Uniform Invariance Principles for Intersection Local Times

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1 Introduction

Let S_n be a strongly aperiodic stable random walk, i.e. in the domain of attraction of a non-degenerate stable random variable U of index β in $\mathbf{R}^d$. Thus

$$\frac{S_n}{b_n} \stackrel{\mathcal{L}}{\Longrightarrow} U$$

for some b_n which is regularly varying of order β.

Given k independent copies $S_n^{(1)}, \dots, S_n^{(k)}$ of S_n we define their k-fold intersection local time by

$$(1.1) \quad I_k(x,t) = \sum_{i_1,\dots,i_k=1}^{t} \delta(S_{i_2}^{(2)} - S_{i_1}^{(1)}, x_1) \cdots \delta(S_{i_k}^{(k)} - S_{i_{k-1}}^{(k-1)}, x_{k-1})$$

where

$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

is the usual Kronecker delta function and $x = (x_1, \dots, x_{k-1}) \in (\mathbf{Z}^d)^{k-1}$, $t \in \mathbf{Z}_+$. This definition is extended to $x \in (\mathbf{R}^d)^{k-1}$, $t \in \mathbf{R}_+$ by linear interpolation.

Let Z_t denote the stable Levy process of index β in $\mathbf{R}^d$ with $Z_1 = U$. We use $p_t(x)$ for the transition density of Z_t. If $Z_t^{(1)}, \dots, Z_t^{(k)}$ denote k independent copies of Z_t, we set

$$\alpha_k(\epsilon, x, t) = \int_{D_k} p_\epsilon(Z_{s_2}^{(2)} - Z_{s_1}^{(1)} - x_1) \cdots p_\epsilon(Z_{s_k}^{(k)} - Z_{s_{k-1}}^{(k-1)} - x_{k-1})\, ds_1 \dots ds_k$$

where $D_k = \{(s_1, \dots, s_k) \in \mathbf{R}^k \mid 0 \le s_1 \le \dots \le s_k \le t\}$. It is known that if $\beta > d - d/k$ then $\alpha_k(\epsilon, x, t)$ converges, as $\epsilon \to 0$, to a random variable, called the k-fold intersection local time, and denoted $\alpha_k(x,t)$. Convergence is locally uniform both a.s. and in all
L^p spaces. The k-fold intersection local time $\alpha_k(x,t)$ is jointly continuous in (x,t).

Theorem 1 *If $\beta > d - d/k$ then*

$$I_k(b_n x, nt)\; b_n^{(k-1)d}/n^k \Longrightarrow \alpha_k(x,t)$$

as $n \to \infty$, where we have weak convergence of processes in $C(\mathbf{R}^{d(k-1)} \times \mathbf{R}_+)$.

Such a theorem is referred to as a uniform invariance principle for intersection local times. It "uniformizes" our work in [5] where the convergence in Theorem 1 is proven for fixed x and t. Our present theorem was inspired by the work of B ass and Khoshnevisan [1, 2] who establish Theorem 1 for random walks with finite variance, in which case $\beta = 2$ and $b_n = \sqrt{n}$. Their work, in turn, was motivated by the uniform invariance principles of Perki ns [4] and Borodin [3] for ordinary local times. We should also mention that in [2], Bass and Khoshnevisan obtain a strong invariance principle for intersection local times of certain random walks. More preci sely, they show that if a random walk converges almost surely to Brownian motion at a certain rate, then this will also hold for their intersection local times.

2 Proof of Theorem 1

The proof of a uniform invariance principle consists of two parts: a proof that the finite dimensional distributions converge, and a proof of tightness. The proof that the finite dimensional distributions converge proceeds almost exactly as in [5] where convergence of the marginal distributions is established. We shall only recall the basic ideas and mention the necessary modifications.

By a change of variables, for $x \in (\mathbf{Z}^d)^{k-1}/b_n$, $t \in \mathbf{Z}_+/n$ we have

$$(2.1)\qquad L_k(n,x,t) \stackrel{\text{def}}{=} I_k(b_n x, nt)\; b_n^{(k-1)d}/n^k$$
$$= \sum_{i_1,\ldots,i_k=1}^{nt} \frac{1}{n^k} \prod_{j=2}^{k} \frac{1}{(2\pi)^d} \int_{|p_j|_0 \le \pi b_n} \exp(ip_j \frac{S_{i_j}^{(j)} - S_{i_{j-1}}^{(j-1)} - b_n x_{j-1}}{b_n})\, dp_j$$

where for a vector $y = (y_1, \ldots, y_d) \in \mathbf{R}^d$ we use $|y|_0 \stackrel{\text{def}}{=} \max_i |y_i|$.

We then define a 'link'

$$(2.2)\quad L_k(\epsilon, n, x, t) \stackrel{\text{def}}{=} \sum_{i_1,\ldots,i_k=1}^{nt} \frac{1}{n^k} \prod_{j=2}^{k} \frac{1}{(2\pi)^d} \int_{|p_j|_0 \le \pi b_n}$$
$$\exp(ip_j \frac{S_{i_j}^{(j)} - S_{i_{j-1}}^{(j-1)} - b_n x_{j-1}}{b_n}) \Phi_\epsilon(p_j)\, dp_j$$

where $\Phi_\epsilon(p)$ is the characteristic function of Z_ϵ, i.e. the Fourier transform of $p_\epsilon(x)$.

In lemma 1 of [5] we essentially prove that

$$||L_k(n,x,t) - L_k(\epsilon,n,x,t)||_2 \le c\epsilon^\gamma$$

for some $c < \infty, \gamma > 0$ uniformly in $(x,t) \in \mathbf{R}^{d(k-1)} \times [0,T]$ for any $T < \infty$. Hence for any fixed $\lambda_i, z_i, t_i \quad i = 1,\dots,m$, if we set

$$L(\epsilon,n) \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda_i L_k(\epsilon,n,z_i,t_i)$$

and

$$L(n) \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda_i L_k(n,z_i,t_i)$$

we have that

(2.3) $$|E(e^{iL(n)}) - E(e^{iL(\epsilon,n)})| \le c\epsilon^\gamma.$$

On the other hand, it follows from the locally uniform convergence of $\alpha_k(\epsilon,x,t)$ to $\alpha_k(x,t)$ that if we set

$$\alpha(\epsilon) \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda_i \alpha_k(\epsilon,z_i,t_i)$$

and

$$\alpha \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda_i \alpha_k(z_i,t_i)$$

we can choose $\epsilon_0 > 0$ such that for any given $\delta > 0$ we have both $c\epsilon_0^\gamma \le \delta$ and

(2.4) $$|E(e^{i\alpha}) - E(e^{i\alpha(\epsilon_0)})| \le \delta.$$

From (2.2) we see that

(2.5) $$L_k(\epsilon,n,x,t) = \sum_{i_1,\dots,i_k=1}^{nt} \frac{1}{n^k} \prod_{j=2}^k p_{\epsilon_0}\left(\frac{S^{(j)}_{i_j} - S^{(j-1)}_{i_{j-1}} - b_n x_{j-1}}{b_n}\right) dp_j + \bigcirc(e^{-\epsilon_0 n^{1/2}}).$$

We see from this, using [6], that we can find n_0 such that for all $n \ge n_0$ we have

$$|E(e^{iL(\epsilon_0,n)}) - E(e^{i\alpha(\epsilon_0)})| \le \delta.$$

Together with (2.4) and (2.5) this shows the convergence of finite dimensional distributions.

To prove tightness it suffices to show that we can find some $\gamma > 0$ such that for any even m

(2.6) $$E\{(L_k(n,x,t) - L_k(n,x',t'))^m\} \le c|(x,t) - (x',t')|^{m\gamma}$$

uniformly over $n \in \mathbf{Z}_+$, $x, x' \in \mathbf{R}^d$ and $t, t' \in [0,T]$.

We begin by showing how to get a bound on

$$E\{(L_k(n,x,t))^m\}$$

which is uniform in $n \in \mathbf{Z}_+$, $x \in \mathbf{R}^d$ and $t \in [0,T]$. Once we see how to accomplish this, it will be easy to establish (2.6).

Recalling (2.1) we have

$$\begin{aligned}(2.7)\qquad & E\{(L_k(n,x,t))^m)\} \\ &= E\{\prod_{h=1}^{m} \sum_{i_{1,h},\dots,i_{k,h}=1}^{nt} \frac{1}{n^k} \prod_{j=2}^{k} \frac{1}{(2\pi)^d} \\ &\qquad \int_{|p_{j,h}|_0 \le \pi b_n} \exp(ip_{j,h} \frac{S^{(j)}_{i_{j,h}} - S^{(j-1)}_{i_{j-1,h}} - b_n x_{j-1}}{b_n})\, dp_{j,h}\} \\ &= \frac{1}{(2\pi)^{d(k-1)m}} \int_{|p_{j,h}| \le \pi b_n} F(p,x) \prod_{j=1}^{k} \frac{1}{n^{km}} \\ &\qquad \sum_{i_{j,1},\dots,i_{j,m}=1}^{nt} E\{\exp(i\sum_{h=1}^{m} \frac{(p_{j,h} - p_{j+1,h})}{b_n} S^{(j)}_{i_{j,h}})\}\, dp\end{aligned}$$

where

$$F(p,x) = \exp(i\sum_{h=1}^{m}\sum_{j=2}^{k} p_{j,h} x_{j-1})$$

and we have set $p_{1,h} = p_{k+1,h} = 0$.

Let $\pi^1, \dots, \pi^k$ be k not necessarily distinct permutations of $\{1,\dots,m\}$ and let

$$\Delta(\pi^1,\dots,\pi^k) = \{i_{j,h} \,|\, i_{j,\pi^j_l} \le i_{j,\pi^j_{l+1}}\, ; j = 1,\dots,k;\ l = 1,\dots,m\}$$

and note that on $\Delta(\pi^1,\dots,\pi^k)$ we have

$$\begin{aligned}(2.8)\qquad & E\{\exp(i\sum_{h=1}^{m} \frac{(p_{j,h} - p_{j+1,h})}{b_n} S^{(j)}_{i_{j,h}})\} \\ &= E\{\exp(i\sum_{h=1}^{m} u_{j,h}(S^{(j)}_{i_{j,\pi^j_h}} - S^{(j)}_{i_{j,\pi^j_{h-1}}})/b_n)\} \\ &= \prod_{h=1}^{m} \varphi^{\pi^j_h - \pi^j_{h-1}}(u_{j,h}/b_n)\end{aligned}$$

where

$$\varphi(u) = E(\exp(iuS_1))$$

and

$$u_{j,h} = \sum_{l=1}^{h} (p_{j,\pi_l^j} - p_{j+1,\pi_l^j}).$$

Note that

(2.9) $span\{u_{j,h} | h = 1, \ldots, m\} = span\{p_{j,h} - p_{j+1,h} | h = 1, \ldots, m\}$

and that if $\tilde{\pi}^j$ denotes the inverse of the permutation π^j we have

(2.10) $$p_{j,h} - p_{j+1,h} = u_{j,\tilde{\pi}_h^j} - u_{j,\tilde{\pi}_h^j - 1}$$

If we define

$$G_r(n,u) = \frac{1}{n} \sum_{j=0}^{nT} |\varphi(u/b_n)|^{jr}$$

then it is clear from the above that (2.7) can be bounded by a sum over regions $\Delta(\pi^1, \ldots, \pi^k)$ of integrals of the form

(2.11) $$\int_{|p_{j,h}| \le \pi b_n; j=2,\ldots,k} \prod_{j=1}^{k} \prod_{h=1}^{m} G_1(n, u_{j,h})\, dp$$
$$\le \prod_{i=1}^{k} \{ \int_{|p_{j,h}| \le \pi b_n; j=2,\ldots,k} \prod_{j=1; j \ne i}^{k} \prod_{h=1}^{m} G_1^{k/(k-1)}(n, u_{j,h})\, dp \}^{(k-1)/k}.$$

By [5] we know that for any $\epsilon, r > 0$ and $T < \infty$ we have

$$G_r(n,u) \le \frac{c}{1 + ((u))^{\beta - \epsilon}}$$

where $((u))$ denotes the norm of the smallest vector which equals u mod 2π. Since, by (2.9), for any i

$$span\{u_{j,h} \,|\, h = 1, \ldots, m;\ j = 1, \ldots, k;\ j \ne i\}$$
$$= span\{p_{j,h} | h = 1, \ldots, m;\ j = 2, \ldots, k\}$$

we have that (2.11) and hence (2.7) is bounded uniformly if $k/(k-1)\beta > d$.

We now show how to modify the above estimates to get (2.6). First , we have

(2.12) $$E\{(L_k(n,x,t) - L_k(n,x',t'))^m\}$$
$$\le E\{(L_k(n,x,t) - L_k(n,x',t))^m\} + E\{(L_k(n,x',t) - L_k(n,x',t'))^m\}$$

and we can handle the x and t variation seperately.

The first term in (2.12), the x variation, can be written as in (2.7) except that the factor $F(p,x)$ will be replaced by

$$H(p,x,x') = \prod_{h=1}^{m} \{\exp(i \sum_{j=2}^{k} p_{j,h} x_{j-1}) - \exp(i \sum_{j=2}^{k} p_{j,h} x'_{j-1})\}.$$

Since for any $0 \leq \delta \leq 1$ we have

$$|H(p,x,x')| \leq c \prod_{h=1}^{m} (\sum_{j=2}^{k} |p_{j,h}|)^{\delta} |x-x'|^{\delta}$$

and using (2.10) for any $i = 1, \ldots, k$ we have

$$\begin{aligned}
&\prod_{h=1}^{m} \sum_{j=2}^{k} |p_{j,h}| \\
&\leq c \prod_{h=1}^{m} \sum_{j=1; j\neq i}^{k} |p_{j,h} - p_{j-1,h}| \\
&\leq c \prod_{h=1}^{m} \prod_{j=1; j\neq i}^{k} 1 + |p_{j,h} - p_{j-1,h}| \\
&\leq c \prod_{h=1}^{m} \prod_{j=1; j\neq i}^{k} 1 + |u_{j,\tilde{\pi}_h^j} - u_{j-1,\tilde{\pi}_h^j - 1}| \\
&\leq c \prod_{h=1}^{m} \prod_{j=1; j\neq i}^{k} 1 + |u_{j,h}|^2
\end{aligned}$$

it is clear that by choosing $\delta > 0$ sufficiently small we can achieve the desired bound.

The second term in (2.12), the t variation, gives rise to a term similar to (2.7) except that for each l; $l = 1, \ldots, m$ the indices $\{i_{1,l}, \ldots, i_{k,l}\}$ run through the set $A = [0, nt]^k - [0, nt']^k$. Using (2.8), we can bound the t variation by a sum over regions $\Delta = \Delta(\pi^1, \ldots, \pi k)$ of integrals of the form (2.11), except that the integrand is replaced by

$$\begin{aligned}
&\frac{1}{n^{km}} \sum_{A^m \cap \Delta} \prod_{j=1}^{k} \prod_{h=1}^{m} |\varphi(u_{j,h}/b_n)|^{\pi_h^j - \pi_{h-1}^j} \\
&\leq (\frac{1}{n^{km}} |A|^m)^{1/q} \{\prod_{j=1}^{k} \prod_{h=1}^{m} G_{q'}(n, u_{j,h})\}^{1/q'} \\
&\leq c|t-t'|^{m/q} \prod_{j=1}^{k} \prod_{h=1}^{m} \frac{1}{1 + ((u_{j,h}))^{(\beta-\epsilon)/q'}}
\end{aligned}$$

for any q, q' satisfying $1/q + 1/q' = 1$. It is now clear that by taking q' suficiently close to 1 we can obtain the desired bound on the t variation in (2.12). This completes the proof of our theorem. □

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