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Vol. 20, No. 3, May 1976
Printed in Belgium

Note

The Number of Product-Weighted Lead Codes for Ballots and Its Relation to the Ursell Functions of the Linear Ising Model

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Communicated by J. Riordan

Received May 28, 1975

The number of lead codes (defined below) for ballots ending in a tie after $2n$ votes, and weighted by the product of code elements is shown to be the tangent number T_n. The proof also yields a simple expression for the Ursell functions of the linear Ising model, which is used to show $(-1)^{n+1}U_{2n}(\cdot) \geqslant 0$ in the ferromagnetic case.

1 INTRODUCTION

The lead code for a two-candidate ballot is the record of the lead as the votes are cast. For ballots ending in a tie after $2n$ votes, the lead codes can be characterized as the mappings

$$l\colon [0, 1, 2,\dots, 2n] \to [0, 1,\dots, n]$$

such that $l(0) = l(2n) = 0$ and $l(j+1) = l(j) \pm 1$ (so that $l(1) = l(2n-1) = 1$). The set of such mappings is denoted, by $L(2n)$. For example, $L(4) = 01010$, 01210 and $L(6) = 0101010, 0121010, 0101210, 0121210, 0123210$. The number of elements in $L(2n)$ is, of course, the Catalan number, $c_n = (2n)!/(n+1)!\,n!$.

An n element subset of $[1, 2,\dots, 2n]$ is called a chording code if, for any $p \leqslant 2n$ the number of integers $\leqslant p$ in the code, is greater than or equal to the number not in the code. For $n = 2, 3$ the chording codes are 12, 13 and 123, 124, 125, 134, 135. The term "chording" code comes from association with the problem of determining the number of ways $2n$ points

* Supported in part by the National Science Foundation under Grant MPS 74-13252.

on a circle may be joined in pairs of nonintersecting chords, for details see [1]. The chording code for a pairing of $[1, 2, \ldots, 2n]$ into n pairs is the set obtained by choosing the smallest integer in each pair.

If $l \in L(2n)$, it is easily seen that the n integers j with $l(j) - l(j-1) = 1$, form a chording code. Similarly, given a chording code, if we let $l(0) = 0$, and $l(j) = l(j-1) + 1$ or $l(j) = l(j-1) - 1$ according as j is or is not in the code, it is seen that $l \in L(2n)$. Hence, this correspondence of lead codes and chording codes is bijective.

If P is a partition of $[1, 2, \ldots, 2n]$ with even blocks [= parts], P gives rise to a pairing of $[1, 2, \ldots, 2n]$ into n pairs by pairing consecutive integers in each block. The lead code corresponding to this pairing is denoted by $A(P)$. For example, the partition $P = (1356)(24)$ gives rise to the pairing $(13)(56)(24)$, so that $A(P) = 0121010$.

The main result of this paper is

THEOREM 1.

$$\sum_{l \in L(2n)} \prod_{j=1}^{2n-1} l(j) = T_n \tag{1}$$

where

$$\tan(x) = \sum_{1}^{\infty} T_n \frac{x^{2n-1}}{(2n-1)!}.$$

The term summed on the left is a product-weighted lead code. The tangent numbers T_n appear in [7, seq. 829] for $n = 1(1)12$. The first few are 1, 2, 16, 272. Theorem 1 is proven in Section 2.

Product-weighting arose from consideration of the linear Ising model. It will be seen that the Ursell functions for this model have a simple expression in terms of product weighted lead codes. It has been conjectured [2, 3] that $(-1)^{n+1} U_{2n}(\cdot) \geqslant 0$ for ferromagnetic Ising models. The simple expression mentioned establishes these inequalities for the linear model.

2. The Number of Product-Weighted Lead Codes for Ballots

Let $E(2n)$ denote the set of even partitions of $[1, 2, \ldots, 2n]$. If $P \in E(2n)$, $|P|$ denotes the number of blocks in P. The mapping $A: E(2n) \to L(2n)$ has been defined in the introduction. The proof of Theorem 1 depends on

LEMMA 2. *For any* $l \in L(2n)$,

$$\sum_{P \in E(2n): A(P) = l} (-1)^{|P|-1}(|P| - 1)! = (-1)^{n+1} \prod_{j=1}^{2n-1} l(j). \tag{2}$$

Theorem 1 follows by summing (2) over all $l \in L(2n)$ and using

LEMMA 3.

$$\sum_{P \in E(2n)} (-1)^{|P|-1}(|P| - 1)! = (-1)^{n+1} T_n. \tag{3}$$

Proof of Lemma 2. The proof is by induction. The lemma is trivial for $n = 1$, so assume it is true for all $n < m$. Fix $l \in L(2m)$. It is clear that there is a unique integer $2m - h$, called the last peak of l, such that

$$\begin{aligned} l(2m - h + j) &= h - j; \qquad 0 \leqslant j \leqslant h. \\ l(2m - h - 1) &= h - 1. \end{aligned}$$

h is called the height of the last peak. The last peak, $2m - h$, is the largest integer in the chording code corresponding to l; hence, if $A(P) = l$, (i) the integers $2m - h + j$, $1 \leqslant j \leqslant h$ must be terminal integers in their blocks, which are therefore distinct, and (ii) the integer $2m - h$ must appear in a block with one of the last h integers.

To use induction, let $C(l)$ $(\in L(2m - 2))$ be the lead code obtained by removing from l the last peak and the integer $2m - h + 1$ immediately succeeding it and filling in the gap in the natural way. Explicitly

$$\begin{aligned} C(l)(j) &= l(j); & 0 \leqslant j \leqslant 2m - h - 1 \\ C(l)(2m - h - 1 + k) &= h - 1 - k; & 0 \leqslant k \leqslant h - 1. \end{aligned}$$

There is a map $B: E(2m) \to E(2m - 2)$ which covers C, in the sense that the following diagram is commutative

$$\begin{array}{ccc} E(2m) & \xrightarrow{B} & E(2m-2) \\ \big\downarrow A & & \big\downarrow A \\ L(2m) & \xrightarrow{C} & L(2m-2) \end{array}$$

If $A(P) = l$, $B(P)$ is obtained by removing $2m - h$ together with the largest integer in its block from P, and filling in the gap in the natural way. More precisely, by (i) and (ii) there is a unique integer among the last h integers, say, $2m - h + i$, appearing in the same block as $2m - h$. Remove $2m - h + i$ and subtract 1 from each of the integers succeeding it. Repeat this process for $2m - h$ to obtain the partition $B(P) \in E(2m - 2)$. Clearly $A(B(P)) = C(l)$. Write

$$\sum_{P: A(P) = l} (-1)^{|P|-1}(|P| - 1)! = \sum_{R: A(R) = C(l)} \left\{ \sum_{P: \left\{ \substack{A(P) = l \\ B(P) = R} \right.} (-1)^{|P|-1}(|P| - 1)! \right\}. \tag{4}$$

Fix $R \in E(2m-2)$ with $A(R) = C(l)$. To compute the inner sum in (4), it is first necessary to specify all $P \in E(2m)$ with $A(P) = l$ and $B(P) = R$.

First, specify the largest integer in the block of P containing $2m - h$. Since P is required to satisfy $A(P) = l$, by (i) and (ii), this integer may be any one of the last h integers, $2m - h + j$, $1 \leqslant j \leqslant h$. Say it is $2m - h + i$. Add 1 to the last $h - 1$ integers in R, and then add 1 more to the last $h - i$ integers to obtain a partition, $\tilde{R}$, of $[1,..., 2m - h - 1, 2m - h + 1,..., 2m - h + i - 1, 2m - h + i + 1,..., 2m]$.

To complete the specification of P, since P is required to satisfy $B(P) = R$, it only remains to specify in which, if any, block of $\tilde{R}$ the pair $2m - h$, $2m - h + i$ is to be placed.

If this pair is placed in one of the blocks of $\tilde{R}$, it must by (i) avoid any block containing one of the $h - 1$ integers $2m - h + j$; $1 \leqslant j \leqslant h, j \neq i$. Since $A(R) = C(l)$, as in (i) these $h - 1$ integers appear in distinct blocks of $\tilde{R}$. Hence the pair $2m - h$, $2m - h + i$ can be placed in any one of $|R| - (h - 1)$ blocks, and then $|R| = |P|$.

However, the pair need not be placed in a block of $\tilde{R}$. It can form a two element block of P, and then $|P| = |R| + 1$ so that $(-1)^{|P|-1}(|P| - 1)! = -R(-1)^{|R|-1}(|R| - 1)!$.

Therefore,

$$\begin{aligned}\sum_{P:\left\{\substack{A(P)=l\\ B(P)=R}\right.} (-1)^{|P|-1}(|P| - 1)! &= h[|R| - (h - 1) - |R|](-1)^{|R|-1}(|R| - 1)! \\ &= -h(h - 1)(-1)^{|R|-1}(|R| - 1)!\end{aligned}$$

By (4) and the induction hypothesis (2)

$$\begin{aligned}\sum_{P:A(P)=l} (-1)^{|P|-1}(|P| - 1)! &= -h(h - 1) \sum_{R:A(R)=C(l)} (-1)^{|R|-1}(|R| - 1)! \\ &= -h(h - 1)(-1)^{m} \prod_{j=1}^{2m-3} C(l)(j) \\ &= (-1)^{m-1} \prod_{j=1}^{2m-1} l(j),\end{aligned}$$

completing the proof of Lemma 2.

Proof of Lemma 3. It is known [4] that the enumerator of $E(2n)$ by number of blocks, $E_n(x)$, has the generating function

$$\begin{aligned}\sum_{0} E_n(x) \frac{y^{2n}}{(2n)!} &= \exp(x(\cosh(y) - 1)) \\ &= \sum_{0} \frac{x^j}{j!} (\cosh(y) - 1)^j.\end{aligned} \tag{5}$$

Hence, with $x^j = \alpha_j = (-1)^{j-1}(j - 1)! = s(j, 1)$, the Stirling number of the first kind

$$\begin{aligned}\sum_{1} E_n(\alpha) \frac{y^{2n}}{2n!} &= \sum_{1} (-1)^{j-1} \frac{(\cosh(y) - 1)^j}{j} \\ &= \log(\cosh(y)).\end{aligned}$$

Hence, with $i^2 = -1$,

$$\begin{aligned}\sum_{1} E_n(\alpha) \frac{y^{2n-1}}{(2n - 1)!} &= \frac{d}{dy} (\log(\cosh(y))) \\ &= \tanh(y) = -i \tan(iy) \\ &= \sum_{1} (-1)^{n+1} T_n \frac{y^{2n-1}}{(2n - 1)!} .\end{aligned}$$

Therefore,

$$\sum_{P \in E(2n)} (-1)^{|P|-1}(|P| - 1)! = E_n(\alpha) = (-1)^{n+1} T_n . \quad \| \tag{6}$$

I am grateful to John Riordan for the following observation. If $T_n(x)$ is the polynomial for the central difference numbers $T(n, k)$, it follows from the known identity [5]

$$\exp(yT(x)) = \exp[2x \sinh(y/2)], \qquad T^n(x) = T_n(x)$$

that

$$\begin{aligned}\cosh(yT(x)) &= \cosh\left[2x \sinh\left(\frac{y}{2}\right)\right] \\ &= \sum_{0} \frac{x^{2j}}{(a_j)\, j!} (\cosh(y) - 1)^j\end{aligned}$$

with $a_j = (2j)!/2^j j!$.

Comparison with (5) shows

$$E_n(x) = \sum_{k=1}^{n} T(2n, 2k)\, a_k x^k$$

so that by (6)

$$(-1)^{n+1} T_n = \sum_{1}^{n} T(2n, 2k)\, a_k s(k, 1)$$

an identity to be compared with [6, (2)] for the Genocchi numbers.

3. The Ursell Functions of the Linear Ising Model

This section describes the linear Ising model and its Ursell functions. Lemma 2 yields a simple expression for these functions from which many of their important properties can be read off.

Let σ_i, i an integer, be a sequence of independent, identically distributed spin-$\frac{1}{2}$ random variables, i.e., $\text{Prob}(\sigma_i = 1) = \text{Prob}(\sigma_i = -1) = \frac{1}{2}$. If σ_i are random variables on a probability space $(X, A, d\mu_0)$, given a sequence of real numbers g_i, i an integer, a new probability measure, $d\mu$, is defined on X by

$$d\mu = \frac{\exp(\sum_i g_i \sigma_i \sigma_{i+1})\, d\mu_0}{\int \exp(\sum_i g_i \sigma_i \sigma_{i+1})\, d\mu_0}.$$

Since $n_i = \sigma_i \sigma_{i+1}$ is also a sequence of independent spin-$\frac{1}{2}$ random variables, it is easy to see that

$$\int \exp\Big(\sum_i g_i \sigma_i \sigma_{i+1}\Big)\, d\mu_0 = \prod_i \cosh(g_i)$$

$$\int \sigma_k \sigma_{k+1}\, d\mu = \tanh(g_k)$$

$$\int \sigma_i \sigma_j\, d\mu = \prod_{k=i}^{j-1} \Big(\int \sigma_k \sigma_{k+1}\, d\mu\Big) \quad \text{for} \quad i < j$$

$$\text{and for} \quad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{2n} \tag{7}$$

$$\int \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{2n}}\, d\mu = \prod_{j=1}^{n} \Big(\int \sigma_{i_{2j-1}} \sigma_{i_{2j}}\, d\mu\Big)$$

$$\int \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{2n-1}}\, d\mu = 0.$$

The Ursell functions $U_{2n}(\sigma_{i_1}, \ldots, \sigma_{i_{2n}})$ [8] are defined by the formula

$$U_{2n}(\sigma_{i_1}, \ldots, \sigma_{i_{2n}}) = \sum_{P \in E(2n)} (-1)^{|P|-1}(|P| - 1)! \prod_{P_j \in P} \Big(\int \prod_{k \in P_j} \sigma_{i_k}\, d\mu\Big) \tag{8}$$

where the P_j are the blocks of P. $U_{2n}(\cdot)$ is a symmetric function of its arguments, and we set $U_{2n+1}(\cdot) = 0$.

If $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{2n}$ note that by (7)

$$\prod_{P_j \in P} \Big(\int \prod_{k \in P_j} \sigma_{i_k}\, d\mu\Big) = \prod_{k=1}^{2n-1} \Big(\int \sigma_{i_k} \sigma_{i_{k+1}}\, d\mu\Big)^{A(P)(k)}$$

where $A(P) \in L(2n)$ has been defined in the introduction. Equation (8) can be rewritten as

$$\begin{aligned} U_{2n}(\sigma_{i_1}, \ldots, \sigma_{i_{2n}}) &= \sum_{P \in E(2n)} (-1)^{|P|-1}(|P| - 1)! \prod_{j=1}^{2n-1} \Big(\int \sigma_{i_j} \sigma_{i_{j+1}}\, d\mu\Big)^{A(P)(j)} \\ &= \sum_{l \in L(2n)} \Big(\sum_{P: A(P)=l} (-1)^{|P|-1}(|P| - 1)!\Big) \prod_{j=1}^{2n-1} \Big(\int \sigma_{i_j} \sigma_{i_{j+1}}\, d\mu\Big)^{l(j)}. \end{aligned}$$

Lemma 2 then yields

Theorem 2. *Let $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{2n}$. Then*

$$U_{2n}(\sigma_{i_1}, \ldots, \sigma_{i_{2n}}) = (-1)^{n+1} \sum_{l \in L(2n)} \prod_{j=1}^{2n-1} l(j) \Big(\int \sigma_{i_j} \sigma_{i_{j+1}}\, d\mu\Big)^{l(j)}. \tag{9}$$

Consider first the ferromagnetic case where all $g_i \geqslant 0$, so that all $\int \sigma_i \sigma_j\, d\mu \geqslant 0$. Since all $l(j) \geqslant 0$, Theorem 2 shows that $(-1)^{n+1} U_{2n}(\cdot) \geqslant 0$.

Next, specialize to the translation invariant case of all $g_i = g > 0$, so that $\int \sigma_i \sigma_j\, d\mu = (\tanh(g))^{|i-j|}$, with $0 < \tanh(g) < 1$. The product weighting in Theorem 2 ensures that all $l(j) \geqslant 1$, so that (9) exhibits the "cluster property"

$$\lim_{\substack{(i_1, \ldots, i_k) \to -\infty \\ (i_{k+1}, \ldots, i_{2n}) \to \infty}} U_{2n}(\sigma_{i_1}, \ldots, \sigma_{i_{2n}}) = 0$$

of the Ursell functions in a transparent form.

ACKNOWLEDGMENTS

I would like to thank John Riordan for his patient assistance in preparing this paper, and Bob Donaghey, James Glimm, and Oliver McBryan for helpful suggestions.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.