{i\in A^c}H_i\times_{j\in A}2L_j)(\epsilon,\mu.)\right)\prod_{j=1}^n f(X_j(\lambda_j))\right)\\ &\qquad = E_{G_{(1)},\dots,G_{(n)}}\left(F\left((\times_{j=1}^n H_j)(\epsilon,\mu.)\right)\prod_{j=1}^n G_{(j),\rho_j}G_{(j),f\cdot dx}\right).\end{aligned} \tag{A.11}$$

where f is as given in Theorem 4.2. Note that

$$(\times_{i\in A^c} H_i \times_{j\in A} L_j)(\epsilon,\mu) = \int \prod_{i\in A^c} H_i(x,\epsilon)\left(\int_0^{\lambda_1}\cdots\int_0^{\lambda_n}\prod_{j\in A} f_\epsilon(X_j(t_j)-x)\prod_{j\in A} dt_j\right) d\mu(x). \tag{A.12}$$

In particular

$$\begin{aligned}(\times_{j=1}^n L_j)(\epsilon,\mu) &= \int \times_{j=1}^n L_j^{x,\epsilon}\, d\mu(x)\\ &= \int \left(\int_0^{\lambda_1}\cdots\int_0^{\lambda_n}\prod_{j=1}^n f_\epsilon(X_j(t_j)-x)\prod_{j=1}^n dt_j\right) d\mu(x)\end{aligned} \tag{A.13}$$

The same reasoning that leads to (A.10) shows that, more generally, for any $B\subseteq\{1,\dots,n\}$

$$E_{G_{(1)},\dots,G_{(n)}} E_{\lambda_B}^{\rho_B}\left(F\left(\sum_{A\subseteq B}(\times_{i\in A^c} H_i\times_{j\in A} 2L_j)(\epsilon,\mu.)\right)\prod_{j\in B} f(X_j(\lambda_j))\right) = E_{G_{(1)},\dots,G_{(n)}}\left(F\left((\times_{j=1}^n H_j)(\epsilon,\mu.)\right)\prod_{j\in B} G_{(j),\rho_j} G_{(j),f\cdot dx}\right) \tag{A.14}$$

where A^c is the complement of A in $\{1,\dots,n\}$ and $P_{\lambda_B}^{\rho_B}$ denotes the product measure $\times_{j\in B} P_{j,\lambda_j}^{\rho_j}$.

This allows us to show inductively, using (A.5), that

$$(\times_{i\in A^c} H_i\times_{j\in A} 2L_j)(\mu) \stackrel{def}{=} \lim_{\epsilon\to 0}(\times_{i\in A^c} H_i\times_{j\in A} 2L_j)(\epsilon,\mu) \tag{A.15}$$

exists as a limit in $L^2(dP_{G_{(1)},\dots,G_{(n)}}\bigotimes f\cdot dP_{\lambda_1,\dots,\lambda_n}^{\rho_1,\dots,\rho_n})$ where $f\cdot dP_{\lambda_1,\dots,\lambda_n}^{\rho_1,\dots,\rho_n} \stackrel{def}{=} \prod_{j=1}^n f(X_j(\lambda_j))\cdot dP_{\lambda_1,\dots,\lambda_n}^{\rho_1,\dots,\rho_n}$. In particular

$$(\times_{j=1}^n L_j)(\mu) = \lim_{\epsilon\to 0}\int\left\{\int_0^{\lambda_1}\cdots\int_0^{\lambda_n}\prod_{j=1}^n f_\epsilon(X_j(t_j)-x)\prod_{j=1}^n dt_j\right\} d\mu(x) \tag{A.16}$$

is a (total) intersection local time for the independent processes $X_1,\dots,X_n$.

We can now state an isomorphism theorem for n independent processes.

THEOREM A.2. *Let $\{\mu_i\}_{i=1}^\infty$ be finite measures in $\mathcal{G}^{2n}$. Then, for any compactly supported measures $\rho_1,\dots,\rho_n\in\mathcal{G}^1$, $\mathcal{C}$ measurable non-negative function F on R^∞,and for any $B\subseteq\{1,\dots,n\}$ we have*

$$E_{G_{(1)},\dots,G_{(n)}} E_{\lambda_B}^{\rho_B}\left(F\left(\sum_{A\subseteq B}(\times_{i\in A^c} H_i\times_{j\in A} 2L_j)(\mu.)\right)\prod_{j\in B} f(X_j(\lambda_j))\right) = E_{G_{(1)},\dots,G_{(n)}}\left(F\left((\times_{j=1}^n H_j)(\mu.)\right)\prod_{j\in B} G_{(j),\rho_j} G_{(j),f\cdot dx}\right). \tag{A.17}$$

where A^c is the complement of A in $\{1,\dots,n\}$ and $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞.

Clearly Theorem A.2 is easier to prove than Theorem 4.1. It does not require that the Lévy processes are in Class A. In fact this theorem is valid for strongly symetric Markov processes, as defined in [**17**].

Theorem A.2 refers to n independent Markov processes starting at arbitrary points. We will need version of this theorem in which the independent processes all start from the same point. To accomplish this we return to (A.11) which we write as

$$\int\int E_{G_{(1)},\dots,G_{(n)}} E^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n} \left(F\left(\sum_A (\times_{i\in A^c} H_i \times_{j\in A} 2L_j)(\epsilon,\mu)\right) \prod_{j=1}^n f(X_j(\lambda_j))\right) \prod_{j=1}^n \rho_j(dy_j) \tag{A.18}$$
$$= E_{G_{(1)},\dots,G_{(n)}} \left(F\left((\times_{j=1}^n H_j)(\epsilon,\mu.)\right) \prod_{j=1}^n G_{(j),\rho_j} G_{(j),f\cdot dx}\right).$$

Let $\rho_j=\rho_{x,\delta}$, $j=1,\dots,n$ and integrate the resulting equation with respect to a measure $\nu\in\mathcal{G}^n$. This gives us

$$\int\int E_{G_{(1)},\dots,G_{(n)}} E^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n} \left(F\left(\sum_A (\times_{i\in A^c} H_i \times_{j\in A} 2L_j)(\epsilon,\mu.)\right) \prod_{j=1}^n f(X_j(\lambda_j)) \prod_{j=1}^n \rho_{x,\delta}(dy_j)\, d\nu(x)\right) \tag{A.19}$$
$$= E_{G_{(1)},\dots,G_{(n)}} \left(F\left((\times_{j=1}^n H_j)(\epsilon,\mu.)\right) \left(\int \prod_{j=1}^n G_{(j),\rho_{x,\delta}}\, d\nu(x)\right) \prod_{j=1}^n G_{(j),f\cdot dx}\right).$$

Let

$$(\times_{j=1}^n G_{(j)})(\delta,\nu) \stackrel{def}{=} \int \prod_{j=1}^n G_{(j),\rho_{x,\delta}}\, d\nu(x) \tag{A.20}$$

for $\delta>0$ and $\nu\in\mathcal{G}^n$. Similarly to (A.3) and (A.4) we see that

$$E_{G_{(1)},\dots,G_{(n)}}((\times_{j=1}^n G_{(j)})(\delta,\mu)(\times_{j=1}^n G_{(j)})(\delta',\nu))$$
$$= \int\int \prod_{j=1}^n u^1(v_j,w_j) \prod_{j=1}^n \rho_{x,\delta}(dv_j)\rho_{y,\delta'}(dw_j)\, d\mu(x)\, d\nu(y) \tag{A.21}$$
$$= \int\int \prod_{j=1}^n u^1(x+v_j,y+w_j) \prod_{j=1}^n \rho_{\delta}(dv_j)\rho_{\delta'}(dw_j)\, d\mu(x)\, d\nu(y)$$

and

$$(v_1,w_1,\dots,v_n,w_n) \mapsto \int\int \prod_{j=1}^n u^1(x+v_j,y+w_j)\, d\mu(x)\, d\nu(y) \tag{A.22}$$

is continuous. Consequently

$$(\times_{j=1}^{n} G_{(j)})(\nu) \stackrel{def}{=} \lim_{\delta\to 0} (\times_{j=1}^{n} G_{(j)})(\delta,\nu) \tag{A.23}$$

exists as a limit in L^2.

Let $\mu_1,\dots,\mu_N \in \mathcal{G}^{2n}$ and let F be a bounded continuous function on R^N. Recall (A.12) and the fact that the functions f_ϵ and f are bounded and uniformly continuous. We see that

$$E^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n}\left(F\left(\sum_A (\times_{i\in A^c} H_i \times_{j\in A} 2L_j)(\epsilon,\mu.)\right) \prod_{j=1}^{n} f(X_j(\lambda_j))\right) \tag{A.24}$$

is a bounded uniformly continuous function of $(y_1,\dots,y_n)$. Also note that the following set of measures on R^n, indexed by δ

$$\int \prod_{j=1}^{n} \rho_{x,\delta}(dy_j)\, d\nu(x) \tag{A.25}$$

converge weakly as $\delta\to 0$, to a measure which we denote by $\Delta_\star\nu$, defined by the equations

$$\int h(y_1,\dots,y_n)\, d\Delta_\star\nu(y_1,\dots,y_n) = \int h(y,\dots,y)\, d\nu(y) \tag{A.26}$$

for all bounded uniformly continuous functions h on R^n.

Thus we can take the limit of (A.19) as $\delta\to 0$ and obtain

$$\begin{aligned} &\int E_{G_{(1)},\dots,G_{(n)}} E^{y,\dots,y}_{\lambda_1,\dots,\lambda_n}\left(F\left(\sum_A (\times_{i\in A^c} H_i \times_{j\in A} 2L_j)(\epsilon,\mu.)\right) \prod_{j=1}^{n} f(X_j(\lambda_j))\, d\nu(y)\right) \\ &\quad = E_{G_{(1)},\dots,G_{(n)}}\left(F\left((\times_{j=1}^{n} H_j)(\epsilon,\mu.)\right) (\times_{j=1}^{n} G_{(j)})(\nu) \prod_{j=1}^{n} G_{(j),f\cdot dx}\right). \end{aligned} \tag{A.27}$$

It is straight forward to take limits in (A.27) as $\epsilon\to 0$ and to extend the resulting equation so that it holds for a countable collection of measures $\mu_1,\dots \in \mathcal{G}^{2n}$, all $\mathcal{C}$ measurable non-negative function F on R^∞, and arbitrary subsets $B\subseteq\{1,\dots,n\}$. Doing this gives the next theorem. (When the Lévy processes $X_1,\dots,X_n$ all have the same initial point $y\in S$, we abbreviate $P^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n}$ by $P^{y}_{\lambda_1,\dots,\lambda_n}$, and use analogous notation when considering arbitrary $B\subseteq\{1,\dots,n\}$.)

THEOREM A.3. *Let $\{\mu_i\}_{i=1}^{\infty}$ be finite measures in $\mathcal{G}^{2n}$. Then for any compactly supported $\nu\in\mathcal{G}^n$, $\mathcal{C}$ measurable non-negative function F on R^∞,and for any $B\subseteq\{1,\dots,n\}$ we have*

$$\begin{aligned} &\int E_{G_{(1)},\dots,G_{(n)}} E^{y}_{\lambda_B}\left(F\left(\sum_{A\subseteq B} (\times_{i\in A^c} H_i \times_{j\in A} 2L_j)(\mu.)\right) \prod_{j\in B} f(X_j(\lambda_j))\right) d\nu(y) \\ &\quad = E_{G_{(1)},\dots,G_{(n)}}\left(F\left((\times_{j=1}^{n} H_j)(\mu.)\right) (\times_{j=1}^{n} G_{(j)})(\nu) \prod_{j\in B} G_{(j),f\cdot dx}\right) \end{aligned} \tag{A.28}$$

where A^c is the complement of A in $\{1,\dots,n\}$ and $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞.

We now need a simple extension of Lemma 3.1. Let

$$I_{\mu,\nu}(z_1,\dots,z_k)=\int\!\!\int\prod_{j=1}^k u^1(x-y+z_j)\,d\mu(x)\,d\nu(y) \tag{A.29}$$

$$J_{\mu,\nu}(v_1,\dots,v_k)=\int\prod_{j=1}^k u^1(x+v_j)\,d(\mu*\nu)(x) \tag{A.30}$$

and

$$\begin{aligned}&K_{\mu,\nu,\rho}(v_1,\dots,v_n,z_1,\dots,z_n)\\&\qquad=\int\!\!\int\prod_{j=1}^n u^1(x+v_j)\prod_{j=1}^n u^1(x-y+z_j)\,d(\mu*\nu)(x)\,d\rho(y).\end{aligned} \tag{A.31}$$

LEMMA A.1. (i) *When $\mu,\nu\in\mathcal{G}^k$, $I_{\mu,\nu}(z_1,\dots,z_k)$ is bounded and uniformly continuous.*

(ii) *When $\mu,\nu\in\mathcal{G}^k$, $J_{\mu,\nu}(v_1,\dots,v_k)$ is bounded and uniformly continuous.*

(iii) *When $\mu,\nu\in\mathcal{G}^{2n}$, and ρ is a finite measure, $K_{\mu,\nu,\rho}(v_1,\dots,v_n,z_1,\dots,z_n)$ is bounded and uniformly continuous.*

PROOF. That $I_{\mu,\nu}(z_1,\dots,z_k)$ is bounded and uniform continuous follows from the proof of Lemma 3.1. For (ii) note that

$$\begin{aligned}J_{\mu,\nu}(v_1,\dots,v_k)&=\int\int\prod_{j=1}^k u^1(x+y+v_j)\,d\mu(x)\,d\nu(y)\\&=I_{\bar\mu,\nu}(v_1,\dots,v_k)\end{aligned} \tag{A.32}$$

where $\bar\mu(A)=\mu(-A)$. Since $\mu\in\mathcal{G}^k$ implies $\bar\mu\in\mathcal{G}^k$, (ii) follows from (i).

Finally, by (A.30)

$$\begin{aligned}&K_{\mu,\nu,\rho}(v_1,\dots,v_n,z_1,\dots,z_n)\\&\qquad=\int\left(J_{\mu,\nu}(v_1,\dots,v_n,z_1-y,\dots,z_n-y)\right)\,d\rho(y).\end{aligned} \tag{A.33}$$

Consequently we obtain (iii) because $J_{\mu,\nu}(v_1,\dots,v_n,z_1-y,\dots,z_n-y)$ is bounded and uniform continuous and ρ is a finite measure. □

We now use Theorem A.3 to obtain the results of Theorem A.1 but for the Gaussian chaos $\{(\times_{j=1}^n H_j)(\mu_x),\, x\in R^m\}$.

THEOREM A.4. *Let $\mu\in\mathcal{G}_F^{2n}$.*

(i) *If $\{(\times_{j=1}^n H_j)(\mu_x)\, x\in R^m\}$ is locally bounded almost surely then*

$$\int(u^1(y-x))^n\,d\mu(x) \tag{A.34}$$

is bounded on R^m.

(ii) *If $\{(\times_{j=1}^n H_j)(\mu_x),\, x\in R^m\}$ is continuous almost surely then (A.34) is continuous on R^m.*

PROOF. For simplicity we prove this with R^m replaced by the torus T^m. The easy modifications necessary to adapt the proof to R^m are carefully explained in [**15**]. Note that by working on the torus we can set $f \equiv 1$ in Theorem A.3. We first assume only that $\{(\times_{j=1}^n H_j)(\mu_x),\, x \in T^m\}$ is bounded almost surely. Let $\mu^\tau = f_\tau * \mu$, that is, $d\mu^\tau(x) = (\int f_\tau(x-y)\, d\mu(y))\, d^m x$. Note that by (A.8)

$$E_{G_{(1)},\ldots,G_{(n)}} \left((\times_{j=1}^n H_j)(\mu_z) - (\times_{j=1}^n H_j)(\mu_v) \right)^2 = 2^n \iint (u^1(x-y))^{2n} d(\mu_z(x) - \mu_v(x)) d(\mu_z(y) - \mu_v(y)). \tag{A.35}$$

Therefore, by Lemma A.1 we have that $\{(\times_{j=1}^n H_j)(\mu_x),\, x \in T^m\}$ is continuous in L^2. This enables us to take a measureable and separable version of $\{(\times_{j=1}^n H_j)(\mu_x),\, x \in T^m\}$. We see that the two processes

$$\{(\times_{j=1}^n H_j)(\mu_y^\tau),\, y \in T^m\} \tag{A.36}$$

and

$$\{\int f_\tau(y-x)(\times_{j=1}^n H_j)(\mu_y)\, d^m x,\, y \in T^m\} \tag{A.37}$$

are stochastically equivalent. This implies that

$$\| \sup_{y \in T^m} (\times_{j=1}^n H_j)(\mu_y^\tau) \|_2 \le \| \sup_{y \in T^m} (\times_{j=1}^n H_j)(\mu_y) \|_2. \tag{A.38}$$

We claim that the convergence $(\times_{j=1}^n L_j)(\mu^\tau) = \lim_{\epsilon \to 0} (\times_{j=1}^n L_j)(\epsilon, \mu^\tau)$ in (A.16) holds in $L^2(P^z_{\lambda_1,\ldots,\lambda_n})$ for each $z \in T^m$, and in fact is uniform in $z \in T^m$. To see this we compute

$$\begin{aligned}
&E^z_{\lambda_1,\ldots,\lambda_n} \left((\times_{j=1}^n L_j)(\epsilon, \mu^\tau)(\times_{j=1}^n L_j)(\epsilon', \mu^\tau) \right) \\
&= \iint \left(E^z_\lambda \int_0^\lambda \int_0^\lambda f_\epsilon(X(s)-x) f_{\epsilon'}(X(t)-y)\, ds\, dt \right)^n d\mu^\tau(x)\, d\mu^\tau(y) \\
&= 2^n \iint \left(\iint u^1(v-z) u^1(w-v) f_\epsilon(v-x) f_{\epsilon'}(w-y)\, dv\, dw \right)^n \\
&\qquad d\mu^\tau(x)\, d\mu^\tau(y) \qquad\qquad \text{(A.39)} \\
&= 2^n \iint \left(\iint u^1(x+v-z) u^1(y-x+w-v) f_\epsilon(v) f_{\epsilon'}(w)\, dv\, dw \right)^n \\
&\qquad d\mu^\tau(x)\, d\mu^\tau(y) \\
&= 2^n \iint \left(\iint \prod_{j=1}^n u^1(x+v_j-z) u^1(y-x+w_j-v_j)\, d\mu^\tau(x)\, d\mu^\tau(y) \right) \\
&\qquad \prod_{j=1}^n f_\epsilon(v_j) f_{\epsilon'}(w_j)\, dv_j\, dw_j.
\end{aligned}$$

Since $\mu^\tau = f_\tau * \mu \in \mathcal{G}^{2n}$ and is also a finite measure, our assretion follows from Lemma A.1.

Convergence in $L^2(P^z_{\lambda_1,\ldots,\lambda_n})$ implies convergence in $L^1(P^z_{\lambda_1,\ldots,\lambda_n})$. Therefore

$$E^z_{\lambda_1,\ldots,\lambda_n}((\times_{j=1}^n L_j)(\mu^\tau)) = \lim_{\epsilon \to 0} E^z_{\lambda_1,\ldots,\lambda_n}((\times_{j=1}^n L_j)(\epsilon, \mu^\tau))$$

$$= \lim_{\epsilon\to 0} E^z_{\lambda_1,\ldots,\lambda_n} \int \left(\int_0^{\lambda_1} \cdots \int_0^{\lambda_n} \prod_{j=1}^n f_\epsilon(X_j(t)-x) \prod_{j=1}^n dt_j \right) d\mu^\tau(x) \tag{A.40}$$

$$= \lim_{\epsilon\to 0} \int \left(\int \prod_{j=1}^n u^1(v_j - z) f_\epsilon(v_j - x)\, dv_j \right) d\mu^\tau(x)$$

$$= \lim_{\epsilon\to 0} \int \left(\int \prod_{j=1}^n u^1(v_j + x - z) f_\epsilon(v_j)\, dv_j \right) d\mu^\tau(x)$$

$$= \lim_{\epsilon\to 0} \int \left(\int \prod_{j=1}^n u^1(v_j + x - z)\, d\mu^\tau(x) \right) \prod_{j=1}^n f_\epsilon(v_j)\, dv_j$$

$$= \int (u^1(x-z))^n \, d\mu^\tau(x)$$

for all $z \in T^m$. The last line follows from Lemma A.1 since $\mu^\tau = f_\tau * \mu \in \mathcal{G}^{2n}$.

Furthermore, calculations similar to those in (A.39) show that

$$E^z_{\lambda_1,\ldots,\lambda_n} \left(\{ (\times_{j=1}^n L_j)(\mu^\tau_w) - (\times_{j=1}^n L_j)(\mu^\tau_v) \}^2 \right) = 2^n \iint (u^1(x-z))^n$$
$$(u^1(y-x))^n d(\mu^\tau_w(x) - \mu^\tau_v(x)) d(\mu^\tau_w(y) - \mu^\tau_v(y)). \tag{A.41}$$

By Lemma A.1 this shows that $\{(\times_{j=1}^n L_j)(\mu^\tau_y),\ y \in T^m\}$ is continuous in $L^2(P^z_{\lambda_1,\ldots,\lambda_n})$ for each $z \in T^m$. Therefore we can choose a separable version $\{(\times_{j=1}^n \tilde{L}_j)(\mu^\tau_y),\ y \in T^m\}$ of $\{(\times_{j=1}^n L_j)(\mu^\tau_y),\ y \in T^m\}$ in $L^2(P^z_{\lambda_1,\ldots,\lambda_n})$ for each $z \in T^m$. When $D \subseteq T^m$ is a countable dense set we then have

$$\sup_{y\in T^m} (\times_{j=1}^n \tilde{L}_j)(\mu^\tau_y) = \sup_{w\in D} (\times_{j=1}^n \tilde{L}_j)(\mu^\tau_w) \tag{A.42}$$

P^z_λ almost surely, for each $z \in T^m$.

The fact that the convergence of

$$(\times_{j=1}^n L_j)(\mu^\tau) = \lim_{\epsilon\to 0} (\times_{j=1}^n L_j)(\epsilon, \mu^\tau) \tag{A.43}$$

in $L^2(P^z_{\lambda_1,\ldots,\lambda_n})$, is uniform in $z \in T^m$ shows that the convergence also holds in $L^2(P^\rho_{\lambda_1,\ldots,\lambda_n})$ for any probability measure ρ. This implies that in Theorem A.3 we can replace $\{(\times_{j=1}^n L_j)(\mu^\tau_y),\ y \in T^m\}$ by $\{(\times_{j=1}^n \tilde{L}_j)(\mu^\tau_y),\ y \in T^m\}$. Using Theorem A.3, and an argument similar to the one used in (6.6), and (A.38) we see that for any probability measure $\nu \in \mathcal{G}^n$ and any $y, z \in R^m$ we have

$$\begin{aligned} \int E^z_{\lambda_1,\ldots,\lambda_n}((\times_{j=1}^n \tilde{L}_j)(\mu^\tau_y))\, d\nu(z) &\le \int E^z_{\lambda_1,\ldots,\lambda_n}(\sup_{w\in D} (\times_{j=1}^n \tilde{L}_j)(\mu^\tau_w))\, d\nu(z) \\ &\le C \| \sup_{w\in D} (\times_{j=1}^n H_j)(\mu_w) \|_2^2 \\ &\le C \| \sup_{w\in T^m} (\times_{j=1}^n H_j)(\mu_w) \|_2^2 \end{aligned} \tag{A.44}$$

where C is a constant independent of z and τ. In particular this holds when ν is normalized Lebesgue measure on T^m. Using (A.40) we see that for any $z \in T^m$

$$\int (u^1(y-x))^n \, d\mu^\tau(x) = \int (u^1(z-x))^n \, d\mu^\tau_{z-y}(x)$$

$$= E^z_{\lambda_1,\dots,\lambda_n}((\times_{j=1}^n \tilde{L}_j)(\mu^\tau_{z-y})) \tag{A.45}$$
$$\le E^z_{\lambda_1,\dots,\lambda_n}(\sup_{w\in D}(\times_{j=1}^n \tilde{L}_j)(\mu^\tau_w)).$$

Integrating (A.45) with respect to normalized Lebesgue measure, which we denote by $d^m z$, and using (A.44) we see that

$$\int (u^1(z-x))^n \, d\mu^\tau(x) \le C\| \sup_{w\in T^m} (\times_{j=1}^n H_j)(\mu_w)\|_2^2 \tag{A.46}$$

where C is a constant independent of z and τ. Since $x \in T^m$ and both μ and $d^m z$ are contained in $\mathcal{G}^{2n}$, (and recalling (3.1)) we see that $(u^1)^n * \mu \in L^1(d^m z)$. Consequently

$$\int (u^1(z-x))^n \, d\mu^\tau(x) = (u^1)^n * \mu * f_\tau(z) \to \int (u^1(z-x))^n \, d\mu(x)$$

as $\tau \to 0$ for almost all $z \in T^m$. Hence for almost all $z \in T^m$ we have

$$\int (u^1(z-x))^n \, d\mu(x) \le C\| \sup_{w\in T^m} (\times_{j=1}^n H_j)(\mu_w)\|_2^2 \tag{A.47}$$

where C is a constant independent of z. Finally, since $u^1(z-x)$ is continuous in x except when $z = x$, and each $\mu \in \mathcal{G}_F^{2n}$ is non-atomic, (i) follows by Fatou's lemma. The result on continuity is treated similarly. □

In order to prove Theorem A.1 we prove a lemma which is both an application and a generalization of Theorem 2.2, [**1**]. Consider the Gaussian process $G_{x,\delta}$ the building block of the Wick power chaos. This has a Karhunen-Loeve expansion, as in (3.17). To simplify the interchange of limits we truncate the expansion and define

$$G_{x,\delta,N} = \sum_{|i|\le N} g_i \phi_i(x,\delta). \tag{A.48}$$

Next, similarly to (3.20), we define the Wick power Gaussian chaos

$$: G^m\mu(\delta, N) := \int : G^m_{x,\delta,N} : \, d\mu(x)$$
$$= \sum_{i_1,\dots,i_m \le N} \int \phi_{i_1}(x,\delta)\cdots\phi_{i_m}(x,\delta)\, d\mu(x) \prod_{j\ge 1} H_{m_j(i_1,\dots,i_m)}(g_j). \tag{A.49}$$

When $\mu \in \mathcal{G}^m$, $: G^m\mu(\delta,N) :$ converges in L^2, as $\delta \to 0$ and $N \to \infty$, to $: G^m\mu :$, the basic m-th Wick power chaos. We associate with $: G^m\mu(\delta,N) :$ the decoupled chaos

$$: G^m_{dec}\mu(\delta, N) := \sum_{i_1,\dots,i_m \le N} \int \phi_{i_1}(x,\delta)\cdots\phi_{i_m}(x,\delta)\, d\mu(x) g^{(1)}_{i_1}\cdots g^{(m)}_{i_m} \tag{A.50}$$

where, in general, $\{g_i^{(k)}\}_{i=0}^\infty$, $k \ge 1$, are independent copies of $\{g_i\}_{i=0}^\infty$. This is just

$$\int \left(\prod_{p=1}^m G^{(p)}_{x,\delta,N}\right) d\mu(x) \tag{A.51}$$

where

$$G^{(p)}_{x,\delta,N} = \sum_{i_p \le N} g^{(p)}_{i_p} \phi_{i_p}(x,\delta). \tag{A.52}$$

In order to use Theorem 2.2, [**1**] we note that the real valued coefficients in the series representation of $:G^m\mu(\delta,N):$ in (A.49) are symmetric in $i_1,\ldots,i_m$.

Let $\mathcal{N}$ be a finite set of finite measures and let $\|\cdot\|_{\mathcal{N}}$ be a norm on $\mathcal{N}$. Theorem 2.2, [**1**] states that for all $u>0$

$$\begin{aligned} 2^{-6m-2}P\left(c_m\|:G^m\mu(\delta,N):\|_{\mathcal{N}}\geq u\right) &\leq P\left(\|:G^m_{dec}\mu(\delta,N):\|_{\mathcal{N}}\geq u\right)\\ &\leq 2^{m-1}P\left(c_m^{-1}\|:G^m\mu(\delta,N):\|_{\mathcal{N}}\geq u\right). \end{aligned} \tag{A.53}$$

were $c_m=(m!/m^m)^{1/2}$.

We need a version of (A.53) for the integral of products of independent Wick powers. Consider the product of r independent Wick powers

$$\prod_{a=1}^r :G^{m_a}_{a,x,\delta,N}: \tag{A.54}$$

where $\{m_a\}_{a=1}^r$ are integers and each term $:G^{(m_a)}_{x,\delta,N}:$ is generated, as above, by a Gaussian process

$$G_{a,x,\delta,N}=\sum_{|i|\leq N} g_{a,i}\phi_{a,i}(x,\delta) \tag{A.55}$$

where the sequences $\{\{g_{a,i}\}\}_{a=1}^r$ are independent sequences of independent identically distributed N(0,1) random variables. The sequences of functions $\{\{\phi_{a,i}(x,\delta)\}\}_{a=1}^r$ are not necessarily the same for different values of a.

Let

$$Q_{N,\delta}(\mu)\stackrel{def}{=}\int\prod_{a=1}^r :G^{m_a}_{a,x,\delta,N}:\,d\mu(x). \tag{A.56}$$

Similarly

$$Q_{N,dec}(\mu)=\int\prod_{a=1}^r\left(\prod_{p=1}^{m_a}G^{(p)}_{a,x,\delta,N}\right)d\mu(x) \tag{A.57}$$

where

$$G^{(p)}_{a,x,\delta,N}=\sum_{|i_p|\leq N} g^{(p)}_{a,i_p}\phi_{a,i_p}(x,\delta) \tag{A.58}$$

and $\{\{g^{(p)}_{a,i_p}\}\}_{p=1}^{m_a}$ are independent copies of $\{\{g_{a,i}\}\}$, $a=1,\ldots,r$. In the next Lemma, by decoupling each Wick power separately, while holding the other terms fixed on the appropriate probability product space, we see that (A.53) holds for the more general processes $Q_{N,\delta}(\mu)$ and $Q_{N,\delta,dec}(\mu)$ except that the constants are now products of the constants in (A.53) over $m_1,\ldots,m_r$.

LEMMA A.2. *Let* $b_m=2^{-6m-2}$, $c_m=(m!/m^m)^{1/2}$, $d_m=2^{m-1}$, $b_{\hat{m}}=\prod_{a=1}^r b_{m_a}$ *and similarly for* $c_{\hat{m}}$ *and* $d_{\hat{m}}$. *Then for* $Q_{N,\delta}(\mu)$ *and* $Q_{N,\delta,dec}(\mu)$ *as defined in (A.56) and (A.57) and for all* $u>0$

$$\begin{aligned} b_{\hat{m}}P\left(c_{\hat{m}}\|Q_{N,\delta}(\cdot)\|_{\mathcal{N}}\geq u\right) &\leq P\left(\|Q_{N,\delta,dec}(\cdot)\|_{\mathcal{N}}\geq u\right)\\ &\leq d_{\hat{m}}P\left(c_{\hat{m}}^{-1}\|Q_{N,\delta}(\cdot)\|_{\mathcal{N}}\geq u\right). \end{aligned} \tag{A.59}$$

PROOF. The proof is straight forward. We indicate how it goes by obtaining the left-hand side inequality in (A.59) in the case $a=2$. Let Ω_a be the product probability space of $\{\{g^{(p)}_{a,i_p}\}\}_{p=1}^{m_a}$ and $\omega_a\in\Omega_a$, $a=1,2$. To emphasize the dependence on the probability space we write

$$Q_{N,\delta}(\mu,\omega_1,\omega_2)=\int :G^{m_1}_{1,x,\delta,N}(\omega_1)::G^{m_2}_{2,x,\delta,N}(\omega_2):\, d\mu(x) \tag{A.60}$$

and

$$Q_{N,\delta,dec}(\mu,\omega_1,\omega_2)=\int \prod_{p=1}^{m_1} G^{(p)}_{1,x,\delta,N}(\omega_1)\prod_{p=1}^{m_2} G^{(p)}_{2,x,\delta,N}(\omega_2)\, d\mu(x) \tag{A.61}$$

where $\omega_a=(\omega_a^{(1)},\ldots,\omega_a^{(m_a)})$, $a=1,2$. By (A.53), with ω_2 fixed

$$b_{m_1}P_{\omega_1}\left(c_{m_1}\|Q_{N,\delta}(\,\cdot\,,\omega_1,\omega_2)\|_{\mathcal{N}}\geq u\right)\leq P_{\omega_1}\left(\|I_{N,dec}(\,\cdot\,,\omega_1,\omega_2)\|_{\mathcal{N}}\geq u\right) \tag{A.62}$$

where

$$\begin{aligned} I_{N,dec}(\mu,\omega_1,\omega_2) &= \int \prod_{p=1}^{m_1} G^{(p)}_{1,x,\delta,N}(\omega_1)\, :G^{m_2}_{2,x,\delta,N}(\omega_2):\, d\mu(x) \\ &= \int :G^{m_2}_{2,x,\delta,N}(\omega_2): \prod_{p=1}^{m_1} G^{(p)}_{1,x,\delta,N}(\omega_1)\, d\mu(x). \end{aligned} \tag{A.63}$$

Taking the expectation of each side of (A.62) over Ω_2 we get

$$\begin{aligned} b_{m_2}b_{m_1}P\left(c_{m_2}c_{m_1}\|Q_{N,\delta}(\cdot)\|_{\mathcal{N}}\geq u\right) &\leq b_{m_2}P\left(c_{m_2}\|I_{N,dec}(\cdot)\|_{\mathcal{N}}\geq u\right) \\ &= b_{m_2}E_{\omega_1}P_{\omega_2}\left(c_{m_2}\|I_{N,dec}(\,\cdot\,,\omega_1,\omega_2)\|_{\mathcal{N}}\geq u\right). \end{aligned} \tag{A.64}$$

Using the representation of $I_{N,dec}$ given in the last line of (A.63) and using (A.53) again we see that

$$\begin{aligned} &b_{m_2}P_{\omega_2}\left(c_{m_2}\|I_{N,dec}(\,\cdot\,,\omega_1,\omega_2)\|_{\mathcal{N}}\geq u\right) \\ &\quad\leq P_{\omega_2}\left(\|Q_{N,\delta,dec}(\,\cdot\,,\omega_1,\omega_2)\|_{\mathcal{N}}\geq u\right). \end{aligned} \tag{A.65}$$

Taking the expectation of (A.65) over Ω_1 and combinning the result with (A.64) we obtain the left-hand side inequality in (A.59) in the case $a=2$. The right-hand side inequality in this case follows similarly as does the general case stated in the theorem. □

Proof of Theorem A.1. Theorem A.1 (i) follows from Theorem A.4 (i), because there exist constants $0<C_1,C_2<\infty$, depending only on n such that

$$\begin{aligned} E\left(\sup_{x\in[-1,1]^m}|:G^{2n}\mu_x:|\right) &\leq C_1E\left(\sup_{x\in[-1,1]^m}|(\times_{j=1}^n H_j)(\mu_x)|\right) \\ &\leq C_2E\left(\sup_{x\in[-1,1]^m}|:G^{2n}\mu_x:|\right) \end{aligned} \tag{A.66}$$

and the fact that a Gaussian chaos process has all moments. We now show that the relationship in (A.66) follows from Lemma A.2. We note that the bounds in Lemma A.2 are independent of N and δ and hence they continue to hold for limits in L^2 of the processes $Q_{N,\delta}(\mu)$ and $Q_{N,\delta,dec}(\mu)$. When $\mu\in\mathcal{G}^{2n}$, $:G^{2n}\mu:$ and $(\times_{j=1}^n H_j)(\mu)$ are limits of

$$\int :G^{2n}_{x,\delta,N}:\, d\mu(x) \quad\text{and}\quad \int\prod_{a=1}^n :G^2_{a,x,\delta,N}:\, d\mu(x) \tag{A.67}$$

respectively. These are two different realizations of $Q_{N,\delta}(\mu)$. However, the corresponding $Q_{N,\delta,dec}(\mu)$ is the same for both of them. Therefore the moments of both of these processes are comparable. Since (A.59) is also independent of the cardinality of $\mathcal{N}$ it can be extended to a countable set of measures $\{\mu_x\}$, where x is dense in $[-1,1]^m$. The extension to all $x \in [-1,1]^m$ follows from the separability of the processes. This completes the proof of Theorem A.1 (i). The proof of Theorem A.1 (ii) is the same except that the sup-norm is replaced by a norm which measures the modulus of continuity. $\square$

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