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RENORMALIZATION AND LIMIT THEOREMS FOR SELF-INTERSECTIONS OF SUPERPROCESSES[1]

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In this paper, we study a renormalized self-intersection local time for superprocesses over stable processes and classical diffusions. When the renormalization breaks down, we obtain limit theorems.

1. Introduction. With any nice Markov process z_t in R^d we associate a new Markov Z_t taking values in the space of finite measures on R^d. The process Z_t is called the superprocess over z_t, and we refer to Dynkin (1988) for an introduction to superprocesses and for further references.

We will use the notation

$$\langle \varphi, Z_t \rangle = \int \varphi(x) Z_t(dx),$$

$$\langle f(x,y), Z_s(dx) Z_t(dy) \rangle = \iint f(x,y) Z_s(dx) Z_t(dy).$$

Throughout this paper we assume that the initial measure $Z_0 = \mu$ has a bounded and integrable density with respect to Lebesgue measure. Also we use $|\nu|$ for the mass of a measure ν. Our starting point is the formal expression

$$\iint_B \langle \delta(x-y), Z_s(dx) Z_t(dy) \rangle \, ds\, dt, \tag{1.1}$$

which intuitively should measure the "self-intersections" of Z_s. In (1.1), δ is the Dirac delta function and $B \subseteq R_+^2$. In an attempt to make (1.1) rigorous, we replace δ by an approximate delta function. Let $f(x) \geq 0$ be a continuous symmetric function with support in the unit ball and such that $\int f(x)\, dx = 1$. Set

$$f_\varepsilon(x) = \frac{1}{\varepsilon^d} f\left(\frac{x}{\varepsilon}\right)$$

and replace (1.1) by

$$\iint_B \langle f_\varepsilon(x-y), Z_s(dx) Z_t(dy) \rangle \, ds\, dt. \tag{1.2}$$

We will describe the behavior of (1.2) as $\varepsilon \to 0$.

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To be specific, let us first take Z_t to be the superprocess over Brownian motion in R^d. Dynkin (1988) has shown that if $d \le 7$ and if B is bounded with $B \subseteq \{(x,y) | |x-y| \ge \gamma\}$ for some $\gamma > 0$, then (1.2) has a limit as $\varepsilon \to 0$. This limit is called the self-intersection local time. In this paper, we study what happens when we lift the restriction that B be separated from the diagonal. If $d \le 3$ there are no problems with the $\varepsilon \to 0$ limit [and no real interest in self-intersections; Z_t itself has a local time; see Dynkin (1988)]. However, if $d \ge 4$, the case we study here, (1.2) will typically diverge as $\varepsilon \to 0$. In Theorem 1 we will show that if $d = 4$ or 5, then (1.2) can be renormalized; that is, by subtracting a singular term which does not involve intersections, we can obtain a nontrivial limit.

This is the analogue of a result for Brownian motion in R^2 which goes back to Varadhan (1969); see also Le Gall (1985), Rosen (1986), Yor (1985a) and Dynkin (1988b). The renormalized intersection local time for Brownian motion in R^2 turns out to be the right tool for analyzing fluctuations of the Wiener sausage [see Le Gall (1986b), Chavel, Feldman and Rosen (1991), Wienryb (1988)] and the range of random walks [Le Gall (1986a) and Le Gall and Rosen (1991)]. It is our hope that the renormalized intersection local time of Theorem 1 will find similar applications to the study of measure-valued processes.

When $d = 6$, we can no longer obtain a renormalized intersection local time. However, Theorem 1 shows that a suitably scaled version converges in distribution. This is the analogue of Yor's theorem for Brownian motion in R^3 [Yor (1985b); Rosen (1988)].

We use B_t to denote a real Brownian motion independent of our superprocess.

THEOREM 1. *Let Z_t denote the superprocess over Brownian motion in R^d and set*

$$\gamma_\varepsilon(T) = \int_0^T \int_0^T \langle f_\varepsilon(x-y), Z_s(dx) Z_t(dy) \rangle \, ds \, dt - \int_0^T \varphi_\varepsilon(T-s) |Z_s| \, ds, \tag{1.3}$$

where

$$\varphi_\varepsilon(t) = \int_0^t \int_0^t \left(\iint p_r(x) f_\varepsilon(x-y) p_s(y) \, dx \, dy \right) dr \, ds \tag{1.4}$$

and

$$p_s(y) = \frac{e^{-y^2/2s}}{(2\pi s)^{d/2}}$$

is the transition density for Brownian motion in R^d.

If $d = 4$ or 5, then $\gamma_\varepsilon(T)$ converges in L^2 as $\varepsilon \to 0$.

If $d = 6$, then $\gamma_\varepsilon(T)/\log(1/\varepsilon)$ converges weakly to B_{M_T} where

$$M_T = \frac{1}{2\pi^6} \int_0^T |Z_s| \, ds. \tag{1.5}$$

REMARK. More generally, if for $h \in C_0^\infty(R^d)$ we set

$$\gamma_\varepsilon(T,h) = \int_0^T\int_0^T \langle h(x) f_\varepsilon(x-y), Z_s(dx)Z_t(dy)\rangle - \int_0^T \langle \varphi_{\varepsilon, T-s}, Z_s\rangle\, ds,$$

where now

$$\varphi_{\varepsilon,t}(z) = \int_0^t\int_0^t\left(\iint h(z+x)p_r(x) f_\varepsilon(x-y)p_s(x)\, ds\, dy\right) dr\, ds,$$

then $\gamma_\varepsilon(T,h)$ converges in L^2 for $d = 4,5$ while if $d = 6$,

$$\frac{\gamma_\varepsilon(T,h)}{\log(1/\varepsilon)}$$

converges weakly to $B_{M_T(h)}$ where

$$M_T(h) = \frac{1}{2\pi^6}\int_0^T \langle h^2, Z_s\rangle\, ds.$$

Theorem 1 will be derived with the aid of the following very explicit theorem.

THEOREM 2. *Let x_t be Brownian motion in R^d killed at an independent exponential time and let X_t be the superprocess over x_t.*

(a) *If $d = 4$, then*

$$(1.6)\qquad \int_0^\infty\int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\, ds\, dt - \frac{1}{\pi^2}\log\left(\frac{1}{\varepsilon}\right)\int_0^\infty |X_s|\, ds$$

converges in L^2 as $\varepsilon \to 0$.

(b) *If $d = 5$, then*

$$(1.7)\qquad \int_0^\infty\int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\, ds\, dt - \frac{1}{2\pi^2}\frac{c(f)}{\varepsilon}\int_0^\infty |X_s|\, ds$$

where $c(f) = \int f(x)(1/|x|)\, dx$, converges in L^2 as $\varepsilon \to 0$.

(c) *If $d = 6$ and*

$$(1.8)\qquad \gamma_\varepsilon = \int_0^\infty\int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\, ds\, dt - a(\varepsilon)\int_0^\infty |X_s|\, ds,$$

where

$$(1.9)\qquad a(\varepsilon) = \frac{1}{2\pi^3}\left(\frac{1}{\varepsilon^2}\int f(y)\frac{1}{y^2}\, dy - \log\left(\frac{1}{\varepsilon}\right)\right),$$

then $\gamma_\varepsilon/\log(1/\varepsilon)$ converges in distribution and we have

$$(1.10)\qquad E_\mu\left(\exp\left[-\lambda\left(\frac{\gamma_\varepsilon}{\log(1/\varepsilon)}\right)\right]\right) \to \exp\left(|u|\frac{1}{2}\left(1-\sqrt{1-\frac{\lambda^2}{\pi^6}}\right)\right)$$

for λ small, as $\varepsilon \to 0$.

REMARK. X_t is not the same as Z_t killed at an independent exponential time.

Theorem 1 can be generalized to nice diffusions in R^d. Let Z_t be a diffusion with generator

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}. \tag{1.11}$$

If a_{ij}, b_i are smooth and uniformly bounded together with their derivatives and

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \geq \delta \sum_{i=1}^{d} \lambda_i^2 \tag{1.12}$$

for some $\delta > 0$, uniformly in x and λ_i, we will say that z_t is a smooth uniformly elliptic diffusion.

THEOREM 3. *Let Z_t denote the superprocess over z_t, a smooth uniformly elliptic diffusion in R^d and set*

$$\gamma_\varepsilon(T) = \int_0^T\int_0^T \langle f_\varepsilon(x-y), Z_s(dx)Z_t(dy)\rangle\, ds\, dt - \int_0^T \langle \varphi_{\varepsilon, T-s}, Z_s\rangle\, ds, \tag{1.13}$$

where

$$\varphi_{\varepsilon,t}(z) = \int_0^t\int_0^t\left(\int\int p_r(z,x)\, f_\varepsilon(x-y)p_s(y,z)\, dx\, dy\right) dr\, ds \tag{1.14}$$

and $p_s(x,y)$ is the transition density for z_t.

If $d = 4$ or 5, then $\gamma_\varepsilon(T)$ converges in L^2 as $\varepsilon \to 0$.

If $d = 6$, then $\gamma_\varepsilon(T)/\log(1/\varepsilon)$ converges weakly to B_{M_T} where

$$M_T = \frac{1}{2\pi^6}\int_0^T \langle \psi, Z_s\rangle\, ds \tag{1.15}$$

and

$$\psi(x) = \frac{1}{\det a_{ij}(x)}. \tag{1.16}$$

We now generalize Theorem 1 to symmetric stable processes of order β in R^d. As before, only the case $\beta \leq d/2$ is of interest, since if $\beta > d/2$, the superprocess has a local time, Dynkin (1988a).

THEOREM 4. *Let Y_t denote the superprocess over the symmetric stable process y_t of order β in R^d and*

$$\gamma_\varepsilon(T) = \int_0^T \int_0^T \langle f_\varepsilon(x-y), Y_s(dx)Y_t(dy)\rangle \, ds\, dt - \int_0^T \varphi_\varepsilon(T-s)|Y_s|\, ds, \tag{1.17}$$

where

$$\varphi_\varepsilon(t) = \int_0^t \int_0^t \left(\iint p_r(x) f_\varepsilon(x-y) p_s(y)\, dx\, dy \right) dr\, ds \tag{1.18}$$

and $p_s(y)$ denotes the transition density for y_t.

If $d/3 < \beta \le d/2$, then $\gamma_\varepsilon(T)$ converges in L^2 as $\varepsilon \to 0$.

If $\beta = d/3$, then $\gamma_\varepsilon(T)/\log(1/\varepsilon)$ converges weakly to B_{M_T} where

$$M_T = 2a(d)\int_0^T |Y_s|\, ds \tag{1.19}$$

and

$$a(d) = \frac{2^{6-2d}}{\pi^d} \cdot \frac{1}{\Gamma^2(d/2)}.$$

Sections 2–6 are devoted to Theorem 2, that is, the superprocess over killed Brownian motion. In Section 7, we derive Theorem 1 with the aid of Theorem 2. Because of space limitations, proofs of Theorems 3 and 4 are omitted. They follow, in general, the lines of the proofs of Theorems 1 and 2.

2. Theorem 2: Preliminaries. Our proofs involve the calculation of moments, and in this section we derive a formula for moments of the approximate renormalized intersection local time. Our starting point is Dynkin's formula (1988a):

$$\begin{aligned} E_\mu\left(\prod_{i=1}^n \langle f_i, X_{t_i}\rangle \right) &= \sum_{D_n} \int \prod_{v \in V_-} \mu(dy_v) \prod_{a \in A} p_{s_{f(a)} - s_{i(a)}}(y_{f(a)} - y_{i(a)}) \\ &\quad \times \prod_{v \in V_0} ds_v\, dy_v \prod_{i=1}^n f_i(z_i)\, dz_i. \end{aligned} \tag{2.1}$$

In (2.1),

$$p_s(x) = e^{-s} \frac{e^{-x^2/2s}}{(2\pi s)^{d/2}} \tag{2.2}$$

is the transition density for exponentially killed Brownian motion in R^d, where by convention $p_s(x) = 0$ if $s < 0$. D_n is the set of directed binary graphs with n exits marked $1, 2, \dots, n$. Given such a graph, A is the set of arrows, and if the arrow $a \in A$ goes from the vertex v to w, we write $v = i(a)$, $w = f(a)$. To each vertex v we associate two variables

$$(s_v, y_v) \in R_+ \times R^d,$$

which we refer to as the time and space coordinates of v. V_- denotes the set of entrances for our graph, and if $v \in V_-$, we set $s_v \equiv 0$. If v is the exit labelled by j, $i \le j \le n$, we set

$$(s_v, y_v) \doteq (t_j, z_j).$$

Finally, V_0 denotes the set of internal vertices; that is, those vertices which are neither entrances nor exits.

Let

$$G(x) = \int_0^\infty p_s(x)\, ds \tag{2.3}$$

denote the Green's function for exponentially killed Brownian motion in R^d. From (2.1) we see that

$$\begin{aligned} &E_\mu\left(\prod_{i=1}^n \int_0^\infty \langle f_i, X_{t_i}\rangle\, dt_i\right) \\ &\quad = \sum_{D_n} \int \prod_{v \in V_-} d\mu(dy_v) \prod_{a \in A} G(y_{f(a)} - y_{i(a)}) \prod_{v \in V_0} dy_v \prod_{i=1}^n f_i(z_i)\, dz_i. \end{aligned} \tag{2.4}$$

From this it follows that

$$\begin{aligned} &E_\mu\left(\left[\int_0^\infty \int_0^\infty \langle f_\varepsilon(x-y), X_s(dx) X_t(dy)\rangle\, ds\, dt\right]^n\right) \\ &\quad = \sum_{D_{2n}} \int \prod_{v \in V_-} d\mu(y_v) \prod_{a \in A} G(y_{f(a)} - y_{i(a)}) \\ &\qquad \times \prod_{v \in V_0} dy_v \prod_{i=1}^n f_\varepsilon(z_{2i} - z_{2i-1})\, dz_{2i}\, dz_{2i-1}. \end{aligned} \tag{2.5}$$

We will say that the pair of exits v, w are coupled if for some k we have

$$z_{2k} = y_v, \qquad z_{2k-1} = y_w$$

or

$$z_{2k} = y_w, \qquad z_{2k-1} = y_v. \tag{2.6}$$

We will say that a pair of exits v, w are twins if they have the same immediate predecessor; that is, if we can find $a, b \in A$ and a vertex u such that

$$i(a) = i(b) = u \tag{2.7}$$

and

$$f(a) = v, \qquad f(b) = w.$$

If twins v, w are coupled, and for example, $z_{2K} = y_v$, $z_{2K-1} = y_w$ and (2.7) holds, then we get a factor in (2.5) of the form

$$\begin{aligned}
&\iint G(y_v - y_u)G(y_w - y_u)\, f_\varepsilon(y_v - y_w)\, dy_v\, dy_w \\
(2.8)\qquad &= \iint G(y_v)G(y_w)\, f_\varepsilon(y_v - y_w)\, dy_v\, dy_w \\
&= \int f_\varepsilon(y) G * G(y)\, dy.
\end{aligned}$$

Set

$$c(\varepsilon) = \int f_\varepsilon(y) G * G(y)\, dy. \tag{2.9}$$

Then it is easy to check that

$$\begin{aligned}
&E_\mu\left(\left[\int_0^\infty \int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\, ds\, dt - 2c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\, ds\right]^n\right) \\
(2.10)\qquad &= \sum_{C_{2n}} \int \prod_{v \in V_-} d\mu(y_v) \prod_{a \in A} G(y_{f(a)} - y_{i(a)}) \\
&\quad \times \prod_{v \in V_0} dy_v \prod_{i=1}^n f_\varepsilon(z_{2i} - z_{2i-1})\, dz_{2i}\, dz_{2i-1},
\end{aligned}$$

where C_{2n} is the set of binary graphs with $2n$ labelled exits; $1, 2, \dots, 2n$, such that no twin exits are coupled, that is, no twin exits are labelled $2i-1, 2i$ for any i.

Thus, the effect of the subtraction term in (2.9) is to eliminate all coupled twins. The factor 2 comes from the two possibilities in (2.6).

We now calculate the asymptotics of $c(\varepsilon)$. We first note that

$$\begin{aligned}
G * G(y) &= \int dx \int_0^\infty \int_0^\infty p_s(y-x)p_t(x)\, ds\, dt \\
&= \int_0^\infty \int_0^\infty p_{s+t}(y)\, ds\, dt \\
&= \int_0^\infty \int_0^t p_t(y)\, ds\, dt \\
(2.11)\qquad &= \int_0^\infty t p_t(y)\, dt \\
&= \frac{1}{2\pi} \int_0^\infty e^{-t} \frac{e^{-y^2/2t}}{(2\pi t)^{(d-2)/2}}\, dt \\
&\doteq \frac{1}{2\pi} g(y),
\end{aligned}$$

where $g(y)$, with obvious notation, corresponds to the Green's function for killed Brownian motion in $d-2$ dimensions.

If $d=4$, it is known that for $|y| \le \frac{1}{2}$,

$$g(y) = \frac{1}{\pi}\left[\log\left(\frac{1}{|y|}\right) + \log(\sqrt{2}) - \kappa\right] + O(|y|), \tag{2.12}$$

where κ is Euler's constant; see, for example, Knight [(1981), page 38]. Hence

$$\begin{aligned} c(\varepsilon) &= \int f_\varepsilon(y) G * G(y)\, dy \\ &= \frac{1}{2\pi^2}\int f_\varepsilon(y)\left(\log\left(\frac{1}{|y|}\right) dy + \log(\sqrt{2}) - \kappa) + O(|y|)\right) dy \\ &= \frac{1}{2\pi^2}\left(\log\left(\frac{1}{\varepsilon}\right) + \int f(y)\log\left(\frac{1}{|y|}\right) dy + \log(\sqrt{2}) - \kappa\right) + O(\varepsilon). \end{aligned} \tag{2.13}$$

If $d=5$, it is known that

$$g(y) = \frac{1}{2\pi}\frac{e^{-|y|}}{|y|}; \tag{2.14}$$

see again Knight [(1981), page 38]. Hence

$$\begin{aligned} c(\varepsilon) &= \frac{1}{4\pi^2}\int f_\varepsilon(y)\frac{e^{-|y|}}{|y|}\, dy \\ &= \frac{1}{4\pi^2\varepsilon}\int \frac{e^{-\varepsilon|y|}}{|y|} f(y)\, dy \\ &= \frac{1}{4\pi^2\varepsilon}\int f(y)\frac{dy}{|y|} - \frac{1}{4\pi^2} + O(\varepsilon). \end{aligned} \tag{2.15}$$

Finally, for $d=6$ let us analyze $u^1(x)$, the one-potential for Brownian motion in R^4. Iterating the resolvent equation we find

$$\begin{aligned} u^0(x) - u^1(x) &= u^1 * u^0(x) \\ &= u^1 * u^1(x) + u^1 * u^1 * u^0(x). \end{aligned} \tag{2.16}$$

By (2.11) and (2.12), we know that $u^1 * u^1(x) = (1/2\pi^2)\log(1/|x|) + O(1)$, $|x| < 1$, and it is easy to see that

$$u^1 * u^1 * u^0(x) = O(1),$$

so that for $|x| < \frac{1}{2}$,

$$\begin{aligned} G * G(x) &= \frac{1}{2\pi}u^1(x) = \frac{1}{2\pi}u^0(x) - \frac{1}{4\pi^3}\log\left(\frac{1}{|x|}\right) + O(1) \\ &= \frac{1}{4\pi^3}\frac{1}{x^2} - \frac{1}{4\pi^3}\log\left(\frac{1}{|x|}\right) + O(1). \end{aligned} \tag{2.17}$$

Hence

$$
\begin{aligned}
c(\varepsilon) &= \int f_\varepsilon(y) G * G(y)\, dy \\
&= \frac{1}{4\pi^3}\frac{1}{\varepsilon^2}\int \frac{f(y)}{y^2}\, dy - \frac{1}{4\pi^3}\log\left(\frac{1}{\varepsilon}\right) + O(1), \qquad d = 6.
\end{aligned} \tag{2.18}
$$

We also note for future reference that, as in (2.11),

$$
\begin{aligned}
G * G * G(y) &= \int_0^\infty \cdot \int_0^\infty \int_0^\infty p_{r+s+t}(y)\, dr\, ds\, dt \\
&= \int_0^\infty \int_0^t \int_0^s p_t(y)\, dr\, ds\, dt \\
&= \int_0^\infty \frac{t^2}{2} p_t(y)\, dt \\
&= \frac{1}{8\pi^2}\int_0^\infty e^{-t}\frac{e^{-y^2/2t}}{2\pi t}\, dt \\
&= \frac{1}{8\pi^2}\mathscr{G}(y) \quad \text{(the Green's function in } R^2\text{)} \\
&= \frac{1}{8\pi^3}\log\left(\frac{1}{|y|}\right) + O(1).
\end{aligned} \tag{2.19}
$$

We also find that, as in (2.16),

$$
\begin{aligned}
G^0(x) - G(x) &= G * G^0(x) \\
&= G * G(x) + G * G * G^0 \\
&= G * G(x) + G * G * G(x) + G * G * G * G^0(x),
\end{aligned}
$$

so that

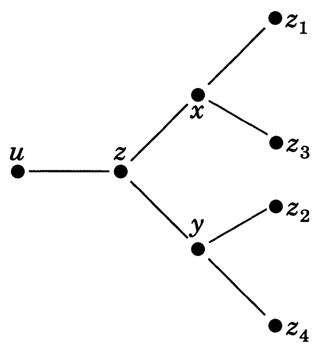
$$
\begin{aligned}
G(x) &= G^0(x) + O\left(\frac{1}{x^2}\right) \\
&= \frac{1}{2\pi^3}\frac{1}{x^4} + O\left(\frac{1}{x^2}\right).
\end{aligned} \tag{2.20}
$$

3. Theorem 2: The second moment. In this section, we compute the asymptotics of

$$
\begin{aligned}
I(\varepsilon) = E_\mu\Bigg[\Bigg(\int_0^\infty\int_0^\infty &\langle f_\varepsilon(x, -y), X_s(dx)X_t(dy)\rangle\, ds\, dt \\
&- 2c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\, ds\Bigg)^2\Bigg].
\end{aligned} \tag{3.1}
$$

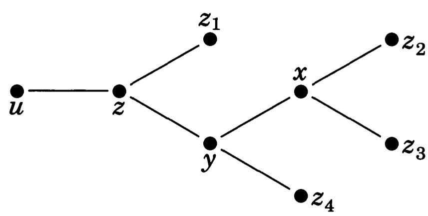
By (2.10) we obtain a contribution from each binary graph with four exits, such that no twin exits are coupled.

We first sketch the possible graphs and write down their contribution. Later we will work out the combinatoric factors.



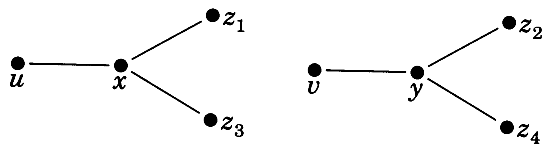
Graph 1

$$
\begin{aligned}
&\int \mu(du) G(z-u) G(x-z) G(y-z) G(x-z_1) G(x-z_3) G(y-z_2)\\
&\qquad \times G(y-z_4) f_\varepsilon(z_1-z_2) f_\varepsilon(z_3-z_4)\, dx\, dy\, d\bar{z}\\
(3.2)\qquad &= \int G(z-u)\, dz\, d\mu(u) \int G(x) G(y) \bigl(G * G * f_\varepsilon(x-y)\bigr)^2\, dx\, dy\\
&= |\mu| \int G(x) G(y) \bigl(G * G * f_\varepsilon(x-y)\bigr)^2\, dx\, dy\\
&= |\mu| \int G * G(x) \bigl(G * G * f_\varepsilon(x)\bigr)^2\, dx.
\end{aligned}
$$



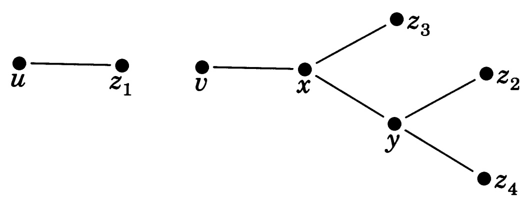
Graph 2

$$
\begin{aligned}
&\int \mu(du) G(z-u) G(z_1-z) G(y-z) G(x-y) G(x-z_2) G(x-z_3)\\
&\qquad \times G(y-z_4) f_\varepsilon(z_1-z_2) f_\varepsilon(z_3-z_4)\, dx\, dy\, d\bar{z}\\
(3.3)\qquad &= |\mu| \int G(y) G(x-y) G * G * f_\varepsilon(x-y) G * G * f_\varepsilon(x)\, dx\, dy\\
&= |\mu| \int G(x) G * G * f_\varepsilon(x) G * G * G * f_\varepsilon(x)\, dx.
\end{aligned}
$$



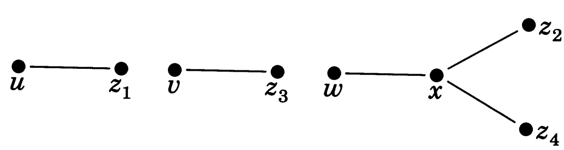
Graph 3

$$\begin{aligned}(3.4)\quad &\int \mu(du)\mu(dv)G(u-x)G(v-y)G(x-z_1)G(x-z_3)G(y-z_2)\\ &\qquad \times G(y-z_4)\,f_\varepsilon(z_1-z_2)\,f_\varepsilon(z_3-z_4)\,dx\,dy\,d\bar{z}\\ &\quad = \int \mu(du)\mu(dv)G(u-x)G(v-y)\big(G*G*f_\varepsilon(x-y)\big)^2\,dx\,dy\\ &\quad = \int \mu(du)\mu(dv)G*G(x-(u-v))\big(G*G*f_\varepsilon(x)\big)^2\,dx.\end{aligned}$$



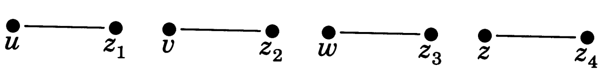
Graph 4

$$\begin{aligned}(3.5)\quad &\int \mu(du)\mu(dv)G(u-z_1)G(v-x)G(x-z_3)G(x-y)G(y-z_2)\\ &\qquad \times G(y-z_4)\,f_\varepsilon(z_1-z_2)\,f_\varepsilon(z_3-z_4)\,dx\,dy\,d\bar{z}\\ &\quad = \int \mu(du)\mu(dv)G(v-x)G*G*f_\varepsilon(u-y)G(x-y)G*G*f_\varepsilon(x-y)\,dx\,dy\\ &\quad = \int \mu(du)\mu(dv)G*G*G*f_\varepsilon(x-(u-v))G(x)G*G*f_\varepsilon(x)\,dx.\end{aligned}$$



Graph 5

$$(3.6)\quad \int \mu(du)\mu(dv)\mu(dw)G(x-w)G*G*f_\varepsilon(x-u)G*G*f_\varepsilon(x-v)\,dx.$$



Graph 6

$$(3.7)\quad \int \mu(du)\mu(dv)\mu(dw)\mu(dz)G*G*f_\varepsilon(u-w)G*G*f_\varepsilon(v-z).$$

We will use the following simple lemma:

LEMMA 4. *Let $u(x)$ be a measurable function with exponential rate of decay as $|x| \to \infty$ and*

$$|u(x)| \leq c \frac{1}{|x|^a}, \qquad a < d.$$

Then $u * f_\varepsilon(x)$ has exponential rate of decay as $|x| \to \infty$ and

$$|u * f_\varepsilon(x)| \leq \begin{cases} \bar{c} \cdot \dfrac{1}{|x|^a}, & |x| \geq \varepsilon, \\ \bar{c} \cdot \dfrac{1}{\varepsilon^a}, & |x| \leq \varepsilon. \end{cases} \tag{3.8}$$

PROOF. The exponential rate of decay as $|x| \to \infty$ is clear. If $|x| \geq 2\varepsilon$,

$$\begin{aligned} u * f_\varepsilon(x) &= \int u(x-y) f_\varepsilon(y)\, dy \\ &\leq c \int \frac{1}{|x-y|^a} f_\varepsilon(y)\, dy \\ &\leq \bar{c} \frac{1}{|x|^a} \int f_\varepsilon(y)\, dy \\ &= \bar{c} \frac{1}{|x|^a}, \end{aligned}$$

while if $|x| \leq 2\varepsilon$,

$$\begin{aligned} u * f_\varepsilon(x) &= \int u(x-y) f_\varepsilon(y)\, dy \\ &= \int u(x - \varepsilon y) f(y)\, dy \\ &\leq c \int_{|y| \leq 1} u(x - \varepsilon y)\, dy \\ &\leq \bar{c} \frac{1}{\varepsilon^d} \int_{|y| \leq \varepsilon} \frac{1}{|x-y|^a}\, dy \\ &\leq \bar{c} \frac{1}{\varepsilon^d} \int_{|y| \leq 3\varepsilon} \frac{1}{|y|^a}\, dy \leq \bar{c} \frac{1}{\varepsilon^a}. \end{aligned}$$

Similarly, if $|u(x)| \leq c \log(1/|x|)$ for $|x| < \frac{1}{2}$, then

$$|u * f_\varepsilon(x)| \leq \begin{cases} \bar{c} \log\left(\dfrac{1}{|x|}\right), & |x| \geq \varepsilon, \\ c \log\left(\dfrac{1}{\varepsilon}\right), & |x| \leq \varepsilon. \end{cases} \tag{3.9}$$

The functions G, $G * G$ and $G * G * G$ are of the above form as we saw in Section 2. They all have exponential rate of decay as $|x| \to \infty$, while for small x we have the following bounds:

$$
\text{(3.10)}\qquad
\begin{array}{llll}
 & d = 4 & d = 5 & d = 6 \\
\hline
G(x) & cx^{-2} & cx^{-3} & cx^{-4} \\
G * G(x) & c\log\left(\frac{1}{|x|}\right) & cx^{-1} & cx^{-2} \\
G * G * G(x) & c & c & c\log\left(\frac{1}{|x|}\right)
\end{array}
$$

Using (3.10) and Lemma 4, it is easy to check that all the integrals (3.2)–(3.7) are uniformly bounded as $\varepsilon \to 0$ when $d = 4$ or 5.

We thus concentrate on $d = 6$. The integrals for Graphs 3, 5 and 6 are uniformly bounded as $\varepsilon \to 0$, while the above shows that the integrals for Graphs 1 and 4 are $O(\log(1/\varepsilon))$. We now carefully compute the integral (3.3) corresponding to Graph 2. We will show that it is $\sim c(\log(1/\varepsilon))^2$.

Using (2.17), (2.19) and (2.20), we find

$$
\begin{aligned}
J(\varepsilon) &= \int G(x) G * G * f_\varepsilon(x) G * G * G * f_\varepsilon(x)\,dx \\
&= \int_{|x| \le 1/2} G(x) G * G * f_\varepsilon(x) G * G * G * f_\varepsilon(x)\,dx + O(1) \\
&= \int_{|x| \le 1/2} \frac{1}{2\pi^3} x^{-4} \frac{1}{4\pi^3} (x^{-2} * f_\varepsilon)(x) \frac{1}{8\pi^3} \left(\log\left(\frac{1}{|x|}\right) * f_\varepsilon\right)(x) \\
&\quad + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) \qquad\qquad (3.11) \\
&= \frac{1}{64\pi^9} \int_{2\varepsilon \le |x| \le 1/2} x^{-4} (x^{-2} * f_\varepsilon)(x) \left(\log\left(\frac{1}{|x|}\right) * f_\varepsilon\right)(x) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) \\
&= \frac{1}{64\pi^9} \int_{2 \le |x| \le 1/2\varepsilon} \int \frac{1}{x^4} \frac{1}{|x - z|^2} \log\left(\frac{1}{\varepsilon|x - y|}\right) f(y) f(z)\,dx\,dy\,dz \\
&\quad + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) \\
&= \frac{1}{64\pi^9} \int_{2 \le |x| \le 1/2\varepsilon} \frac{1}{x^6} \left(\log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{1}{|x|}\right)\right) dx + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) \\
&= \frac{1}{64\pi^9} \frac{2\pi^3}{\Gamma(3)} \int_2^{1/2\varepsilon} \frac{1}{r} \left(\log\left(\frac{1}{\varepsilon}\right) - \log(r)\right) dr + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) \\
&= \frac{1}{64\pi^6} \frac{1}{2} \log^2\left(\frac{1}{\varepsilon}\right) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right).
\end{aligned}
$$

We now count the number of graphs in C_4 which give rise to a contribution (3.3).

We first consider those graphs which have four unlabelled exits and which are of the form of Graph 2. With obvious notation they are

$$\begin{array}{cccc} & a & (b & (cd)) \\ & a & ((b & c)d) \\ (3.12) & (a & (b & c))d \\ & ((a & b) & c)d. \end{array}$$

Consider the unlabelled graph

$$a(b(cd)).$$

We can assign any of the four labels, $1, 2, 3, 4$ to c, but of the three remaining labels, one will be forbidden to d by our restriction that no twins are coupled. The remaining two labels can be assigned arbitrarily to a, b. Thus, to each graph in (3.12) there are 4^2 labelling schemes, for a total of 4^3 terms of the form (3.3).

Thus

$$(3.13)\quad I(\varepsilon) = 4^3|\mu|J(\varepsilon) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right) = \frac{1}{2\pi^6}|\mu|\log^2\left(\frac{1}{\varepsilon}\right) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right).$$

4. Proof of Theorem 2(a) and (b). Let

$$(4.1)\quad \gamma_\varepsilon = \int_0^\infty\int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\, ds\, dt - 2c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\, ds,$$

where

$$c(\varepsilon) = \begin{cases} \dfrac{1}{2\pi^2}\log\left(\dfrac{1}{\varepsilon}\right), & d = 4, \\[2ex] \dfrac{1}{4\pi^2}\dfrac{1}{\varepsilon}\displaystyle\int \frac{f(y)}{|y|}\, dy, & d = 5. \end{cases}$$

We will show that

$$(4.2)\quad E_\mu\left[(\gamma_\varepsilon - \gamma_{\bar\varepsilon})^2\right] \to 0 \quad \text{as } \varepsilon, \bar\varepsilon \to 0.$$

By the discussion in Section 3, we know that (4.2) can be written as a sum of terms which are of the form (3.2)–(3.7), except that in those formulas we replace f_ε by $f_\varepsilon - f_{\bar\varepsilon}$.

Let us define

$$(4.3)\quad H_\alpha(x) = \int_0^\infty e^{-t}\frac{e^{-x^2/2t}}{t^{(\alpha/2)+1}}\, dt.$$

It is clear that $H_\alpha(x)$ falls off exponentially as $|x| \to \infty$, and by scaling in the region $0 \le t \le 1$, we find that for $|x| \le \frac{1}{2}$,

$$H_\alpha(x) \le \begin{cases} c|x|^{-\alpha}, & \alpha > 0, \\ c \log\left(\dfrac{1}{|x|}\right), & \alpha = 0, \\ c, & \alpha < 0. \end{cases}$$

We note that for $0 \le \beta \le 1$,

$$\left|e^{-(x+z)^2/2t} - e^{-x^2/2t}\right| \le c|z|^\beta t^{-\beta/2}\left(e^{-(x+z)^2/2t} + e^{-x^2/2t}\right), \tag{4.4}$$

as follows by arguing separately for $|z|^2 \le t$ and $|z|^2 \ge t$.

From this it follows that

$$\begin{aligned} &|H_\alpha * f_\varepsilon(x) - H_\alpha * f_{\bar\varepsilon}(x)| \\ &\quad\le \int f(y)\,dy\left(\int_0^\infty \frac{e^{-t}}{t^{(\alpha/2)+1}}\left|e^{-(x-\varepsilon y)^2/2t} - e^{-(x-\bar\varepsilon y)^2/2t}\right| dt\right) \\ &\quad\le c|\varepsilon - \bar\varepsilon|^\beta\left(H_{\alpha+\beta} * f_\varepsilon(x) + H_{\alpha+\beta} * f_{\bar\varepsilon}(x)\right). \end{aligned} \tag{4.5}$$

We can easily check that the integrals (3.2)–(3.7) which were uniformly bounded in ε in dimensions 4, 5 will remain uniformly bounded if we replace the factors G, $G * G$ and $G * G * G$ which are of the form H_α, by corresponding factors of the form $H_{\alpha+\beta}$ with β small. Then (4.5) allows us to extract a factor $|\varepsilon - \bar\varepsilon|^\beta$. □

5. Proof of Theorem 2(c): Combinatorial aspect. Our proof is by the method of moments.

Recall that

$$\gamma_\varepsilon = \int_0^\infty\int_0^\infty \langle f_\varepsilon(x-y), X_s(dx)X_t(dy)\rangle\,ds\,dt - 2c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\,ds, \tag{5.1}$$

where

$$\begin{aligned} c(\varepsilon) &= \int f_\varepsilon(x) G * G(x)\,dx \\ &= \frac{1}{4\pi^3}\frac{1}{\varepsilon^2}\int \frac{f(y)}{y^2}\,dy - \frac{1}{2\pi^3}\log\left(\frac{1}{\varepsilon}\right) + O(1). \end{aligned} \tag{5.2}$$

By (2.10) we know that

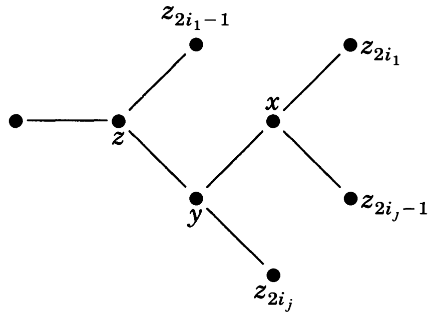
$$E_\mu(\gamma_\varepsilon^{2m}) \tag{5.3}$$

is a sum of contributions from the graphs of C_{4m}, that is, the set of binary graphs with $4m$ labelled exits, $1, 2, \ldots, 4m$ with no twin exits coupled; that is, no twin exits are ever labelled $2i-1, 2i$ for any i.

The basic idea which we explain in this and the next section is that the dominant contribution to (5.3) comes from graphs which effectively break (5.3) up into a product of m second moments.

Let $A_{4m} \subset C_{4m}$ denote those binary graphs in C_{4m} for which there is a complete pairing $(i_1, j_1), \dots, (i_m, j_m)$ of the $2m$ integers $1, 2, \dots, 2m$ and such that for each such pair (i_l, j_l) the exits labelled $2i_l - 1, 2i_l, 2i_l - 1, 2j_l$ are arranged as in Graph 2 of Section 3:

(5.4)



or one of its 4^3 variants as described at the end of Section 3.

We will see later that the dominant contribution to (5.3) comes from the graphs in A_{4m} and is of order $\log^{2m}(1/\varepsilon)$, while any other graph in C_{4m} will give a contribution which is $O(\log^{2m-1}(1/\varepsilon))$.

Let us compute the contribution from the graphs in A_{4m}. Consider the subgraph (5.4). The partial integral with respect to

$$dx\, dy\, dz_{2i_{l-1}}\, dz_{2i_l}\, dz_{2j_{l-1}}\, dz_{2j_{l-1}}$$

is described in (3.3). It is crucial that this partial integral is independent of z (a consequence of the translation invariance of Brownian motion), and is simply the constant [see (3.11)]

$$\begin{aligned} J(\varepsilon) &= \int G(x) G * G * f_\varepsilon(x) G * G * G * f_\varepsilon(x)\, dx \\ &= \frac{1}{2} \frac{1}{4^3 \pi^6} \log^2\left(\frac{1}{\varepsilon}\right) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right). \end{aligned} \tag{5.5}$$

As we saw at the end of Section 3, there are 4^3 variants of (5.4). Thus the partial integration corresponding to all m pairs (i_l, j_l) and all the 4^3 variants for each pair gives rise to the factor

$$\left(\frac{1}{2\pi^6} \log^2\left(\frac{1}{\varepsilon}\right)\right)^m + O\left(\log^{2m-1}\left(\frac{1}{\varepsilon}\right)\right). \tag{5.6}$$

After this partial integration, we are simply left with a binary graph with m exits. Since any graph in D_m can arise in this fashion and since there are $(2m)!/m!2^m$ ways to pair the integers $1, 2, \dots, 2m$, we see that [see (2.4)] the contribution to (5.3) from A_{4m} is

$$\frac{(2m)!}{m!2^m} \left(\frac{1}{2\pi^6} \log^2\left(\frac{1}{\varepsilon}\right)\right)^m E_\mu\left(\left(\int_0^\infty \langle 1, X_s\rangle\, ds\right)^m\right) + O\left(\log^{2m-1}\left(\frac{1}{\varepsilon}\right)\right). \tag{5.7}$$

We will show in the next section that the contribution of all graphs in $C_{4m} \setminus A_{4m}$ is $O(\log^{2m-1}(1/\varepsilon))$. This will give

$$E_\mu\left[\left(\frac{\gamma_\varepsilon}{\log(1/\varepsilon)}\right)^{2m}\right] \to \frac{(2m)!}{m!}\left(\frac{1}{4\pi^6}\right)^m E_\mu\left(\left(\int_0^\infty \langle 1, X_s\rangle\, ds\right)^m\right) \quad \text{as } \varepsilon \to 0. \tag{5.8}$$

Furthermore, the next section will show that

$$E_\mu\left[\left(\frac{\gamma_\varepsilon}{\log(1/\varepsilon)}\right)^{2m-1}\right] \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.9}$$

Let M_{2m} denote the right-hand side of (5.8). We now show that for $|\lambda|$ small,

$$\sum_{m=0}^\infty \frac{\lambda^{2m} M_{2m}}{(2m)!} = \exp\left(|\mu|\frac{1}{2}\left(1 - \sqrt{1 - \frac{\lambda^2}{\pi^6}}\right)\right). \tag{5.10}$$

This will show at once that any limit distribution of $\gamma_\varepsilon/\log(1/\varepsilon)$ is determined by its moments, hence is unique, and will also show that its Laplace transform is given by (5.10), which will establish Theorem 2(c).

We give a simple combinatoric argument for (5.10). Let us calculate the moments

$$E_\mu\left(\left(\int_0^\infty \langle 1, X_s\rangle\, ds\right)^m\right) \tag{5.11}$$

via (2.4). We integrate successively over the exit and internal variables, using

$$\int G(x)\, dx = 1,$$

so that (5.11) is equal to

$$\sum_{D_m} \int \prod_{v \in V_-} \mu(dy_v) = \sum_{r=1}^m |D_{m,r}||\mu|^r, \tag{5.12}$$

where $D_{m,r}$ denotes the set of labelled binary graphs with m exits and r entrances, and $|A|$ denotes the cardinality of a set A. Hence

$$\sum_{m=0}^\infty \frac{\lambda^{2m}}{(2m)!} M_{2m} = 1 + \sum_{m=1}^\infty \frac{1}{m!} \sum_{r=1}^m |\mu|^r |D_{m,r}| \left(\frac{\lambda^2}{4\pi^6}\right)^m. \tag{5.13}$$

Let $d(n) = |D_{n,1}|$, the number of binary graphs with n labelled exits and one entrance. We have

$$|D_{m,r}| = \frac{1}{r!} \sum_{i_1, \dots, i_r \neq 0} \binom{m}{i_1, i_2, \dots, i_r} d(i_1) \cdots d(i_r) \tag{5.14}$$

(since components of $D_{m,r}$ corresponding to different entrances are unordered).

Hence

$$\sum_{m=0}^{\infty}\frac{\lambda^{2m}}{(2m)!}M_{2m}$$

$$\begin{aligned}
&= 1+\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\lambda^2}{4\pi^6}\right)^m\sum_{r=1}^{m}\frac{|\mu|^r}{r!}\sum_{i_1,\dots,i_r\neq 0}\binom{m}{i_1,\dots,i_r}d(i_1)\cdots d(i_r)\\
&= 1+\sum_{m=1}^{\infty}\sum_{r=1}^{m}\frac{|\mu|^r}{r!}\sum_{\substack{i_1,\dots,i_r\neq 0\\ \Sigma i_j=m}}\prod_{j=1}^{r}\left(\frac{\lambda^2}{4\pi^6}\right)^{i_j}\frac{1}{i_j!}d(i_j)\\
(5.15)\qquad &= 1+\sum_{r=1}^{\infty}\frac{|\mu|^r}{r!}\sum_{m=r}^{\infty}\sum_{\substack{i_1,\dots,i_r\neq 0\\ \Sigma i_j=m}}\prod_{j=1}^{r}\left(\frac{\lambda^2}{4\pi^6}\right)^{i_j}\frac{1}{i_j!}d(i_j)\\
&= 1+\sum_{r=1}^{\infty}\frac{|\mu|^r}{r!}\left(\sum_{n=1}^{\infty}\frac{d(n)}{n!}\left(\frac{\lambda^2}{4\pi^6}\right)^n\right)^r\\
&= \exp\left(|\mu|\sum_{n=1}^{\infty}\frac{d(n)}{n!}\left(\frac{\lambda^2}{4\pi^6}\right)^n\right).
\end{aligned}$$

However, it is known that

$$(5.16)\qquad \sum_{n=1}^{\infty}\frac{d(n)}{n!}\left(\frac{\lambda^2}{4\pi^6}\right)^n=\frac{1}{2}\left(1-\sqrt{1-\frac{\lambda^2}{\pi^6}}\right)$$

[$(d(n))/n!$ is the number of unlabelled binary graphs with n exits; see Comtet (1974), page 52 for (5.16)].

(5.15) and (5.16) now establish (5.10). □

6. Proof of Theorem 2(c): Analytic aspect. We recall from (2.10) that

$$(6.1)\qquad \begin{aligned}E_\mu(\gamma_\varepsilon^n) &= \sum_{C_{2n}}\int\prod_{v\in V_-}du(y_v)\prod_{a\in A}G(y_{f(a)}-y_{i(a)})\\ &\quad\times\prod_{v\in V_0}dy_v\prod_{i=1}^{n}f_\varepsilon(z_{2i}-z_{2i-1})\,dz.\end{aligned}$$

In this section, we show that unless $n=2m$ and the graph C is in A_{4m}, then the contribution of C to (6.1) is

$$(6.2)\qquad O\left(\log^{n-1}\left(\frac{1}{\varepsilon}\right)\right).$$

As discussed in Section 5, this will complete the proof of Theorem 2(c).

We can think of the integral in (6.1) as obtained by assigning a factor $G(y_{f(a)}-y_{i(a)})$ to each arrow $a\in A$. We must integrate out all internal variables dy_v, $v \in V_0$, all entrances with respect to $d\mu$ and all exits with $\prod_{i=1}^{n} f_\varepsilon(z_{2i} - z_{2i-1})\, dz_{2i-1}\, dz_{2i}$.

Our approach to (6.2) is to successively integrate out the variables, at each stage replacing the graph C by a different graph C' (not necessarily a directed or binary graph).

The arrows of C' are associated with factors described below, such that the contribution of C is bounded by that of C'. In this process we will be able to associate a factor $O(\log(1/\varepsilon))$ to each f_ε in (6.1) in such a way that these factors will bound all divergences as $\varepsilon \to 0$, and we will show that unless $n = 2m$ and $C \subseteq A_{4m}$, at least one of the factors associated to some f_ε will be $O(1)$.

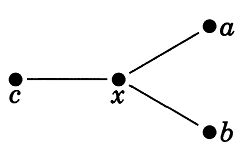
Here are the details:

We begin by integrating the exit variables $z_1, \dots, z_{2n}$. We obtain n factors of the form

$$\begin{aligned}(6.3)\qquad &\int G(a - z_{2i-1})\, f_\varepsilon(z_{2i} - z_{2i-1}) G(b - z_{2i})\, dz_{2i-1}\, dz_{2i} \\ &\quad = G * G * f_\varepsilon(b - a).\end{aligned}$$

We know from the fact that $C \subseteq C_{2m}$, that $a \not\equiv b$. Form a new graph C' obtained by putting an edge between $i(u)$ and $i(v)$ whenever $f(u) = z_{2i-1}$, $f(v) = z_{2i}$; that is, we connect the vertices associated with a, b in (6.3). With this new edge, called a *leading edge*, we associate the factor $G * G * f_\varepsilon$. At this stage, we have not made any estimates so that the contribution of C' is identical to that of C.

Assume that C' has a subgraph of the form

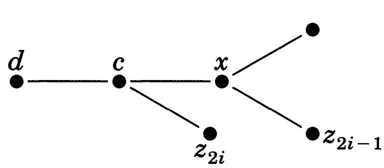
(6.4) 

where (x, a) and (x, b) are both leading edges. We distinguish three possibilities:

(i) $a \equiv c$, or $b \equiv c$ (we cannot have both).
(ii) $a \equiv b$.
(iii) a, b and c are distinct.

We analyze each in turn:

(i) Assume that $b \equiv c$. This can only have occurred if C contained the subgraph

(6.5) 

Since we think of z_{2i}, z_{2i-1} as connected by f_ε, we refer to the situation in (6.5) as a simple loop.

The partial integral over x in this case is

$$\begin{aligned}(6.6)\qquad &\int G(c-x)G*G*f_\varepsilon(c-x)G*G*f_\varepsilon(a-x)\,dx\\ &\quad=\int G(x)G*G*f_\varepsilon(x)G*G*f_\varepsilon(a-c-x)\,dx.\end{aligned}$$

We let $u_{\alpha,\varepsilon}(x)$ denote a generic function which falls off exponentially and monotonically in $|x|$, and such that

$$u_{\alpha,\varepsilon}(x)\le\begin{cases}c|x|^{-\alpha}, & |x|\ge\varepsilon,\\ c\varepsilon^{-\alpha}, & |x|\le\varepsilon.\end{cases}$$

With $u_{0,\varepsilon}$ we associate $\log(1/|x|)$ instead of $|x|^{-\alpha}$.

We know from Lemma 4 that

$$G*G*f_\varepsilon\le u_{2,\varepsilon}.$$

Hence (6.6) is bounded by

$$(6.7)\qquad \int G(x)u_{2,\varepsilon}(x)u_{2,\varepsilon}(a-c-x)\,dx.$$

If $|x|\ge\frac{1}{2}|a-c|$, (6.7) is bounded by

$$(6.8)\qquad u_{2,\varepsilon}(a-c)\int G(x)u_{2,\varepsilon}(a-c-x)\,dx,$$

while if $|x|\le\frac{1}{2}|a-c|$, so that $|a-c-x|\ge|a-c|/2$, (6.7) is bounded by

$$(6.9)\qquad u_{2,\varepsilon}(a-c)\int G(x)u_{2,\varepsilon}(x)\,dx=\log\left(\frac{1}{\varepsilon}\right)u_{2,\varepsilon}(a-c).$$

The integral in (6.8) is easily evaluated by noting that with

$$G^\lambda(x)=\int_0^\infty e^{-\lambda x}\frac{e^{-x^2/2t}}{(2\pi t)^{d/2}}\,dt$$

we have, for some λ small,

$$u_{2,\varepsilon}\le cG^\lambda*G^\lambda*f_\varepsilon$$

and

$$G\le cG^\lambda.$$

Thus the integral in (6.8) is bounded by

$$(6.10)\qquad G^\lambda*G^\lambda*G^\lambda*f_\varepsilon(a-c)=u_{0,\varepsilon}(a-c),$$

by (3.10) and Lemma 4.

In any event, (6.6) is bounded by $\log(1/\varepsilon)u_{2,\varepsilon}(a-c)$. (It is important to recall that we cannot have $a\equiv c$.) We then form a new graph C'', with an edge between the vertices associated with a and c. We consider the factor $\log(1/\varepsilon)$

as associated with $f_\varepsilon(z_{2i} - z_{2i-1})$ and we associate $u_{2,\varepsilon}$ to our new edge, now called a leading edge.

Because (6.5) refers to a binary graph, in C'', aside from our new edge connecting a and c, there is only one other arrow connecting c, with a factor $G(c - d)$. We now integrate

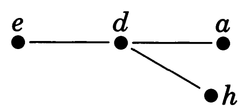
$$\int G(c-d)u_{2,\varepsilon}(a-c)\,dc = u_{0,\varepsilon}(d-a). \tag{6.11}$$

[This integral was already computed in (6.8) and (6.10).]

If $a \equiv d$, we are in the situation of Section 3, Graph 2, that is, our subgraph (6.5) was precisely of the form making up A_{4m}, contributing $\log^2(1/\varepsilon)$, which we associate with the two f_ε factors for that subgraph, which now have no further influence.

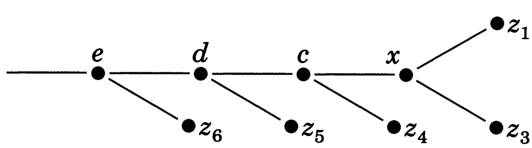
If $a \not\equiv d$, we form a new graph C'' linking a and d, and with the factor $u_{0,\varepsilon}(d-a)$. We have a subgraph

(6.12)



which looks like (6.4), except that instead of the factor $u_{2,\varepsilon}$ associated to (d, a), we have $u_{0,\varepsilon}$.

We will see after analyzing cases (ii) and (iii) that the worst possible case comes from the two-loop subgraph



that is, $h \equiv e$ (so that $a \not\equiv e$) and the partial integral over d is

$$\begin{aligned}\int G(e-d)u_{2,\varepsilon}(e-d)u_{0,\varepsilon}(d-a)d(d)\\ = \int G(x)u_{2,\varepsilon}(x)u_{0,\varepsilon}(a-e-x)\,dx,\end{aligned} \tag{6.13}$$

with $a \not\equiv e$.

As in the analysis of (6.7), we find (6.13) bounded by

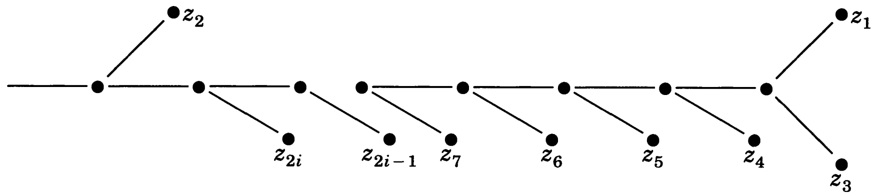
$$\log\left(\frac{1}{\varepsilon}\right)u_{0,\varepsilon}(a-e) + u_{2,\varepsilon}(a-e), \tag{6.14}$$

so that the de integral is

$$O\left(\log\left(\frac{1}{\varepsilon}\right)\right). \tag{6.15}$$

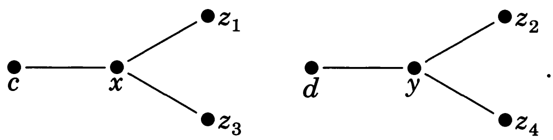
Thus, three factors of f_ε give rise only to a $\log^2(1/\varepsilon)$ contribution [as opposed to $\log^3(1/\varepsilon)$].

In particular, a subgraph of the form

(6.16) 

$\cdots$ with $i - 1$ loops, gives a contribution which is $O(\log^{i-1}(1/\varepsilon))$, unless $i = 2$.

(ii) This case arises from the subgraph

(6.17) 

The partial integral with respect to $z_1, \dots, z_4, x, y$ is

$$\begin{aligned} &\iint G(x-c)(G*G*f_\varepsilon(x-y))^2 G(y-d)\,dx\,dy \\ &\quad = \int G*G(c-d-x)(G*G*f_\varepsilon(x))^2\,dx \\ &\quad \le \int G*G(c-d-x)u_{4,\varepsilon}(x)\,dx \qquad (6.18) \\ &\quad \le c\int G^\lambda * G^\lambda(c-d-x)G^\lambda * f_\varepsilon(x)\,dx \\ &\quad = cG^\lambda * G^\lambda * G^\lambda * f_\varepsilon(c-d) \\ &\quad = u_{0,\varepsilon}(c-d) \end{aligned}$$

as in (6.10).

If $c \equiv d$ (which is the situation of Section 3, Graph 1), we have a $\log(1/\varepsilon)$ for two factors of f_ε, while if $c \not\equiv d$ we have the bound

$$u_{0,\varepsilon}(c-d) \le \log\left(\frac{1}{\varepsilon}\right)u(c-d),$$

where $u(c-d)$ is bounded, and falls off exponentially as $|c-d| \to \infty$. We have a factor $\log(1/\varepsilon)$ for the two f_ε's, a new graph with an edge connecting the vertices associated with c and d and associated factor $u(c-d)$.

(iii) If a, b and c are distinct, the partial x integral is

$$\int G(c-x)G*G*f_\varepsilon(x-a)G*G*f_\varepsilon(x-b)\,dx. \qquad (6.19)$$

If the variable a or b no longer appears in any other factors associated with edges of our graph, we perform the da or db integral. If, for example, we first do the da integration, then (6.19) is bounded by

$$u_{0,\varepsilon}(c-b) \le \log\left(\frac{1}{\varepsilon}\right)u(c-b) \tag{6.20}$$

and as in the discussion of (ii), we associate $\log(1/\varepsilon)$ with two f_ε factors.

If both a and b appear in other factors, we use

$$uv \le \tfrac{1}{2}(u^2+v^2)$$

to bound (6.19) by

$$G * u_{4,\varepsilon}(a-c) + G * u_{4,\varepsilon}(b-c) \le u_{2,\varepsilon}(a-c) + u_{2,\varepsilon}(b-c). \tag{6.21}$$

We now form two new graphs: one with a new edge connecting a and c, with a factor $u_{2,\varepsilon}(a-c)$, and proceed analogously for the other graph.

It suffices to consider the first graph. Notice that the factor $u_{2,\varepsilon}(a-c)$ is the type of factor we obtained from the initial integration over exits—hence we can continue our analysis as if it arose in the latter manner—with the difference that we have actually used up two f_ε factors!

We can now return to the end of our discussion of case (i) and see that indeed the worst possible case for (6.12) is as described there, that is, (6.16).

By iterating (i), (ii) and (iii), applied to leading edges, we see that (6.2) holds. $\square$

7. Proof of Theorem 1. In analogy with (2.10) we find

$$\begin{aligned} E_\mu(\gamma_\varepsilon^n(T)) &= \sum_{C_{2n}} \int \prod_{v\in V_-} \mu(dy_v) \prod_{a\in A} p_{s_{f(a)}-s_{i(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v\in V_0} ds_v\, dy_v \\ &\quad \times 1_{\{t_j\le T,\text{ for all } j\}} \prod_{i=1}^n f_\varepsilon(z_{2i-1} - z_{2i})\, dz_{2i-1}\, dz_{2i}\, dt_{2i-1}\, dt_{2i}, \end{aligned} \tag{7.1}$$

where now

$$p_s(y) = \frac{e^{-(y^2/2s)}}{(2\pi s)^{d/2}}$$

is the transition density for Brownian motion in R^d.

Note that by inserting factors $\exp(-(s_{f(a)} - s_{i(a)})) \ge e^{-T}$, we can bound the contribution to (7.1) of any graph C, by its contribution to (2.10). This immediately shows that if $d = 4, 5$ and $n = 2$, then (7.1) is uniformly bounded in ε, while if $n = 6$, we can bound the contribution of each graph to (7.1) by $c \log^n(1/\varepsilon)$, and in fact, unless $n = 2m$ and our graph belongs to A_{4m}, then its contribution can be bounded by $c \log^{n-1}(1/\varepsilon)$.

The L^2 convergence for $d = 4, 5$ follows easily by using such a domination together with (4.5). The case of $d = 6$ is more subtle.

We consider in detail the contribution of a subgraph of the type described by Graph 2 of Section 3. This contribution is

$$
\begin{aligned}
J(s,\varepsilon) &\doteq \int p_{t_1-s}(z_1 - z) p_{s_1-s}(y - z) p_{s_2-s_1}(x - y) p_{t_2-s_2}(z_2 - x) \\
&\qquad \times p_{t_3-s_2}(z_3 - x) p_{t_4-s_1}(z_4 - y) f_\varepsilon(z_1 - z_2) f_\varepsilon(z_3 - z_4)\, dx\, dy\, dz_i\, dt_i\, ds_j \\
&= \int p_{s_1-s}(y) p_{s_2-s_1}(x-y) p_{t_1-s} * p_{t_2-s_2} * f_\varepsilon(x) p_{t_3-s_2} \\
&\qquad \times * p_{t_4-s_1} * f_\varepsilon(x-y)\, dx\, dy\, dt_i\, ds_j \\
&= \int p_{s_2-s_1}(x) p_{t_3-s_2} * p_{t_4-s_1} * f_\varepsilon(x) p_{s_1-s} * p_{t_1-s} * p_{t_2-s_2} * f_\varepsilon(x)\, dx\, dt_i\, ds_j \\
&= \int p_{s_2-s_1}(x) G_0^{T-s_2} * G_0^{T-s_1} * f_\varepsilon(x) p_{s_1-s} * G_0^{T-s} * G_0^{T-s_2} * f_\varepsilon(x)\, dx\, ds_j,
\end{aligned}
\tag{7.2}
$$

where

$$
G_0^r(x) = \int_0^r p_s(x)\, ds. \tag{7.3}
$$

Recall from (3.11) that

$$
\begin{aligned}
J(\varepsilon) &\doteq \int G(x) G * G * f_\varepsilon(x) G * G * G * f_\varepsilon(x)\, dx \\
&= \frac{1}{64\pi^6} \cdot \frac{1}{2} \log^2\left(\frac{1}{\varepsilon}\right) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right).
\end{aligned}
\tag{7.4}
$$

We now show that for any fixed $\delta > 0$,

$$
J(s,\varepsilon) = J(\varepsilon) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right), \qquad T - s \ge 3\delta. \tag{7.5}
$$

Let

$$
q_t = e^{-t} p_t(x), \qquad G^r(x) = \int_0^r e^{-t} p_t(x)\, dt.
$$

Using

$$
|1 - e^{-t}| \le 2t,
$$

we easily check that

$$
\begin{aligned}
J(s,\varepsilon) &= \int q_{s_2-s_1}(x) G^{T-s_1} * G^{T-s_2} * f_\varepsilon(x) q_{s_1-s} \\
&\qquad \times * G^{T-s} * G^{T-s_2} * f_\varepsilon(x)\, dx\, ds_j + O\left(\log\left(\frac{1}{\varepsilon}\right)\right).
\end{aligned}
\tag{7.6}
$$

Using the bound

$$
q_r(x) \le \frac{c}{\delta^3}
$$

if

$$r \geq \delta,$$

we see that in (7.6) we can assume that the integral is over the region

$$\{T - s_1 \geq 2\delta, T - s_2 \geq \delta\}$$

and using the bound

$$|G(x) - G^r(x)| \leq c\int_r^\infty \frac{1}{t^3}\,dt = \frac{c}{r^2},$$

we see that

$$\int_{\left\{\substack{T-s_1\geq 2\delta \\ T-s_2\geq\delta}\right\}} q_{s_2-s_1}(x) G^{T-s_1} * G^{T-s_2} * f_\varepsilon(x) q_{s_1-s} * G^{T-s} * G^{T-s_2} * f_\varepsilon(x)\,dx\,ds_j$$

$$= \int_{\left\{\substack{T-s_1\geq 2\delta \\ T-s_2\geq\delta}\right\}} q_{s_2-s_1}(x) G * G * f_\varepsilon(x) q_{s_1-s} * G * G * f_\varepsilon(x)\,dx\,ds_j + O\left(\log\left(\frac{1}{\varepsilon}\right)\right)$$

$$(7.7)\qquad = \int_{\{T-s_1\geq 2\delta\}} G^{T-\delta-s_1}(x) G * G * f_\varepsilon(x) q_{s_1-s} * G * G * f_\varepsilon(x)\,dx\,ds_1 + O\left(\log\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \int_{\{T-s_1\geq 2\delta\}} G(x) G * G * f_\varepsilon(x) q_{s_1-s} * G * G * f_\varepsilon(x)\,dx\,ds_1 + O\left(\log\left(\frac{1}{\varepsilon}\right)\right)$$

$$= J(\varepsilon) + O\left(\log\left(\frac{1}{\varepsilon}\right)\right),$$

which proves (7.5).

Let us now consider the contribution $E_\mu^C(\gamma_\varepsilon^{2m}(T))$ of a graph $C \subseteq A_{4m}$ to (7.1). C has m subgraphs of the form we have just considered. Let $s_1, \dots, s_m$ denote the variables corresponding to s in $J(s, \varepsilon)$. Considering first the contribution from $\{T - s_i \geq 3\delta$, for all $i\}$ we find that

$$(7.8)\quad E_\mu^C(\gamma_\varepsilon^{2m}(T)) = \left(J^{2m}(\varepsilon) + O\left(\log^{2m-1}\left(\frac{1}{\varepsilon}\right)\right)\right) E_\mu^{C_0}\left\{\left(\int_0^{T-3\delta} |Z_s|\,ds\right)^m\right\}$$

plus a term which comes from

$$\{T - s_i \geq 3\delta \text{ for all } i\}^c.$$

In (7.8), C_0 is the graph obtained from C by "amputating" the m subgraphs just described.

Since $J(s,\varepsilon) \le c\log^2(1/\varepsilon)$ for some c independent of s_1, we see that the error term to (7.8) coming from $\{T - s_i \ge 3\delta \text{ for all } i\}^c$ is bounded by

$$c\log^{2m}\left(\frac{1}{\varepsilon}\right) E_\mu^{C_0}\left(\left(\int_0^T |Z_s|\,ds\right)^{m-1}\left(\int_{T-3\delta}^T |Z_s|\,ds\right)\right), \tag{7.9}$$

where c is independent of δ.

Combining (7.8) and (7.9) we find that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{E_m^C(\gamma_\varepsilon^{2m}(T))}{\log^{2m}(1/\varepsilon)} &= E_\mu^{C_0}\left(\left(\int_0^T |Z_s|\,ds\right)^m\right)\\ &\quad + O\left\{E_\mu^{C_0}\left(\left(\int_0^T |Z_s|\,ds\right)^{m-1}\left(\int_{T-3\delta}^T |Z_s|\,ds\right)\right)\right\},\end{aligned} \tag{7.10}$$

where $O\{\ \}$ means that we can bound the error term by a multiple (which is independent of δ) of the expectation inside the brackets. Since the left-hand side of (7.10) is independent of δ, we in fact have

$$\lim_{\varepsilon\to 0} E_\mu^C\left(\left(\frac{\gamma_\varepsilon(T)}{\log(1/\varepsilon)}\right)^{2m}\right) = E_\mu^{C_0}\left(\left(\int_0^T |Z_s|\,ds\right)^m\right). \tag{7.11}$$

In a similar manner, we can show that

$$\begin{aligned} &E_\mu\left(\exp\left(-\sum_{i=1}^n \lambda_i\gamma_\varepsilon(t_i)\Big/\log\left(\frac{1}{\varepsilon}\right)\right)\right)\\ &\quad\to E_\mu\left(\exp\left(\frac{1}{4\pi^6}\sum_{i,j=1}^n \lambda_i\lambda_j\int_0^{t_i\wedge t_j}|Z_s|\,ds\right)\right),\end{aligned} \tag{7.12}$$

which shows that the limit process for $\gamma_\varepsilon(t)/\log(1/\varepsilon)$ has the same finite dimensional distributions as B_{Mt} where B_s is a real Brownian motion independent of Z and

$$M_t = \frac{1}{2\pi^6}\int_0^t |Z_s|\,ds. \tag{7.13}$$

To complete the proof of Theorem 1 it suffices to prove tightness for $\gamma_\varepsilon(t)/\log(1/\varepsilon)$; for this we show that for all m,

$$E_\mu\left(\left(\frac{\gamma_\varepsilon(t)-\gamma_\varepsilon(s)}{\log(1/\varepsilon)}\right)^{2m}\right) \le cm\frac{|t-s|^{\alpha m}}{s^{2m}} \tag{7.14}$$

and

$$E_\mu\left(\left(\frac{\gamma_\varepsilon(s)}{\log(1/\varepsilon)}\right)^{2m}\right) \le c_m s^{\alpha m} \tag{7.15}$$

for some $\alpha > 0$, for all s, t on any bounded interval. We can assume $s < t$.

(7.15) follows easily from the fact that all time integrals in (7.1) are now over times less than s, and in our analysis in Section 6, there were a number (at least proportional to m) of factors which we bounded by

$$\int\left(\int_0^s p_r(x,y)\,dr\right)dy = s.$$

To prove (7.14), we note that since $\gamma_\varepsilon(t) - \gamma_\varepsilon(s)$ involves a double integral over $[0,t]^2 - [0,s]^2 = [0,s] \times [s,t] \cup [s,t] \times [0,s] \cup [s,t]^2$, we can easily see from (7.1) and our analysis that it suffices to show that

$$\int_s^t p_r(x)\,dr \le \frac{c|t-s|^\alpha}{s^{1+\alpha}} G\left(\frac{x}{b}\right) \tag{7.16}$$

for some α, b, c. This follows from Hölder's inequality:

$$\begin{aligned}
\int_s^t \frac{e^{-x^2/2r}}{r^2}\,dr &\le |t-s|^{1/p'}\left(\int_s^t \frac{e^{-px^2/2r}}{r^{2p}}\,dr\right)^{1/p} \\
&\le \frac{|t-s|^{1/p'}}{s^{2-1/p}}\left(\int_s^t \frac{e^{-px^2/2r}}{r^2}\,dr\right)^{1/p} \\
&\le \frac{|t-s|^{1/p'}}{s^{1+1/p'}}\left(G(p^{1/2}x)\right)^{1/p} \\
&\le c\frac{|t-s|^\alpha}{s^{1+\alpha}} G\left(\frac{x}{b}\right),
\end{aligned} \tag{7.17}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1. \qquad \square$$

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