Φ-Variation of the Local Times of Symmetric Lévy Processes and Stationary Gaussian Processes

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1 Introduction

Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a symmetric real-valued Lévy process with characteristic function

$$Ee^{i\lambda X(t)} = e^{-t\psi(\lambda)} \tag{1.1}$$

and Lévy exponent

$$\psi(\lambda) = 2\int_0^\infty (1 - \cos u\lambda)\, d\nu(u) \tag{1.2}$$

for ν a Lévy measure, i.e. $\int_0^\infty (1 \wedge u^2)\, d\nu(u) < \infty$. We also include the case $\psi(\lambda) = \lambda^2/2$ which gives us standard Brownian motion.

In [4] we used the Dynkin Isomorphism Theorem to study the almost sure variation in the spatial variable of the local time of the symmetric stable processes of index $1 < \beta \leq 2$, i.e. $\psi(\lambda) = c|\lambda|^\beta$. In this note we will show how the proofs in [4] can be modified so as to generalize the almost sure variation results obtained there to a large class of symmetric Lévy processes with Lévy exponent *psi* regularly varying at infinity of order $1 < \beta \leq 2$.

Such Lévy processes X have an almost surely jointly continuous local time which we denote by $L = \{L_t^x, (t, x) \in \mathbf{R}^+ \times \mathbf{R}\}$, and normalize by requiring that

$$E^0\left(\int_0^\infty e^{-t}\, dL_t^x\right) = u^1(x)$$

where

$$u^1(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos x\lambda}{1 + \psi(\lambda)}\, d\lambda$$

is the 1-potential density for X. We set

$$\sigma^2(x) = \frac{1}{\pi}\int_0^\infty \frac{1 - \cos x\lambda}{\psi(\lambda)}\, d\lambda. \tag{1.3}$$

It follows from Pitman [6] that if $\psi(\lambda)$ is regularly varying at infinity of order $1 < \beta \leq 2$, then $\sigma^2(x)$ is regularly varying at zero of order $\beta - 1$, and we have

$$\sigma^2(x) \sim c_\beta \frac{1}{x\psi(\frac{1}{x})} \qquad as\ x \to 0 \tag{1.4}$$

with c_β depending only on β. Throughout this paper we use the notation $f \sim g$ to mean that $\lim f/g = 1$.

Let $\pi = \{0 = x_0 < x_1 \cdots < x_{k_\pi} = a\}$ denote a partition of $[0, a]$, and let $m(\pi) = \sup_{1 \le i \le k_\pi}(x_i - x_{i-1})$ denote the length of the largest interval in π. $m(\pi)$ is called the mesh of π. Let $Q_a(\delta) = \{\text{partitions } \pi \text{ of } [0, a] \mid m(\pi) \le \delta\}$.

To clarify the notation in all that follows we note that in the expression $\sum_{x_i \in \pi} f(x_{i-1}, x_i)$, for some function f, we mean that the sum is taken over all the terms in which both x_{i-1} and x_i are contained in π.

Theorem 1 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a real valued symmetric Lévy process with $\sigma^2(x)$ concave on $[0, \delta]$ and regularly varying at zero of order $\beta - 1$ where $1 < \beta \le 2$, and let $\{L_t^x, (t, x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X.*

i) Let $\Phi(x)$ denote any function which is an inverse for $\sigma(x)$ near $x = 0$. (Thus, $\Phi(x)$ is regularly varying at zero of order $2/(\beta - 1)$). If $\{\pi(n)\}$ is any sequence of partitions of $[0, a]$ such that $m(\pi(n)) = o(\Phi(\sqrt{1/\log n}))$ then

$$\lim_{n \to \infty} \sum_{x_i \in \pi(n)} \Phi(|L_t^{x_i} - L_t^{x_{i-1}}|) = c(\beta) \int_0^a |L_t^x|^{1/(\beta-1)}\, dx \tag{1.5}$$

for almost all $t \in \mathbf{R}^+$ almost surely, where

$$c(\beta) = \frac{2^{2/(\beta-1)}}{\sqrt{\pi}} \Gamma\left(\frac{1}{\beta - 1} + \frac{1}{2}\right) \left(\frac{1}{\Gamma(\beta) \sin(\frac{\pi}{2}(\beta - 1))}\right)^{1/(\beta-1)} \tag{1.6}$$

ii) Let $\Upsilon(x)$ denote any function which is an inverse for $\sigma(x)\sqrt{2 \log\log 1/x}$ near zero. (Thus, $\Upsilon(x)$ is regularly varying at zero of order $2/(\beta - 1)$). Then

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \Upsilon(|L_t^{x_i} - L_t^{x_{i-1}}|) = c'(\beta) \int_0^a |L_t^x|^{1/(\beta-1)}\, dx \tag{1.7}$$

almost surely for each $t \in \mathbf{R}^+$, where

$$c'(\beta) = \left(\frac{1}{\Gamma(\beta) \sin(\frac{\pi}{2}(\beta - 1))}\right)^{1/(\beta-1)}. \tag{1.8}$$

To prove Theorem 1 we use a corollary of the Dynkin Isomorphism Theorem, Lemma 4.3 of [5], which enables us to obtain almost sure results for variations of the local times of symmetric Lévy processes from analogous results about the variation of their associated Gaussian processes. The mean zero Gaussian process $\{G(x), x \in \mathbf{R}\}$ with covariance $g(x, y)$ is said to be associated with the Markov process X if $g(x,y) = u^1(x,y)$, the 1–potential density of X.

In particular, part ii) of Theorem 1 follows immediately by applying the methods of [4] to the results on Gaussian processes of Kawada and Kono [2]. Part i) of Theorem 1 follows from the next theorem on Gaussian processes in the same way that Theorem 1.1 followed from Theorem 1.2 in [4].

Theorem 2 *Let $\{G(x), x \in \mathbf{R}\}$ be a mean zero Gaussian process with stationary increments, such that*

$$\sigma^2(x) = E(G(x) - G(0))^2. \tag{1.9}$$

is concave and regularly varying at $x = 0$ of order $\beta - 1$ where $1 < \beta \le 2$. Let $\Phi(x)$ denote any function which is an inverse for $\sigma(x)$ near $x = 0$. (Thus, $\Phi(x)$ is regularly varying at zero of order $2/(\beta - 1)$). If $\{\pi(n)\}$ is any sequence of partitions of $[0,a]$ such that $m(\pi(n)) = o(\Phi(\sqrt{1/\log n}))$ then

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} \Phi(|G(x_i) - G(x_{i-1})|) = E|\eta|^{2/(\beta-1)} \qquad a.s. \tag{1.10}$$

where η is a normal random variable with mean 0 and variance 1. Also,

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} \Phi(|G^2(x_i) - G^2(x_{i-1})|) = E|\eta|^{2/(\beta-1)} 2^{2/(\beta-1)} \int_0^a |G(x)|^{2/(\beta-1)}\, dx \tag{1.11}$$

almost surely.

As in [4] the almost sure result of Theorem 1 part i) will lead to the following L^r convergence of the variation. Note that in Theorem 3 we do not require any conditions on the rate of convergence of $m(\pi(n))$ to zero. In addition, our results hold for all t.

Theorem 3 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a real valued symmetric Lévy process with $\sigma^2(x)$ concave on $[0,\delta]$ and regularly varying at zero of order $\beta - 1$ where $1 < \beta \le 2$, and let $\{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X.*

Let $\Phi(x)$ denote any function which is an inverse for $\sigma(x)$ near $x = 0$. (Thus, $\Phi(x)$ is regularly varying at zero of order $2/(\beta - 1)$). If $\{\pi(n)\}$ is any sequence of partitions of $[0,a]$ with $\lim_{n\to\infty} m(\pi(n)) = 0$ then

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} \Phi(|L_t^{x_i} - L_t^{x_{i-1}}|) = c(\beta) \int_0^a |L_t^x|^{1/(\beta-1)}\, dx \tag{1.12}$$

in L^r uniformly in t on any bounded interval of R^+, for all $r > 0$, where $c(\beta)$ is given in (1.6).

In Section 2 we give the proof of Theorem 2, and in Section 3 we give the proof of Theorem 3. In Section 4 we provide a large class of examples of Lévy processes satisfying the assumptions of our Theorem 1. In particular, we show in Corollary 1 that if $h(x)$ is any function which is regularly varying and increasing as $x \to \infty$, and if $1 < \beta \le 2$, then we can find a Lévy process with $\sigma^2(x)$ concave such that $\sigma^2(x) \sim |x|^{\beta-1} h(\ln 1/x)$ as $x \to 0$.

2 Almost Sure Variation of Gaussian Processes

In this section we prove Theorem 2 which gives almost sure variation results for a wide class of Gaussian processes. The basic ideas are already contained in the proof of Theorem 1.2 of [4]. The main point of this section is to show how t hat proof can be adapted to the Orlicz space setting.

Proof of Theorem 2: Set $p = 2/(\beta - 1)$. It will suffice to prove the theorem for $\Phi(x)$ chosen to be a polynomially bounded Young's function regularly varying at zero of order $p \ge 2$. We first show that

$$\lim_{n\to\infty} E\left(\sum_{x_i\in\pi(n)} \Phi(G_{x_i} - G_{x_{i-1}})\right) = E(|Z|^p)a. \tag{2.1}$$

where Z is a standard Gaussian random variable. Let $Z_i = \frac{G_{x_i}-G_{x_{i-1}}}{\sigma(x_i-x_{i-1})} \overset{d}{=} N(0,1)$ and write

$$E\left(\sum_{x_i\in\pi(n)} \Phi(G_{x_i} - G_{x_{i-1}})\right) = E\left(\int_0^a F_n(\omega, x)\, dx\right) \tag{2.2}$$

where

$$\begin{aligned} F_n(\omega, x) &= \textstyle\sum_i \frac{\Phi(G_{x_i}-G_{x_{i-1}})}{\Delta x_i} 1_{(x_{i-1},x_i]}(x) \\ &= \textstyle\sum_i \frac{\Phi(Z_i\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))} 1_{(x_{i-1},x_i]}(x) \end{aligned} \tag{2.3}$$

and $\Delta x_i = x_i - x_{i-1}$. We begin by showing that

$$||F_n||_2^2 \le c \tag{2.4}$$

uniformly in $L^2(\Omega \times [0, a], P \times dx)$. This inequality, (2.4), will be used later on, in (2.24), and the method used to prove it will lead to (2.1).

To prove (2.4) we first note that

$$F_n^2 = \sum_i \left(\frac{\Phi(Z_i\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))}\right)^2 1_{(x_{i-1},x_i]}(x). \tag{2.5}$$

If $|Z_i| \le 1$, then by monotonicity

$$\frac{\Phi(Z_i\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))} \le 1. \tag{2.6}$$

If $|Z_i| \geq 1$ is such that for suitable $\delta > 0$ we have $|Z_i|\sigma(\Delta x_i) \leq \delta$, then

$$\frac{\Phi(Z_i\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))} \leq c|Z_i|^{p+\epsilon} \tag{2.7}$$

by the regular variation of Φ. See e.g. [1], Theorem 1.5.6 (iii). Finally, if $|Z_i|\sigma(\Delta x_i) \geq \delta$, we can use the assumption that Φ is polynomially bounded to see that for some c and $k > p+1$

$$\begin{aligned} \frac{\Phi(Z_i\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))} &\leq c|Z_i|^k \frac{\sigma^k(\Delta x_i)}{\Phi(\sigma(\Delta x_i))} \\ &\leq c|Z_i|^k. \end{aligned} \tag{2.8}$$

Using (2.6)-(2.8) in (2.5) we see that

$$F_n^2 \leq c\sum_i \left(1+|Z_i|^k\right)^2 1_{(x_{i-1},x_i]}(x). \tag{2.9}$$

and (2.4) follows, since $Z_i \stackrel{d}{=} N(0,1)$.

We get (2.1) from the same reasoning: Note that we can write

$$E\left(\sum_{x_i\in\pi(n)} \Phi(G_{x_i} - G_{x_{i-1}})\right) = E\left(\int_0^a \tilde{F}_n(\omega,x)\,dx\right) \tag{2.10}$$

where now

$$\tilde{F}_n(\omega,x) = \sum_i \frac{\Phi(Z\sigma(\Delta x_i))}{\Phi(\sigma(\Delta x_i))} 1_{(x_{i-1},x_i]}(x). \tag{2.11}$$

i.e. we replace Z_i by Z. As before, in the proof of (2.4), we find that $\tilde{F}_n$ is uniformly bounded in L^2 , hence uniformly integrable. But by the regular variation of Φ we have

$$\lim_{n\to\infty} \tilde{F}_n(\omega,x) = |Z(\omega)|^p \qquad a.s.$$

Therefore we see that (2.1) follows from (2.10).

With these preliminaries out of the way, the proof of Theorem 2 will proceed analogously to the proof of Theorem 1.2 in [4]. Instead of l_p we will use the Orlicz space l_Φ. Let f, g be sequences of real numbers, $f = \{f_i\}, g = \{g_i\}, i = 1,2,\ldots,\infty$. Recall the definition of the Luxembourg norm:

$$||f||_{(\Phi)} = \inf\{c : \sum_i \Phi\left(\frac{f_i}{c}\right) \leq 1\} \tag{2.12}$$

and note that

$$||f||_{(\Phi)} = \sup_{\{g:||g||_{\Phi^*}\}\leq 1} \sum_i f_i g_i \tag{2.13}$$

see [3], (14.10). In the last equation, Φ^* is the dual Young's function, and $||g||_{\Phi^*}$ is the 'standard' norm which satisfies

$$(2.14)\qquad ||g||_{\Phi^*} \le 1 \Rightarrow \sum_i \Phi^*(g_i) \le 1,$$

see e.g. [3], Lemma 9.2.

We use the notation $\Delta G(\pi)$ to denote the sequence $\{G_{x_i} - G_{x_{i-1}}, i = 1, 2, \ldots, k_\pi\}$, and for any sequence g we set $\langle g, \Delta G(\pi)\rangle = \sum_{i=1}^{k_\pi} g_i(G_{x_i} - G_{x_{i-1}})$. Let

$$M_n = \text{median}\left(||\Delta G(\pi(n))||_{(\Phi)}\right).$$

Then, using Borell's inequality as we did in lemma 2.1 of [4], we have

$$(2.15)\qquad P\left(|\ ||\Delta G(\pi(n))||_{(\Phi)} - M_n \ | > t\right) \le 2\exp(-t^2/2\hat{\sigma}_n^2)$$

and the estimate

$$(2.16)\qquad |\ E\left(||\Delta G(\pi(n))||_{(\Phi)}\right) - M_n \ | \le \hat{\sigma}_n\sqrt{2\pi}$$

where

$$\hat{\sigma}_n^2 \stackrel{\text{def}}{=} \sup_{\{g: ||g||_{\Phi^*} \le 1\}} E\left\{\langle g, \Delta G(\pi(n))\rangle^2\right\}.$$

Using (2.14) we see that

$$\begin{aligned} \hat{\sigma}_n^2 &\le \sup_{\{g: \sum_i \Phi^*(g_i) \le 1\}} E\left\{\langle g, \Delta G(\pi(n))\rangle^2\right\} \\ &\le \sup_{\{g: ||g||_2 \le \tilde{c}\}} E\left\{\langle g, \Delta G(\pi(n))\rangle^2\right\} \\ (2.17)\qquad &\le c \sup_i \sigma^2(x_i - x_{i-1}) \end{aligned}$$

for some $c < \infty$, as in the proof of lemma 2.2 of [4]. Here we have used the fact that for some $\tilde{c} < \infty$

$$\Phi^*(x) \le 1 \Rightarrow x^2 \le \tilde{c}^2\Phi^*(x).$$

This fact is most easily derived from the following chain of implications for x near zero: By the concavity of σ^2, we have $\sigma^2(x) \ge cx$ and therefore $\sigma(x) \ge cx^{1/2}$, so that $\Phi(x) \le cx^2$, which implies that $\Phi^\star(x) \ge cx^2$ b y (2.9) of [3].

Note that

$$||\Delta G(\pi(n))||_{(\Phi)} \le 1 + \sum_i \Phi(G_{x_i} - G_{x_{i-1}}),$$

see e.g. [3], (9.12) and (9.20). Using this, (2.1), (2.16) and (2.17) show that $\{M_n\}, n = 1, \ldots, \infty$ is bounded. Hence some subsequence M_{n_j} has a limit point M_*. We now show that

$$(2.18)\qquad M_*^p = E(|Z|^p)a$$

and that along the subsequence $\{n_j\}$

$$\lim_{j\to\infty} \sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}}) = M_*^p \qquad a.s. \tag{2.19}$$

This will complete the proof of our theorem just as in the proof of Theorem 1.2 of [4], since the uniqueness of the limit point M_* demonstrated in (2.18) will show that we actually have convergence of the full sequence in (2.19).

To prove (2.18) and (2.19) we begin by arguing exactly as in [4] to find that

$$R_{n_j}(\omega) \stackrel{\text{def}}{=} \|\Delta G(\pi(n_j))\|_{(\Phi)} \to M_* \qquad a.s. \quad \text{as } j \to \infty \tag{2.20}$$

By definition we have

$$\sum_{x_i\in\pi(n_j)} \Phi\left(\frac{G_{x_i}(\omega) - G_{x_{i-1}}(\omega)}{R_{n_j}(\omega)}\right) = 1 \tag{2.21}$$

see e.g. [3], chapter II, section 7. Now note that by the regular variation of Φ, for fixed ω in a set of measure 1 and for any $\epsilon > 0$, we can find N so large that for all $n_j \geq N$ we have

$$\begin{aligned} \frac{(1-\epsilon)}{M_*^p} &\sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}}) \\ &\leq \sum_{x_i\in\pi(n_j)} \Phi\left(\frac{G_{x_i} - G_{x_{i-1}}}{R_{n_j}}\right) \\ &\leq \frac{(1+\epsilon)}{M_*^p} \sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}}). \end{aligned} \tag{2.22}$$

Therefore by (2.21) and (2.22) we get (2.19). To obtain (2.18) we observe that in the notation of (2.3)

$$\sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}}) = \int_0^a F_{n_j}(\omega, x)\, dx. \tag{2.23}$$

Hence, using (2.4) and Cauchy-Schwarz inequality we see that

$$\begin{aligned} E\left[\left(\sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}})\right)^2\right] &= E\left[\left(\int_0^a F_{n_j}(x)\, dx\right)^2\right] \\ &\leq aE\left(\int_0^a F_{n_j}^2(x)\, dx\right) \\ &\leq ca \end{aligned} \tag{2.24}$$

uniformly in n_j. Therefore, (2.23) is uniformly integrable, and (2.19) and (2.1) now imply that

$$\begin{aligned} M_*^p &= \lim_{j\to\infty} E\left(\sum_{x_i\in\pi(n_j)} \Phi(G_{x_i} - G_{x_{i-1}})\right) \\ &= E(|Z|^p)a \end{aligned} \tag{2.25}$$

by (2.1). This proves (2.18), and completes the proof of (1.10) of Theorem 2. (1.11) follows from (1.10) as in the proof of Theorem 1.2 of [4].

3 L^r Convergence

Theorem 3 will follow from Theorem 1 as in the proof of Theorem 1.1 of [4] once we establish the next lemma, which is the analogue of Lemma 3.4 of [4].

For a fixed partition π we introduce the notation

$$||L_t||_{\pi,\Phi} = \sum_{x_i\in\pi} \Phi(\triangle L_t^{x_i})$$

where

$$\triangle L_t^{x_i} = L_t^{x_i} - L_t^{x_{i-1}},$$

and we let

$$g(t) = \int_0^t p_s(0)\,ds$$

denote the partial Greens' function.

Lemma 1 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a real valued symmetric Lévy process with $\sigma^2(x)$ concave on $[0,\delta]$ and regularly varying at zero of order $\beta-1$ where $1 < \beta \le 2$, and let $\{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X.*

Let $\Phi(x)$ denote any function which is an inverse for $\sigma(x)$ near $x = 0$. (Thus, $\Phi(x)$ is regularly varying at zero of order $2/(\beta-1)$). Then we can find an $\epsilon > 0$ such that for all partitions π of $[0,a]$ with $m(\pi) \le \epsilon$, and all $s,t,T \in R^+$, with $s \le t \le T < \infty$, and integers $m \ge 1$

$$||\, ||L_t||_{\pi,\Phi} - ||L_s||_{\pi,\Phi}||_m \le C(\beta,m,T)g^{1/2}(t-s)a \tag{3.1}$$

and in particular

$$||\, ||L_t||_{\pi,\Phi}||_m \le C(\beta,m,T)g^{1/2}(t)a. \tag{3.2}$$

Proof: As before, we can take Φ to be convex, so that the monotonicity of Φ' together with the Mean Value Theorem show that

$$|\Phi(u) - \Phi(v)| \le (\Phi'(u) + \Phi'(v))\,|u - v|.$$

Hence

$$(3.3)\qquad \begin{aligned} &\| \, \|L_t\|_{\pi,\Phi} - \|L_s\|_{\pi,\Phi} \|_m \\ &\le \sum_{x_i\in\pi} \|\Phi(\Delta L_t^{x_i}) - \Phi(\Delta L_s^{x_i})\|_m \\ &\le \sum_{x_i\in\pi} (\|\Phi'(\Delta L_t^{x_i})\|_{2m} + \|\Phi'(\Delta L_s^{x_i})\|_{2m})\|\Delta L_t^{x_i} - \Delta L_s^{x_i}\|_{2m} \end{aligned}$$

With the notation

$$Z_i = \frac{\Delta L_t^{x_i}}{\sigma(\Delta x_i)}$$

and using (2.6)-(2.8), but with Φ replaced by Φ' we see that for some fixed c and any $k > p = 2/(\beta - 1)$

$$\begin{aligned} \|\Phi'(\Delta L_t^{x_i})\|_{2m} &= \|\tfrac{\Phi'(Z_i\sigma(\Delta x_i))}{\Phi'(\sigma(\Delta x_i))}\|_{2m}\Phi'(\sigma(\Delta x_i)) \\ (3.4)\qquad &\le c\|1 + Z_i^k\|_{2m}\Phi'(\sigma(\Delta x_i)). \end{aligned}$$

By (3.17) of [4] we see that for any $u \in R$ and any integer $j \ge 1$

$$(3.5)\qquad E^u(\Delta L_t^{x_i})^{2j} \le (2j)!2^j(g^{1/2}(t)\sigma(\Delta x_i))^{2j}$$

so that

$$(3.6)\qquad \|1 + Z_i^k\|_{2m} \le 1 + cg^{k/2}(t).$$

Similarly, using (3.5) together with the Markov property as in the proof of lemma 3.4 of [4] we see that

$$(3.7)\qquad \|\Delta L_t^{x_i} - \Delta L_s^{x_i}\|_{2m} \le cg^{1/2}(t-s)\sigma(\Delta x_i).$$

Using once again the convexity of Φ, the Monotone Density Theorem, (theorem 1.7.2b of [1]) shows that

$$(3.8)\qquad x\Phi'(x) \sim p\Phi(x) \qquad \text{as} \quad x \to 0.$$

Putting these estimates together and remembering that Φ is an inverse for σ near zero finishes the proof of Lemma 1.

4 Examples

We will now give examples of Lévy processses which satisfy the assumptions of Theorem 1.

Lemma 2 *Let μ be a finite positive measure on* $(1, 2]$ *and let*

$$\psi(\lambda) = \int_1^2 \lambda^s \, d\mu(s).$$

Then ψ is a Lévy exponent, and

$$\sigma^2(x) = \int_0^\infty \frac{(1-\cos\lambda x)}{\psi(\lambda)}\,d\lambda$$

is concave on $[0,\infty)$.

Proof: Note that

$$(4.1)\qquad 2\sigma^2(x)-\sigma^2(x-h)-\sigma^2(x+h) = \int_0^\infty (1-\cos v) \left[\frac{2}{x\psi(v/x)} - \frac{1}{(x+h)\psi(v/x+h)} - \frac{1}{(x-h)\psi(v/x-h)}\right]dv$$

for all $|h|\le|x|$. Therefore, to show that σ^2 is concave on $[0,\infty)$ it suffices to show that the term in the bracket is positive, i.e. that $1/x\psi(v/x)$ is concave in x for all $x>0, v>0$. This is clearly equivalent to showing that $g(x)=1/x\psi(1/x)$ is concave for $x>0$.

By definition

$$g(x)=\frac{1}{\int_1^2 x^{1-s}\,d\mu(s)}$$

so that

$$g'(x)=\frac{\int_1^2 (s-1)x^{-s}\,d\mu(s)}{\left(\int_1^2 x^{1-s}\,d\mu(s)\right)^2}$$

and

$$g''(x)=\frac{2\left(\int_1^2 (s-1)x^{-s}\,d\mu(s)\right)^2}{\left(\int_1^2 x^{1-s}\,d\mu(s)\right)^3} - \frac{\int_1^2 s(s-1)x^{-s-1}\,d\mu(s)}{\left(\int_1^2 x^{1-s}\,d\mu(s)\right)^2}.$$

Thus, $g''\le 0$ if

$$2\left(\int_1^2 (s-1)x^{-s}\,d\mu(s)\right)^2 \le \int_1^2 s(s-1)x^{-s-1}\,d\mu(s)\int_1^2 x^{1-s}\,d\mu(s)$$

or equivalently

$$2\left(\int_1^2 (s-1)x^{-s}\,d\mu(s)\right)^2 \le \int_1^2 s(s-1)x^{-s}\,d\mu(s)\int_1^2 x^{-s}\,d\mu(s).$$

This last inequality follows from the Schwartz inequality applied to

$$2\left(\int_1^2 (s-1)x^{-s}\,d\mu(s)\right)^2 = 2\left(\int_1^2 \left[x^{-s}(s-1)/s\right]^{1/2}\left[x^{-s}(s-1)s\right]^{1/2}d\mu(s)\right)^2$$

since $2(s-1)/s \leq 1$.

That $\psi(\lambda)$ is a Lévy exponent follows immediately from the fact that λ^s is a Lévy exponent.□

Lemma 3 *Let $1 < \beta \leq 2$, and let $\rho(s)$ be a bounded, continuous, increasing function such that the measure $d\rho$ is supported on the interval $[0, \beta - 1)$. Then, we can find a Lévy process with exponent ψ such that*

$$\psi(\lambda) = \lambda^\beta \hat{\rho}(\ln \lambda)$$

for all $\lambda > 0$, and such that $\sigma^2(x)$ is concave on $[0, \infty)$. Here

$$\hat{\rho}(\lambda) = \int_0^\infty e^{-\lambda s}\, d\rho(s).$$

Proof: Set

$$\mu(s) = |\rho| - \rho(\beta - s)$$

where $|\rho|$ denotes the mass of $d\rho$. $\mu(s)$ is a continuous, increasing function such that the measure $d\mu$ is supported on the interval $(1, \beta]$. By Lemma 2, $\psi(\lambda) = \int_1^2 \lambda^s\, d\mu(s)$ is a Lévy exponent and the function σ^2 associated with ψ is concave on $[0, \infty)$.

Our present lemma then follows from the fact that

$$\begin{aligned} \psi(\lambda) = \int_1^2 \lambda^s\, d\mu(s) &= -\int_1^\beta \lambda^s\, d\rho(\beta - s) \\ &= \lambda^\beta \int_0^{\beta-1} \lambda^{-s}\, d\rho(s) \\ &= \lambda^\beta \int_0^{\beta-1} e^{-(\ln\lambda)s}\, d\rho(s) \\ &= \lambda^\beta \hat{\rho}(\ln \lambda). \qquad \square \end{aligned} \tag{4.2}$$

Combining Lemma 3, (1.4), and Theorem 1.7.1' of [1], we obtain a large class of Lévy processes which satisfy the hypotheses of our theorems. In particular, if $\hat{\rho}(\ln \lambda)$ is slowly varying at infinity and $1 < \beta \leq 2$ then we can fin d a Lévy process with exponent ψ given by (3.2) and with concave $\sigma^2(x) \sim c_\beta |x|^{\beta-1} \frac{1}{\hat{\rho}(\ln 1/x)}$. By the cited Theorem in [1] we see that we can find a $\hat{\rho}(\lambda)$ asymptotic at infinity to any regular ly varying function of index less than zero, or to any decreasing slowly varying function. Taking $h(x) = c_\beta \frac{1}{\hat{\rho}(\ln 1/x)}$ leads to the following Corollary mentioned at the end of section 1.

Corollary 1. *Let $h(x)$ be any function which is regularly varying and increasing as $x \to \infty$, and let $1 < \beta \leq 2$. Then we can find a Lévy process with $\sigma^2(x)$ concave such that*

$$\sigma^2(x) \sim |x|^{\beta-1} h(\ln 1/x) \qquad \text{as } x \to 0.$$

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