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A JOINTLY CONTINUOUS LOCAL TIME FOR TRIPLE INTERSECTIONS OF A STABLE PROCESS IN THE PLANE

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We study the asymptotics of the triple intersection local time $\alpha(x, y, t)$ for the symmetric stable process X_t of index $\beta > \frac{8}{5}$ in $\mathbb{R}^2$, as $x, y \to 0$. This describes the set of "near-intersections", $\{r \le s \le t \le T \mid X_s - X_r = x, \quad X_t - X_s = y\}$. With $U(x) = c(\beta)/|x|^{2-\beta}$ we show that $\alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T)$ has a continuous extension, where $\hat{\alpha}(x, T)$ is a continuous function, the renormalized local time for double intersections.

KEY WORDS: Stable process, triple intersections, local time.

1 INTRODUCTION

Let X_t be a symmetric stable process of index β in the plane, with density

$$p_t(x) = \frac{1}{(2\pi)^2} \int e^{iq \cdot x} e^{-|q|^\beta} \, d^2 q. \tag{1.1}$$

When $\beta = 2$, X_t is a Brownian motion.

We will use the notation $X(r, s) \doteq X_s - X_r$. If $\beta > \frac{4}{3}$, then the random field

$$Y(r, s, t) = (X(r, s), X(s, t))$$

has a local time over any bounded Borel set $B \subseteq R^3_{\le} \doteq \{(r, s, t) \mid 0 \le r \le s \le t\}$, i.e. there exists a function $\alpha(x, y, B)$ such that

$$\int_B f(Y(r, s, t)) \, dr \, ds \, dt = \int f(x, y) \alpha(x, y, B) \, d^2 x \, d^2 y, \tag{1.2}$$

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for all bounded Borel functions $f: \mathbb{R}^4 \to \mathbb{R}$. In addition, when B lies away from both diagonals $r = s$, $s = t$, we can take $\alpha(x, y, B)$ to be a measure in the set variable B, and weakly continuous in x, y. The measure $\alpha(0, 0, \cdot)$ is supported on

$$\{(r, s, t) | X_r = X_s = X_t\}.$$

Because of this, we call $\alpha(x, y, B)$ a triple intersection local time. The results above are essentially due to Shieh [1986], and in Section 7 we shall review them.

In this paper we remove the restriction that B lies away from the diagonals, and study the behavior of $\alpha(x, y, B)$ for arbitrary bounded Borel sets $B \subseteq R^3_{\leq}$. It will turn out that for $\beta > \frac{8}{5}$, we can choose a version of $\alpha(x, y, B)$ which is a measure in B, weakly continuous in x, $y \neq (0, 0)$, see Section 6. In general, $\alpha(x, y, B)$ "blows up" as x, $y \to 0$. Let

$$B_T = \{(r, s, t) | 0 \leq r \leq s \leq t \leq T\}$$

and with $c(\beta) = (1/2^\beta \pi)\Gamma(2 - \beta/2)/\Gamma(\beta/2)$, let

$$U(x) = \int_0^\infty p_t(x)\, dt = \frac{c(\beta)}{|x|^{2-\beta}}, \qquad \beta < 2,$$

the potential of our process. The main result of this paper, established in Section 5, is that with $\alpha(x, y, T) \doteq \alpha(x, y, B_T)$, $\beta > \frac{8}{5}$,

$$\alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T) \tag{1.3}$$

has a continuous extension to all x, y, T. Here $\hat{\alpha}(x, T)$ is the continuous version of the renormalized intersection local time for double intersections, see Rosen (1988). It is defined by the a.s. limit

$$\hat{\alpha}(x, T) = \lim_{\varepsilon \to 0} \int_0^T \int_0^t p_{\varepsilon, x}(X_t - X_s)\, ds\, dt - TU_\varepsilon(x),$$

where $p_{\varepsilon, x}(a) = p_\varepsilon(x - a)$ and $U_\varepsilon(x) = \int_\varepsilon^\infty p_t(x)\, dt$. Thus (1.3) isolates the singularity of $\alpha(x, y, T)$ as x, $y \to 0$. We note that (1.3) has recently been established for the case $\beta = 2$, Brownian motion in the plane, using stochastic integrals, Rosen-Yor [1991].

To establish (1.3) for all $\beta > \frac{8}{5}$ we study the functional

$$I(\varepsilon, x, y, B) = \int_B \{p_{\varepsilon, x}(X(r, s))\}_0 \{p_{\varepsilon, y}(X(s, t))\}_0\, dr\, ds\, dt, \tag{1.4}$$

where for any random variable Z we use the notation $\{Z\}_0 = Z - E(Z)$. In Sections 2 and 3 we will show that with probability 1, $I(\varepsilon, x, y, B)$ has a limit $I(x, y, B)$ as $\varepsilon \to 0$, which we call the renormalized triple intersection local time. We will show that

$I(x, y, T) \doteq I(x, y, B_T)$ is jointly continuous in (x, y, T) and, modulo a regular term, is equal to (1.3).

Along the way we will find a nice approximation to $I(x, y, B_T)$. Let

$$D(2) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 2), \tag{1.4}$$

$$A(k, n, 2) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]_{\le}^2,$$

where

$$[a, b]_{\le}^2 \doteq \{(s, t) | a \le s \le t \le b\},$$

and

$$D(1) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 1),$$

$$A(k, n, 1) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right]_{\le}^2 \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right],$$

We note that $B_1 = D(1) \cup D(2)$. To see this, if $(r, s, t) \in B_1$, take n to be the smallest place in which the dyadic expansions of r, s, and t differ.

We now describe the approximation we will use for $I(x, y, D(2))$. There will be an analogous approximation for $I(x, y, D(1))$. Let

$$D(2, N, M) = \bigcup_{n=1}^{N} \bigcup_{k=1}^{2^{n-1}} A(k, n, M, 2) \tag{1.5}$$

$$A(k, n, M, 2) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left\{\left(\frac{2k-1}{2^n}, \frac{2k-1}{2^n}\right)\right.$$
$$\left. + \bigcup_{m=1}^{M} \bigcup_{l=1}^{2^{m-1}} \frac{1}{2^n} \left[\frac{2l-2}{2^m}, \frac{2l-1}{2^m}\right] \times \left[\frac{2l-1}{2^m}, \frac{2l}{2^m}\right]\right\}$$

In Section 4 we will show that

$$I(x, y, D(2, N, M)) \to I(x, y, D(2))$$

in all L^p spaces, and a.s., as $N, M \to \infty$.

It is a pleasure to thank J.-F. Le Gall for suggesting the basic idea of using scaling.

2 SCALING

The next two sections will be concerned with proving

THEOREM 1 *Let $\beta > \frac{8}{5}$, then for any bounded Borel set $B \subseteq R^3_{\le}$,*

$$|I(\varepsilon, x, y, B) - I(\varepsilon', x', y', B)| \le c\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^{\gamma}, \tag{2.1}$$

for some $\gamma > 0$, locally in x, x', y', y' and ε, $\varepsilon' > 0$, *a.s.* (2.1) *is uniform over all B in any bounded region of $R^3_{\le}$.*

Since $I(\varepsilon, x, y, B)$ is clearly continuous in x, y, (2.1) will guarantee the existence of a limit

$$I(x, y, B) = \lim_{\varepsilon \to 0} I(\varepsilon, x, y, B)$$

which is continuous in x, y. In Section 4 we will discuss the behavior of $I(x, y, B)$ as a function of the set variable B.

Formula (2.1) will come from applying the multiparameter version of Kolmogorov's lemma, Meyer [1980], p. 116 to the bound

$$E(I(\varepsilon, x, y, B) - I(\varepsilon', x', y', B))^m \le C_m\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^{\gamma m}, \tag{2.2}$$

which we now establish for some $\gamma > 0$ independent of m. Here C_m will be independent of the parameters ε, ε', $x'x'$, y, y'.

It suffices to consider $B \subseteq B_1 = D(1) \cup D(2)$, and by symmetry we can assume $B \subseteq D(2)$. Let us define

$$J(\varepsilon, x, y, B) = \int_B p_{\varepsilon,x}(X(r, s))\{p_{\varepsilon,y}(X(s, t))\}_0 \, dr \, ds \, dt \tag{2.3}$$

$$K(\varepsilon, x, y, B) = \int_B E(p_{\varepsilon,x}(X(r, s)))\{p_{\varepsilon,y}(X(s, t))\}_0 \, dr \, ds \, dt \tag{2.4}$$

so that $I = J - K$. We will establish bounds of the form (2.2) separately for J and K, and here scaling will play a crucial role. We work with J, as K will be similar and actually easier.

In the next section we will show that uniformly for all Borel sets $B \subseteq A(1, 1, 2) = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]^2_{\le}$,

$$\|J(\varepsilon, x, y, B) - J(\varepsilon', x', y', B)\|_m \le c_m(\gamma)\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^{\gamma}, \tag{2.5}$$

for all m, and all γ sufficiently small (independent of m). Now use the scaling

$$X_{ct} \sim c^{1/\beta} X_t \quad \text{where} \sim \text{denotes equivalence in} \tag{2.6}$$

law of stochastic processes in t, and

$$p_\varepsilon(y/k) = k^2 p_{\varepsilon k^\beta}(y),$$

to see that in distribution

$$J(\varepsilon, x, y, B) = 2^{-3n} \int_{2^n B} p_{\varepsilon, x}\left(X\left(\frac{r}{2^n}, \frac{s}{2^n}\right)\right)\left\{p_{\varepsilon, y}\left(X\left(\frac{s}{2^n}, \frac{t}{2^n}\right)\right)\right\}_0 \, dr\, ds\, dt \tag{2.7}$$

$$\overset{\text{dist}}{=} 2^{-3n} \int_{2^n B} p_{\varepsilon, x}\left(\frac{X(r, s)}{2^{n/\beta}}\right)\left\{p_{\varepsilon, y}\left(\frac{X(s, t)}{2^{n/\beta}}\right)\right\}_0 \, dr\, ds\, dt$$

$$= 2^{-3n + 4n/\beta} J(2^n \varepsilon, 2^{n/\beta} x, 2^{n/\beta} y, 2^n B).$$

If $B \subseteq D(2)$, write $B(k, n) \doteq B \cap A(k, n, 2)$ so that

$$B = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} B(k, n).$$

Note that $2^{n-1} B(k, n) \subseteq A(1, 1, 2)$ so that using (2.5) *and* (2.7) we have

$$\begin{aligned}
&\|J(\varepsilon, x, y, B) - J(\varepsilon', x', y', B)\|_m \\
&= \left\| \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} J(\varepsilon, x, y; B(k, n)) - J(\varepsilon', x', y', B(k, n)) \right\|_m \\
&\leq \sum_{n=1}^{\infty} \left\| \sum_{k=1}^{2^{n-1}} J(\varepsilon, x, y, B(k, n)) - J(\varepsilon', x', y', B(k, n)) \right\|_m \\
&\leq \sum_{n=1}^{\infty} 2^{n-1/2} \sup_k \|J(\varepsilon, x, y, B(k, n)) - J(\varepsilon', x', y', B(k, n))\|_m \\
&= \sum_{n=1}^{\infty} 2^{-(n-1)(3-4/\beta-1/2)} \sup_k \|J(2^{n-1}\varepsilon, 2^{(n-1)/\beta} x, 2^{(n-1)/\beta} y, 2^{n-1} B(k, n)) \\
&\qquad - J(2^{n-1}\varepsilon', 2^{(n-1)/\beta} x', 2^{(n-1)/\beta} y', 2^{n-1} B(k, n))\|_m \\
&\leq c_m(\gamma)\left(\sum_{n=1}^{\infty} 2^{-(n-1)(3-4/\beta-1/2-\gamma)}\right)\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^\gamma \\
&\leq \bar{c}_m(\gamma)\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^\gamma,
\end{aligned} \tag{2.8}$$

if we can choose $\gamma > 0$ so small that

$$\frac{4}{\beta} + \frac{1}{2} + \gamma < 3,$$

This holds as long as $\beta > \frac{8}{5}$. In the second inequality of (2.8) we used the fact that for fixed n, the $J(\varepsilon, x, y, B(k, n))$, $0 \leq k \leq 2^{n-1}$ are independent with mean zero.

3 THE MAIN ESTIMATE

In this section we prove (2.5). We begin by establishing a bound on

$$E(J(\varepsilon, x, y, B))^m,$$

where J is defined in (2.5) and which is uniform in $x, y, B \subseteq [0, \frac{1}{2}] \times [\frac{1}{2}, 1]^2_\leq$ and $\varepsilon > 0$.

Recalling (1.1) we write.

$$E(J(\varepsilon, x, y, B)^m) = \frac{1}{(2\pi)^{4m}} \int_{B^m} \int F(\varepsilon, x, y, p, q) E(e^{i\Sigma p_j X(r_j, s_j)} \prod \{e^{iq_j X(s_j, t_j)}\}_0)\, dp\, dq\, dr\, ds\, dt \tag{3.1}$$

with

$$F(\varepsilon, x, y, p, q) = e^{ix \cdot \sum p_j + iy \cdot \sum q_j - \varepsilon(\sum p_j^\beta + \sum q_j^\beta)}. \tag{3.2}$$

where here and in the sequel we use the notation p^β for $|p|^\beta$.

First, by independence of increments.

$$E(e^{i\sum p_j \cdot X(r_j, s_j)} e^{i\sum q_j \cdot X(s_j, t_j)}) = E(e^{i\sum p_j \cdot X(r_j, 1/2)}) E(e^{i\sum p_j \cdot X(1/2, s_j) + i\sum q_j \cdot X(s_j, t_j)}) \tag{3.3}$$

To evaluate (3.3) we rewrite

$$\sum p_j X(r_j, 1/2) = \sum_{k=1}^{m} \bar{p}_k \cdot X(\bar{r}_k, \bar{r}_{k+1}),$$

where $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_{m+1}$ are the points $r_1, r_2, \ldots, r_m$, $1/2$ in their natural order, and $\bar{p}_k = \sum_{\{j = r_j \leq \bar{r}_k\}} p_j$. Note that $\{\bar{p}_k\}$ generates $\{p_j\}$ in the sense of linear algebra, i.e. every p_j can be written as a linear combination of the $\bar{p}_k$'s. Similarly, we rewrite

$$\sum p_j X(1/2, s_j) + \sum q_j X(s_j, t_j) = \sum_{i=1}^{2m} u_k \cdot X(d_k, d_{k+1})$$

where $d_1, \ldots, d_{2m+1}$ are the points $1/2, s_1, \ldots, s_n, t_1, \ldots, t_m$ in their natural order and $u_k = \sum_{\{j : s_j > d_k\}} p_j + \sum_{\{l : s_l \leq d_k < t_l\}} q_l$.

We note that $u_1 = \sum_{j=1}^m p_j$, and

$$u_k - u_{k-1} = \begin{cases} q_l - p_l & \text{if} \quad d_k = s_l, \\ -q_l & \text{if} \quad d_k = t_l \end{cases}$$

while $u_{2m+1} = 0$. This shows that $\{u_k\}$ generates $\{p_j\} \cup \{q_l\}$. The expression in (3.3) can now be rewritten as

$$e^{-\sum \bar{p}_j^\beta \tilde{r}_j} e^{-\sum u_j^\beta \tilde{d}_k}, \tag{3.4}$$

here $\tilde{r}_j = \bar{r}_{j+1} - \bar{r}_j$ and $\tilde{d}_k = d_{k+1} - d_k$.

We will call q_l an isolated variable if $[s_l, t_l]$ is of the form $[d_k, d_{k+1}]$ for some k, in which case the interval will be referred to as an isolated interval. In such a case q_l appears as a summand of u_i if and only if $i = k$. We set $I = \{k \mid [d_k, d_{k+1}] \text{ is isolated}\}$.

Of course, (3.3) differs from the expectation in (3.1) in that all brackets have been removed. It will suffice to restrict our attention in (3.1) to integrating over a subset of B^m in which $r_1, \ldots, r_m, s_1, \ldots, s_m, t_1, \ldots, t_m$ have a fixed relative ordering. We restore the brackets of (3.3) to (3.1) in two steps. In the first step we insert all brackets corresponding to isolated variables, which changes (3.4) to

$$e^{-\sum \bar{p}_j^\beta \tilde{r}_j} e^{-\sum_{I^c} u_k^\beta \tilde{d}_k} \prod_{k \in I} \left(e^{-u_k^\beta \tilde{d}_k} - e^{-(u_{k+1}^\beta + q_{l(k)}^\beta)\tilde{d}_k}\right), \tag{3.5}$$

where $q_{l(k)}$ is the isolated variable corresponding to $[d_k, d_{k+1}]$. Thus $u_k = u_{k+1} + q_{l(k)}$, $u_{k-1} = u_{k+1} + p_{l(k)}$.

In the second step we expand the product in (3.1) over all brackets corresponding to nonisolated variables, and obtain many terms-one of which will be (3.5). We will explain in detail how to handle (3.5)—the other terms can be handled similarly—and in fact will be easier.

In finding a bound for the right-hand member of (3.1), we use the absolute value of (3.5) times $|F| \le 1$ as a bound for the integrand, and replace the domain of integration by a convenient super set.

We first discuss the isolated variables. By arguing separately on the region $(u+q)^\beta \ge u^\beta + q^\beta$ and $(u+q)^\beta \le u^\beta + q^\beta$ we see that

$$\int_0^\infty |e^{-t(u+q)^\beta} - e^{-t(u^\beta + q^\beta)}| \, dt \le \left| \frac{1}{(u+q)^\beta} - \frac{1}{u^\beta + q^\beta} \right|.$$

We then integrate with respect to the isolated variable q. For the present we make use of the following bound in the case $\gamma = 0$.

$$\begin{aligned} \int q^\gamma \left| \frac{1}{(u+q)^\beta} - \frac{1}{u^\beta + q^\beta} \right| d^2 q &\le \int_{|q| \le 4|u|} + \int_{|u| \le |q|/4} \\ &\le c u^\gamma \int_{|q| \le 5|u|} \frac{1}{q^\beta} \, d^2 q + c \int_{|q| \ge 4|u|} q^{\gamma - \beta} \frac{|u|}{|q|} \, d^2 q \\ &\le C u^{2 - \beta + \gamma}, \end{aligned} \tag{3.6}$$

for γ sufficiently small ($\gamma = 0$ included).

After bounding in this fashion all terms involving isolated variables, we integrate out the remaining $\tilde{r}$ and $\tilde{d}$ using

$$\int_0^1 e^{-tu^\beta}\,dt \le \frac{c}{1+u^\beta},$$

so that we can bound (3.1) in terms of

$$\int \prod_{j=1}^{m} (1+|\bar{p}_j|)^{-\beta} \prod_{I^c} (1+|u_k|)^{-\beta} \prod_{k\in I} u_{k+1}^{2-\beta}\,dp\,dq', \tag{3.7}$$

where dq' is over the remaining, i.e. nonisolated, variables.

We now prove that (3.7) is finite if $\beta > \frac{8}{5}$.

Let

$$F = \{k \in I^c \mid d_k = s_j \text{ for some } j\}$$

and

$$Q = \{k \in I^c \mid d_k = t_j \text{ for some } j \text{ with } [s_j, t_j] \text{ nonisolated, i.e. } k-1 \in I^c\}.$$

We use $\{u_k\}_A$ to donote the set $\{u_k | k \in A\}$. We can easily verify that $\{\bar{p}_j\} \cup \{u_k\}_F$ generates $\{p_j\} \cup \{q_l'\}$. This is not generally true for $\{\bar{p}_j\} \cup \{u_k\}_Q$. The latter generates $\{p_j\}$ and all $q_{l's}$ which appear as summands in any u_k, $k \in Q$ (see Rosen [1983] for the simple proof). The only q_l's missed are those whose associated interval $[s_l, t_l]$ contains no nonisolated t_j. Let L denote the set of such l. Since $[s_l, t_l]$ is nonisolated, $s_l = d_k$ implies that $d_{k+1} = s_j$ for some j. If $k+1 \in I^c$, set $w_l = u_{k+1}$, while if $k+1 \in I$ set $w_l = 0$. In any event, set $v_l = u_k$. Noting that I^c is the disjoint union of F, Q, and $I+1$, we can bound (3.7) by

$$\int \hat{P}\hat{F}\hat{Q}\hat{I}\,dp\,dq' \tag{3.8}$$

where

$$\hat{P} = \prod_{j=1}^{m} (1+|\bar{p}_j|)^{-\beta},$$

$$\hat{F} = \prod_F (1+|u_j|)^{-\alpha\beta}, \quad \text{where } \alpha = \begin{cases} 2/3 \text{ if } u_j = v_l \text{ or } w_l & \text{for some } l, \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{Q} = \prod_L (1+|v_k|)^{-\beta/3}(1+|w_k|)^{-\beta/3} \prod_Q (1+|u_j|)^{-\beta},$$

$$\hat{I} = \prod_{k\in I} (1+|u_{k+1}|)^{-(2\beta-2)}.$$

We have

$$\int \hat{P}\hat{F}\hat{Q}\hat{I} = \int (\hat{P}^{1/2}\hat{F}^{3/4})(\hat{P}^{1/2}\hat{Q}^{3/4}\hat{I}^{2/3})(\hat{F}^{1/4}\hat{Q}^{1/4}\hat{I}^{1/3})$$
$$\leq \|\hat{P}^{1/2}\hat{F}^{3/4}\|_{5/2}\|\hat{P}^{1/2}\hat{Q}^{3/4}\hat{I}^{2/3}\|_{5/2}\|\hat{F}^{1/4}\hat{Q}^{1/4}\hat{I}^{1/3}\|_{5}. \tag{3.9}$$

The first norm is finite:

$$\int \prod (1 + |\bar{p}_j|)^{-5\beta/4} \prod_F (1 + |u_j|)^{-5\beta/4}\, dp\, dq' < \infty,$$

since the $\{\bar{p}_j\} \cup \{u_l\}_F$ generate and $\frac{5}{4}\beta > \frac{8}{5}\frac{5}{4} = 2$.

For the second norm we note that both $\{p_j\} \cup \{u_l\}_Q \cup \{v_k\}_L$ and $\{p_j\} \cup \{u_l\}_Q \cup \{v_k^*\}_L$ generate $\{p_j\} \cup \{q_l'\}$ where

$$v_k^* = \begin{cases} w_k = u_{k+1} & \text{if} \quad k + 1 \in I^c \\ u_{k+2} & \text{if} \quad k + 1 \in I \end{cases}.$$

We can thus use the Cauchy-Schwartz inequality, after which we proceed as above. We need only note that $\frac{5}{3}(2\beta - 2) > \frac{5}{3} \cdot \frac{6}{5} = 2$.

Finally we note that $\hat{F}\hat{Q} = \prod_{\{k|k-1, k \in I^c\}} (1 + |u_k|)^{-\beta}$, while $\{u_k\}_{k \in I^c}$ generates $\{p_j\} \cup \{q_l'\}$, and as before $\frac{5}{3}(2\beta - 2) > 2$.

Thus we have bounded (3.1) uniformly in ε, $x \cdot y$, $B \subseteq A(1, 1, 2)$. To prove (2.5) we write

$$\|J(\varepsilon, x, y, B) - J(\varepsilon', x', y', B)\|_m \leq \|J(\varepsilon, x, y, B) - J(\varepsilon', x, y, B)\|_m$$
$$+ \|J(\varepsilon', x, y, B) - J(\varepsilon', x', y, B)\|_m + \|J(\varepsilon', x', y, B) - J(\varepsilon', x', y', B\|_m, \tag{3.10}$$

and bound each term separately. The first term in (3.10), raised to the mth power differs from (3.1) in that instead of $F(\varepsilon, x, y, p, q)$ we have

$$F(\varepsilon, \varepsilon', x, y, p, q) = e^{ix \cdot \sum p_j + iy \cdot \sum q_j} \prod_j (e^{-\varepsilon(p_j^\beta + q_j^\beta)} - e^{-\varepsilon'(p_j^\beta + q_j^\beta)}). \tag{3.11}$$

Instead of bounding $|F|$ by 1 we use the bound

$$|e^{-\varepsilon|a|} - e^{-\varepsilon'|a|}| \leq c|\varepsilon - \varepsilon'|^\delta |a|^\delta,$$

for any $0 \leq \delta \leq 1$. Taking δ sufficiently small, using (3.6) and the proof above of the boundedness of (3.1) we find that the first term in (3.10) is bounded by $c(\delta)|\varepsilon - \varepsilon'|^{\delta m}$ for all δ sufficiently small. The remaining terms in (3.10) are handled similarly using

$$|e^{ix \cdot p} - e^{ix' \cdot p}| \leq c|x|^\delta |p|^\delta,$$

for all $0 \leq \delta \leq 1$. This completes the proof of (2.5), and of Theorem 1.

4 CONTINUITY IN THE SET VARIABLE

PROPOSITION 2 *Let $\beta > \frac{8}{5}$ and $B \subseteq D$ (2), then for some $\gamma > 0$*

$$E(J(\varepsilon, x, y, B)^m) \leq C_m(\gamma)|B|^{\gamma m}, \tag{4.1}$$

with $C_m(\gamma)$ independent of B.

Proof As in the proof of Theorem 1, scaling (see Section 2) reduces this to proving (4.1) for all

$$B \subseteq A(1, 1, 2).$$

We proceed as in Section 3, arriving at (3.5). Now, however, we treat isolated intervals differently. Before taking absolute values, we integrate over each isolated variable q: using Parseval for the first equality

$$\begin{aligned}
&\left| \int (e^{-t(u+q)^\beta} - e^{-t(u^\beta + q^\beta)}) e^{iy\cdot q - \varepsilon q^\beta} \, d^2 q \right| \\
&= \left| \int (e^{iu\cdot z} p_t(z) - e^{-tu^\beta} p_t(z)) p_\varepsilon(z-y) \, d^2 z \right| \\
&\leq \int (|1 - e^{iu\cdot z}| + |1 - e^{-tu^\beta}|) p_t(z) p_\varepsilon(z-y) \, d^2 z \\
&\leq c \int (|u|^\delta |z|^\delta + (tu^\beta)^\gamma) p_t(z) p_\varepsilon(z-y) \, d^2 z \\
&\leq c \sup_z (u^\delta z^\delta p_t(z) + u^{\beta\gamma} t^\gamma p_t(z)) \\
&\leq c u^\delta \sup_z \frac{z^\delta}{t^{2/\beta}} p_1(z/t^{1/\beta}) + u^{\beta\gamma} t^\gamma / t^{2/\beta},
\end{aligned} \tag{4.2}$$

where the last inequality uses the scaling (2.6) and the fact that $p_t(z) \leq p_t(0) = (c/t^{2/\beta})$. The last member of (4.2) is bounded by

$$u^\delta \frac{1}{t^{2/\beta - \delta/\beta}} \sup_z \left(\frac{z}{t^{1/\beta}} \right)^\delta p_1\left(\frac{z}{t^{1/\beta}} \right) + u^{\beta\gamma} \frac{1}{t^{2/\beta - \gamma}}.$$

Note that $\sup_z |z| p_1(z) < \infty$ as follows easily from (1.1) and integration by parts. Take δ so that $(2-\delta)/\beta < 1$, i.e. $\delta > 2 - \beta$, and γ so that $2/\beta - \gamma < 1$, i.e. $\beta\gamma > 2 - \beta$. We see that the last member of (4.2) is bounded by $1/t$ to a power slightly less than one, times u to a power slightly larger than $2 - \beta$. Now, in the proof of Section 3, see (3.7), we had $u^{2-\beta}$—but we had power to spare! Hence if, after bounding all isolated variable integrals as above, we apply Holder's inequality in the form (with $h' > 1$, close to 1)

$$\int_A G(r, s, t, p, q')\, dr\, ds\, dt \le |A|^{1/h}\left(\int_A G(r, s, t, p, q')^{h'}\, dr\, ds\, dt\right)^{1/h'},$$

we can then follow the path which follows (3.7) in Section 3. This establishes Proposition 2. □

This quickly leads to a bound for $I(\varepsilon, x, y, B)$ similar to (4.1).

THEOREM 3 *Let $\beta > \frac{8}{5}$, then $I(x, y, B_T)$ is jointly continuous in x, y, T, with probability one.*

Proof Combining (2.2) with the preceding remark, we have

$$E(I(\varepsilon, x, y, B_T) - I(\varepsilon', x', y', B'_T))^m \le c_m(\gamma)\|(\varepsilon, x, y, T) - (\varepsilon', x', y', T')\|^{\gamma m}, \quad (4.3)$$

for all γ sufficiently small and all m, if T, T' are bounded by some fixed number $\tilde{T}$. Our theorem follows from this as before. □

The approximation described in the introduction follows easily from Proposition 2.

THEOREM 4 *If $\beta > \frac{8}{5}$ then*

$$I(x, y, D(2, n, M)) \to I(x, y, D(2))$$

in all L^p-spaces, $p > 1$, and a.s., as N, $M \to \infty$.

5 ASYMPTOTICS OF THE LOCAL TIME

In this section we establish (1.3).

THEOREM 5 *If $\beta > \frac{8}{5}$, then we can find a version of the local time $\alpha(x, y, T)$ such that*

$$\alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T) \quad (5.1)$$

has a jointly continuous extension to all x, y, T.

Proof We set

$$\alpha(\varepsilon, x, y, T) = \int_{B_T} p_{\varepsilon, x}(X(r, s))p_{\varepsilon, y}(X(s, t))\, dr\, ds\, dt,$$

and recall (1.4); then write out

$$
\begin{aligned}
I(\varepsilon, x, y, T) = \alpha(\varepsilon, x, y, T) &- \int_{B_T} E(p_{\varepsilon,x}(X(r, s)))\{p_{\varepsilon,y}(X(s, t))\}_0 \\
&- \int_{B_T} \{p_{\varepsilon,x}(X(r, s))\}_0 E(p_{\varepsilon,y}(X(s, t)))\, dr\, ds\, dt \\
&- \int_{B_T} E(p_{\varepsilon,x}(X(r, s)))E(p_{\varepsilon,y}(X(s, t)))\, dr\, ds\, dt.
\end{aligned} \tag{5.2}
$$

By closure under convolution

$$
E(p_{\varepsilon,x}(X(r, s))) = \int p_\varepsilon(y - x)p_{s-r}(y)\, d^2y = p_{s-r+\varepsilon}(x) \tag{5.3}
$$

so that (5.2) becomes

$$
\begin{aligned}
I(\varepsilon, x, y, T) = \alpha(\varepsilon, x, y, T) &- \int_{B_T} p_{s-r+\varepsilon}(x)\{p_{\varepsilon,y}(X(s, t)\}_0 \\
&- \int_{B_T} \{p_{\varepsilon,x}(X(r, s))\}_0 p_{t-s+\varepsilon}(y) - \int_{B_T} p_{s-r+\varepsilon}(x)p_{t-s+\varepsilon}(y).
\end{aligned} \tag{5.4}
$$

We rewrite the last term as

$$
\int_0^T \left(\int_0^s p_{s-r+\varepsilon}(x)\, dr\right)\left(\int_s^T p_{t-s+\varepsilon}(y)\, dt\right) ds = \int_0^T \left(\int_0^s p_{r+\varepsilon}(x)\, dr\right)\left(\int_0^{T-s} p_{t+\varepsilon}(y)\, dt\right) ds. \tag{5.5}
$$

Now use

$$
\begin{aligned}
\int_0^s p_{r+\varepsilon}(x)\, dr &= U_\varepsilon(x) - U_{s+\varepsilon}(x) \\
\int_0^{T-s} p_{t+\varepsilon}(y)\, dt &= U_\varepsilon(y) - U_{T-s+\varepsilon}(y)
\end{aligned} \tag{5.6}
$$

where $U_r(x) = \int_r^\infty p_t(x)\, dt$ to rewrite (5.5) as

$$
\begin{aligned}
&\int_{B_T} p_{s-r+\varepsilon}(x)p_{t-s+\varepsilon}(y)\, dr\, ds\, dt \\
&\quad = \int_0^T (U_\varepsilon(x) - U_{s+\varepsilon}(x))(U_\varepsilon(y) - U_{T-s+\varepsilon}(y))\, ds
\end{aligned}
$$

$$= TU_\varepsilon(x)U_\varepsilon(y) - U_\varepsilon(x)\int_0^T U_{T-s+\varepsilon}(y) - U_\varepsilon(y)\int_0^T U_{s+\varepsilon}(x)$$

$$+ \int_0^T U_{s+\varepsilon}(x)U_{T-s+\varepsilon}(y). \tag{5.7}$$

It is easily seen that the last term is continuous in all parameters: this follows from

$$U_s(x) \le \int_s^\infty \frac{1}{t^{2/\beta}}\, dt = \frac{1}{s^{2/\beta-1}}, \tag{5.8}$$

and

$$\int_0^T \frac{1}{s^{2/\beta-1}} \frac{1}{(T-s)^{2/\beta-1}}\, ds < \infty, \tag{5.9}$$

similarly we rewrite the second term of (5.4) as

$$\int_0^T \int_0^t \left(\int_0^s p_{s-r+\varepsilon}(X)\, dr \right) \{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt$$

$$= U_\varepsilon(x) \int_0^T \int_0^t \{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt - \int_0^T \int_0^t U_{s+\varepsilon}(x)\{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt$$

$$= U_\varepsilon(x) \int_0^T \int_0^t (p_{\varepsilon,y}(X(s,t)) - p_{t-s+\varepsilon}(y))\, ds\, dt$$

$$- \int_0^T \int_0^t U_{s+\varepsilon}(x)\{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt$$

$$= U_\varepsilon(x)\left(\int_0^T \int_0^t p_{\varepsilon,y}(X(s,t))\, ds\, dt - TU_\varepsilon(y) \right)$$

$$+ U_\varepsilon(x) \int_0^T U_{T-s+\varepsilon}(y)\, ds - \int_0^T \int_0^t U_{s+\varepsilon}(x)\{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt. \tag{5.10}$$

Introduce

$$\hat{\alpha}(\varepsilon, x, T) \doteq \int_0^T \int_0^t p_{\varepsilon,x}(X(s,t))\, ds\, dt - TU_\varepsilon(x)$$

$$\tilde{\alpha}(\varepsilon, x, y, T) = \int_0^T \int_0^t U_{s+\varepsilon}(x)\{p_{\varepsilon,y}(X(s,t))\}_0\, ds\, dt$$

$$\tilde{\tilde{\alpha}}(\varepsilon, x, y, T) = \int_0^T \int_0^s \{p_{\varepsilon,x}(X(r,s))\}_0 U_{T-s+\varepsilon}(y)\, dr\, ds.$$

Using (5.4), (5.7), (5.10) we can write

$$I(\varepsilon, x, y, T) = \alpha(\varepsilon, x, y, T) - TU_\varepsilon(x)U_\varepsilon(y) - U_\varepsilon(x)\hat{\alpha}(\varepsilon, y, T) - U_\varepsilon(y)\hat{\alpha}(\varepsilon, x, T)$$
$$+ \tilde{\alpha}(\varepsilon, x, y, T) + \tilde{\tilde{\alpha}}(\varepsilon, x, y, T) - \int_0^T U_{s+\varepsilon}(x)U_{T-s+\varepsilon}(y)\, ds. \qquad (5.11)$$

We have already seen that the last term is continuous in all parameters. We will show this now for $\hat{\alpha}$ and $\tilde{\tilde{\alpha}}$ which will complete the proof of Theorem 5—once we identify $\lim_{\varepsilon\to 0} \alpha(\varepsilon, x, y, T)$ as the local time of y.

We concentrate on $\tilde{\alpha}$. The proof will be similar to that of Theorem 3. We introduce

$$\tilde{\alpha}(\varepsilon, x, y, B) = \int_B U_{s+\varepsilon}(x)\{p_{\varepsilon,y}(X(s, t))\}_0\, ds\, dt,$$

and using Section 2 we have the scaling

$$\tilde{\alpha}(\varepsilon, x, y, B) = 2^{-2n} \int_{2^n B} U_{s/2^n+\varepsilon}(x)\{p_{\varepsilon,y}(X(s/2^n, t/2^n))\}\, ds\, dt$$
$$\overset{\text{dist}}{=} 2^{-2n} \int_{2^n B} U_{s/2^n+\varepsilon}(x)\left\{p_{\varepsilon,y}\left(\frac{X(s,t)}{2^{n/\beta}}\right)\right\}_0 ds\, dt$$
$$= 2^{-3n+4n/\beta}\tilde{\alpha}(2^n\varepsilon, 2^{n/\beta}x, 2^{n/\beta}y, 2^n B).$$

Writing

$$\{0 \le s \le t \le 1\} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n),$$

with

$$A(k, n) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right],$$

we see exactly as before that it sufficies to show for some $\gamma > 0$

$$E(\tilde{\alpha}(\varepsilon, x, y, B) - \tilde{\alpha}(\varepsilon', x', y', B))^m \le c_m(\gamma)\|(\varepsilon, x, y) - (\varepsilon', x', y')\|^{\gamma m} \qquad (5.12)$$

and

$$E(\tilde{\alpha}(\varepsilon, x, y, B)^m) \le c_m(\gamma)|B|^{m\gamma}, \qquad (5.13)$$

for all $B \subseteq A(1, 1) = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. This is much simpler than Section 3 since we do not have to worry about isolated intervals.

A typical integral is of the form

$$\iint \prod_{i=1}^{n} U_{s_i}(x) e^{-\sum \tilde{p}_i^\beta \tilde{s}_i} e^{-\sum \tilde{\tilde{p}}_i^\beta \tilde{t}_i}\, d^2p\, ds\, dt,$$

where $\{\tilde{p}_i\}$, $\{\tilde{\tilde{p}}_i\}$ are linear combinations of the p_j, and each generates $\{p_j\}$. We first apply the Cauchy-Schwartz inequality in dp, then use

$$U_s(x) \leq \frac{1}{s^{2/\beta - 1}}$$

to bound the integral displayed above by

$$\begin{aligned}
&c \int_{[0,T]} \prod_{i=1}^{m} \frac{1}{s_i^{2/\beta-1}} \frac{1}{\tilde{s}_i^{1/\beta}} \frac{1}{\tilde{t}_i^{1/\beta}}\, ds\, dt \\
&\leq c \int_{[0,T]^n} \prod_{i=1}^{m} \frac{1}{s_i^{2/\beta-1}} \frac{1}{\tilde{s}_i^{1/\beta}}\, ds, \quad (\text{since } \beta > 1) \\
&\leq c \left[\int_{[0,T]^n} \left(\prod_{i=1}^{m} \frac{1}{s_i^{2/\beta-1}} \right)^3 ds \right]^{1/3} \left[\int_{[0,T]^n} \left(\prod_{i=1}^{m} \frac{1}{\tilde{s}_i^{1/\beta}} \right)^{3/2} ds \right]^{2/3} < \infty
\end{aligned}$$

since

$$3(2/\beta - 1) < 3\left(\frac{10}{8} - 1\right) = \frac{6}{8} < 1$$

and

$$\frac{3}{2}\frac{1}{\beta} < \frac{15}{16} < 1.$$

With these ideas (5.12) and (5.13) are easily proven following the argument in Section 3. $\tilde{\tilde{\alpha}}$ is handled similarly.

We now use (5.11) to identify $\alpha(x, y, T) \doteq \lim_{\varepsilon \to 0} \alpha(\varepsilon, x, y, T)$ as the local time of $Y(r, s, t)$. Let $f(x, y)$: $\mathbb{R}^4 \to \mathbb{R}$ be a continuous function with compact support away from the hyperplanes $x = 0$ and $y = 0$.

All terms in (5.11) except $\alpha(\varepsilon, x, y, T)$ are known to converge locally uniformly in x, y away from $x = 0$ and $y = 0$—hence this must also be true for $\alpha(\varepsilon, x, y, T)$ (the above fact for $\hat{\alpha}(\varepsilon, x, T)$ is proven in Rosen [1987]). Thus we can integrate $f\, dx\, dy$

and interchange the $\varepsilon \to 0$ limit. Therefore

$$\begin{aligned}\int f(x, y)\alpha(x, y, T)\, dx\, dy &= \lim_{\varepsilon\to 0} \int f(x, y)\alpha(\varepsilon, x, y, T)\, dx\, dy \\ &= \lim_{\varepsilon\to 0} \int_{B_T} \left(\int p_{\varepsilon, x}(X(r, s)) p_{\varepsilon, y}(X(s, t)) f(x, y)\, dx\, dy \right) dr\, ds\, dt \\ &= \lim_{\varepsilon\to 0} \int_{B_T} p_\varepsilon^* f(X(r, s), X(s, t))\, dr\, ds\, dt \\ &= \int_{B_T} f(X(r, s), X(s, t))\, dr\, ds\, dt \end{aligned} \tag{5.14}$$

where

$$p_\varepsilon^* f(a, b) = \int p_\varepsilon(x - a) p_\varepsilon(y - b) f(x, y)\, dx\, dy \to f(a, b) \quad \text{uniformly.}$$

Comparing with (1.2) we see that $\alpha(x, y, T) = \lim_{\varepsilon\to 0} \alpha(\varepsilon, x, y, T)$ is a version of the local time for $x, y \to 0$. However, it is clear from (5.11) that $\alpha(x, y, T)$ has an integrable singularity along the hyperplanes—hence $\alpha(x, y, T)$ is a version of the local time for all x, y, T.

6 PROPERTIES OF THE LOCAL TIME $\alpha(x, y, \cdot)$

In this section we will prove the following theorem.

THEOREM 6 *Let $\beta > \frac{8}{5}$, then we can choose a version of the local time $\alpha(x, y, B)$ with the following properties holding simultaneously with probability* 1.

i) $\alpha(x, y, \cdot)$ *is a measure for each* $x, y \neq 0$

ii) $\alpha(x, y, \cdot)$ *is weakly continuous in* $x, y \neq 0$

iii) $\alpha(x, y, \cdot)$ *is supported on* $\{(r, s, t) | X_s - X_r = x, X_t - X_s = y\}$ *for* $x, y \neq 0$

Proof As before we take $\alpha(x, y, B) \doteq \lim_{\varepsilon\to 0} \alpha(\varepsilon, x, y, B)$ for B a rectangle. We will show that this can be extended to a measure with all the properties of Theorem 6. We identify a rectangle

$$B = [a, a'] \times [b, b'] \times [c, c']$$

with its end points—so we often write B for (a, a', b, b', c, c'). Using Theorem 1 and Proposition 2 we verify that for all rectangles $B, B' \subseteq B_1$,

$$\|I(\varepsilon, x, y, B) - I(\varepsilon', x', y', B')\|_m \leq c_m(\gamma) \|(\varepsilon, x, y, B) - (\varepsilon', x', y', B')\|^\gamma, \tag{6.1}$$

for some $\gamma > 0$, and all m. Following the steps in Section 5, and using the results of Rosen [1988] on $\hat{\alpha}(x, \cdot)$ we find that for x, y away from zero $\alpha(\varepsilon, x, y, B)$ satisfies an inequality similar to (6.1), hence via Kolmogorov's lemma, Meyer [1980], p. 116 we have locally

$$|\alpha(\varepsilon, x, y, B) - \alpha(\varepsilon', x', y', B')| \leq c|(\varepsilon, x, y, B) - (\varepsilon', x', y', B')|^{\gamma} \tag{6.2}$$

with probability one for all rational ε, $\varepsilon' > 0$, B, B' and x, x', y, y' away from zero. Using the continuity of $\alpha(\varepsilon, x, y, B)$ for $\varepsilon \neq 0$ we see that in fact (6.2) is true for all B, B', ε, $\varepsilon' > 0$ and x, x', y, y' away from zero. Hence defining α as above, we have locally.

$$|\alpha(x, y, B) - \alpha(x', y', B')| \leq c\|(x, y, B) - (x', y', B')\|^{\gamma} \tag{6.3}$$

for all x, y, x', y' away from 0.

For such x, y this shows that $\alpha(x, y, \cdot)$ can be extended to a measure, also denoted $\alpha(x, y, \cdot)$ and by the argument in Section 5 that this is a local time, i.e. will satisfy (1.2). This proves (i). (ii) is easy using (6.3). Finally to prove (iii) we use an argument of Shieh [1986]. See also LeGall, Rosen and Shieh [1989]. Let $D_\delta = \{(r, s, t) | X_{s\pm} - X_{r\pm} - x| > \delta$ and $|X_{t\pm} - X_{s\pm} - y| > \delta$ simultaneously for all possible choices of the $\pm$'s$\}$. Of course, D_δ is open, and the definition of $\alpha(\varepsilon, x, y, D_\delta)$ shows that for any $\eta > 0$,

$$\alpha(\varepsilon, x, y, D_\delta) \leq \eta \quad \text{for } \varepsilon \text{ small.}$$

By property (ii), weak continuity, this will also hold for $\alpha(x, y, D_\delta)$—hence

$$\alpha(x, y, D_\delta) = 0.$$

Thus $\alpha(x, y, \cdot)$ is supported on $(\bigcup_\delta D_\delta)^c$, hence on the set (r, s, t) such that $X_{s\pm} - X_{r\pm} = x$, $X_{t\pm} - X_{s\pm} = y$ for at least one choice of $\pm$'s. However, it is clear from (6.3) that $\alpha(x, y, \cdot)$ has no hyperplane mass—and since x has only a countable number of discontinuities, $\alpha(x, y, \cdot)$ is in fact supported by

$$\{(r, s, t) | X_s - X_r = x, X_t - X_s = y\}. \qquad \square$$

7 LOCAL TIME AWAY FROM THE DIAGONAL

In the introduction we mentioned various properties of the local time which hold for $\beta > \frac{4}{3}$. These are much easier to verify than the properties we have been dealing with until now, but since they do not appear explicitly in the literature we will indicate the proofs.

THEOREM 7 *If $\beta > \frac{4}{3}$, the $Y(r, s, t)$ has a local time over any bounded Borel set $B \subseteq R^3_{\leq}$.*

Proof The proof is similar to Rosen [1983], and goes back to Berman [1973]. We use Fourier analysis to show that the measure $u_B(\cdot)$ defined by $u_B(A) =$