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SOBOLEV INEQUALITIES FOR WEIGHT SPACES AND SUPERCONTRACTIVITY

BY

JAY ROSEN(1)

ABSTRACT. For $\phi \in C^2(\mathbf{R}^n)$ with $\phi(x) = a|x|^{1+s}$ for $|x| \geqslant x_0$, $a, s > 0$, define the measure $d\mu = \exp(-2\phi)d^nx$ on $\mathbf{R}^n$. We show that for any $k \in \mathbf{Z}^+$

$$\int |f|^2 |\lg(|f|)|^{2sk/(s+1)}\, d\mu \leqslant c\left\{\sum_{|\alpha|=0}^{k} \|D^\alpha f\|^2_{L_2(d\mu)} + \|f\|^2_{L_2(d\mu)} \cdot |\lg(\|f\|_{L_2(d\mu)})|^{2sk/(s+1)}\right\}$$

As a consequence we prove $e^{-t\nabla^*\cdot\nabla}: L_q(\mathbf{R}^n, d\mu) \longrightarrow L_p(\mathbf{R}^n, d\mu)$, $p, q \neq 1, \infty$, is bounded for all $t > 0$.

1. **Introduction.** The classical Sobolev inequalities state

$$\|f\|_p \leqslant c \sum_{|\alpha|=k} \|D^\alpha f\|_q, \qquad f \in C_0^\infty(\mathbf{R}^n), \tag{1}$$

where $p = (1/q - k/n)^{-1}$, $1 \leqslant p < \infty$, α is an n-tuple, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$ [14].

Recently, L. Gross has proven a beautiful analogue of the Sobolev inequalities for the Gaussian measure $d\nu = (2\pi)^{-n/2}\exp(-|x|^2/2)d^nx$ on $\mathbf{R}^n$ [1]. This "logarithmic" Sobolev inequality states

$$\begin{aligned}(2\pi)^{-n/2}&\int |f|^2 \lg(|f|)\exp(-|x|^2/2)\, d^nx \\ &\leqslant \sum_{i=1}^{n} \left\|\frac{\partial f}{\partial x_i}\right\|^2_{L_2(d\nu)} + \|f\|^2_{L_2(d\nu)} \cdot \lg(\|f\|_{L_2(d\nu)}).\end{aligned} \tag{2}$$

Furthermore, Gross has exhibited a function $f \in L_2(d\nu)$ with $\sum_{i=1}^n \|\partial f/\partial x_i\|^2_{L_2(d\nu)} < \infty$ but

$$\int |f|^2 \lg(|f|)\lg^+(\lg^+(|f|))\exp(-|x|^2/2)\, d^nx = \infty,$$

showing how good his inequality (2) is. Similar, higher order inequalities for the

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Gaussian measure have been proved by G. Feissner [2].

If $\phi \in C(\mathbf{R}^n)$ with $\int \exp(-2\phi)\, d^n x < \infty$ let us define the weight space $L_2^{(k)}(\mathbf{R}^n, \phi)$ to be the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm $\Sigma_{|\alpha|=0}^k \|D^\alpha f\|_{2,\phi}$, where

$$\|g\|_{2,\phi}^2 = \frac{\int |g|^2 \exp(-2\phi)\, d^n x}{\int \exp(-2\phi)\, d^n x}.$$

The main aim of this paper is to develop a method for obtaining precise Sobolev inequalities for a large class of weights ϕ.

To illustrate our results assume $\phi \in C^2(\mathbf{R}^n)$, with $\phi = a|x|^{1+s}$ for large $|x| \geqslant x_0$; $a > 0$, $s > 0$. We will show that

$$\int_{\mathbf{R}^n} |f|^2 |\lg(|f|)|^{2ks/(s+1)} \exp(-2\phi)\, d^n x$$

$$\leqslant c \left\{ \sum_{|\alpha|=0}^{k} \|D^\alpha f\|_{2,\phi}^2 + \|f\|_{2,\phi}^2 |\lg(\|f\|_{2,\phi})|^{2ks/(s+1)} \right\}, \tag{3}$$

$$f \in L_2^{(k)}(\mathbf{R}^n, \phi).$$

This result is best possible in the sense that for any $m \in \mathbf{Z}^+$ we exhibit $f \in L_2^{(k)}(\mathbf{R}^n, \phi)$ with

$$\int_{\mathbf{R}^n} |f|^2 |\lg(|f|)|^{2ks/(s+1)} \overbrace{\lg^+(\cdots \lg^+(|f|) \cdots)}^{m} \exp(-2\phi)\, d^n x = \infty. \tag{4}$$

L. Gross has also shown [1] how 'logarithmic' Sobolev inequalities can be used to prove that $e^{-t\nabla^* \cdot \nabla}$, $t > 0$, is a hypercontractive semigroup. Recall that a selfadjoint contraction semigroup e^{-tF} on a probability space $(M, d\mu)$ is called hypercontractive if $e^{-tF}: L_q \to L_p$ is bounded for $p, q \neq 1, \infty$ and $t \geqslant t(p, q)$ [3]. In particular E. Nelson has shown [4] that for the Gaussian measure $d\nu = (2\pi)^{-n/2} \exp(-|x|^2/2) d^n x$ on $\mathbf{R}^n$,

$$e^{-t\nabla^* \cdot \nabla}: L_q(\mathbf{R}^n, d\nu) \longrightarrow L_p(\mathbf{R}^n, d\nu)$$

is bounded, $p, q \neq 1, \infty$, *only* if $t \geqslant \lg([(p-1)/(q-1)]^{1/2})$, in which case it is a contraction. Using our precise Sobolev inequalities, together with Gross's theorem, we show that for a large class of weights ϕ,

$$e^{-t\nabla^* \cdot \nabla}: L_q(\mathbf{R}^n, \phi) \longrightarrow L_p(\mathbf{R}^n, \phi),$$

$p, q \neq 1, \infty$, is bounded for *all* $t > 0$! We call this property of the semigroup $e^{-t\nabla^* \cdot \nabla}$ supercontractivity.

We note that J.-P. Eckmann [5] has independently extended Gross's methods to prove that $e^{-t\nabla^*\cdot\nabla}$ is hypercontractive for many weights ϕ. We have been able to push his technique to prove supercontractivity, but it is not powerful enough to prove our precise Sobolev inequalities.

In §§2 and 3 we prove our basic Sobolev inequalities. Supercontractivity is proven in §4. In §5 we describe some weights which satisfy the general requirements of our theorems, and we show that in many cases our results are best possible.

We remark that our inequalities have also been used to determine the fine fluctuations of paths in the $P(\phi)_1$ Markoff processes [9].

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2. **First order inequalities.** Throughout this paper we assume $\phi \in C^2(\mathbf{R}^n)$ with $\int \exp(-2\phi)\,d^nx < \infty$.

THEOREM 1. *Let $r > 0$ be such that*

$$|\phi(x)|^r \leqslant a(\nabla\phi \cdot \nabla\phi - \Delta\phi + b); \tag{5}$$

then

$$\begin{aligned}\int |f|^2 |\lg(|f|)|^r \exp(-2\phi)\,d^nx \\ \leqslant c\left\{\sum_{i=1}^n \left\|\frac{\partial f}{\partial x_i}\right\|_{2,\phi}^2 + \|f\|_{2,\phi}^2 + \|f\|_{2,\phi}^2 \cdot |\lg(\|f\|_{2,\phi})|^r\right\},\end{aligned} \tag{6}$$

$$f \in L_2^{(1)}(\mathbf{R}^n, \phi).$$

If, in addition, $r \geqslant 1$, then

$$\begin{aligned}\int |f|^2 \lg(|f|)\exp(-2\phi)\,d^nx \\ \leqslant c(f, (\nabla^* \cdot \nabla + 1)^{1/r} f) + \|f\|_{2,\phi}^2 \lg(\|f\|_{2,\phi}),\end{aligned} \tag{7}$$

$$f \in Q((\nabla^* \cdot \nabla)^{1/r})$$

and

$$\int |f|^2 \lg(|f|)\exp(-2\phi)\,d^nx \leqslant c\left(\sum_{i=1}^n \left\|\frac{\partial f}{\partial x_i}\right\|_{2,\phi}^{2/r} + 1\right), \tag{8}$$

$$f \in L_2^{(1)}(\mathbf{R}^n, \phi),\ \|f\|_{2,\phi} = 1.$$

PROOF. We may assume $\int \exp(-2\phi)\, d^n x = 1$. We will first prove our theorem for all f such that $\|f\|_{2,\phi} = 1$. For such an f we have

$$\int \exp(\lg^+(|f|^2)) \exp(-2\phi) d^n x \leqslant 2. \tag{9}$$

Setting $h = (\lg^+(|f|^2))^r \geqslant 0$ we can write this as

$$\int \exp(h^{1/r} - 2\phi) d^n x \leqslant 2. \tag{10}$$

Let

$$U = \{x \in \mathrm{R}^n \,|\, h^{1/r} - 2\phi \leqslant 0\}, \qquad V = U^c = \{x \in \mathrm{R}^n \,|\, h^{1/r} - 2\phi > 0\}.$$

Since $\exp(\cdot) \geqslant 0$, (10) implies

$$\int_V \exp(h^{1/r} - 2\phi) d^n x \leqslant 2. \tag{11}$$

Since, by the definition of V, $h^{1/r}_{|V} - 2\phi_{|V} \geqslant 0$, (11) tells us that $[h^{1/r}_{|V} - 2\phi_{|V}] \in \bigcap L_p(\mathrm{R}^n, d^n x)$ and $\int_V 1\, d^n x \leqslant 2$, hence

$$[h^{1/r}_{|V} - 2\phi_{|V}]^r \in \bigcap L_p(\mathrm{R}^n, d^n x). \tag{12}$$

Now, the classical Sobolev inequality (1) implies [8] that $f \leqslant d\|f\|_n(-\Delta + 1)$ as forms on $L_2(\mathrm{R}^n, d^n x)$ for all $f \in L_n(\mathrm{R}^n, d^n x)$, so that (12) implies

$$[h^{1/r}_{|V} - 2\phi_{|V}]^r \leqslant k(-\Delta + 1).$$

If $r \geqslant 1$, the convexity and monotonicity of x^r now give

$$\begin{aligned} h_{|V} &= (h^{1/r}_{|V})^r \leqslant [h^{1/r}_{|V} - 2\phi_{|V} + 2|\phi_{|V}|]^r \\ &\leqslant 2^{r-1}\{[h^{1/r}_{|V} - 2\phi_{|V}]^r + (2|\phi_{|V}|)^r\} \leqslant k(-\Delta + |\phi_{|V}|^r + 1). \end{aligned} \tag{13}$$

A similar argument works for $0 < r \leqslant 1$, using the monotonicity and subadditivity of x^r.

Since the definition of U requires

$$h_{|U} \leqslant (2|\phi_{|U}|)^r, \tag{14}$$

we have, combining (13) and (14),

$$(\lg^+(|f|^2))^r = h \leqslant k(-\Delta + |\phi(x)|^r + 1). \tag{15}$$

Then by our hypothesis (5)

$$(\lg^+(|f|^2))^r \leqslant k(-\Delta + \nabla\phi \cdot \nabla\phi - \Delta\phi + 1) \tag{16}$$

as forms on $L_2(\mathbf{R}^n, d^nx)$, where k is independent of f, if $\|f\|_{2,\phi} = 1$.

Now, multiplication by $\exp(-\phi)$ is a unitary equivalence from $L_2(\mathbf{R}^n, \phi)$ to $L_2(\mathbf{R}^n, d^nx)$, which takes $\nabla^* \cdot \nabla$ into

$$\exp(-\phi)\,(\nabla^* \cdot \nabla)\exp(\phi) = -\Delta + \nabla\phi \cdot \nabla\phi - \Delta\phi$$

so that (16) is equivalent to

$$(17) \qquad (\lg^+(|f|^2))^r \leqslant k(\nabla^* \cdot \nabla + 1)$$

as forms on $L_2(\mathbf{R}^n, \phi)$.

In particular (17) gives

$$\int |f|^2(\lg^+(|f|^2))^r \exp(-2\phi) d^nx \leqslant k((f, \nabla^* \cdot \nabla f) + 1)$$

which implies (6) for $\|f\|_{2,\phi} = 1$.

Furthermore, if $r \geqslant 1$ we may use Loewner's theorem [10], which tells us that for $r \geqslant 1$ the rth root is a monotone operator function. (17) then implies

$$(18) \qquad \lg(|f|^2) \leqslant \lg^+(|f|^2) \leqslant k(\nabla^* \cdot \nabla + 1)^{1/r}$$

which, as before, yields (7) for $\|f\|_{2,\phi} = 1$.

(6) and (7) now follow for all f from the following lemma.

LEMMA 2. *Let $d\mu$ be an arbitrary probability measure and let F be an operator on $L_2(d\mu)$ with*

$$\int |f|^2 |\lg(|f|)|^r d\mu \leqslant \|Ff\|_2^2$$

for all $f \in D(F)$, $\|f\|_2 = 1$. If $r \geqslant 1$, then for any $p, q \neq 1, \infty$, $1/p + 1/q = 1$, we have

$$\int |f|^2 |\lg(|f|)|^r d\mu \leqslant q^{r-1}\|Ff\|_2^2 + p^{r-1}\|f\|_2^2\,|\lg(\|f\|_2)|^r, \quad \textit{all } f \in D(F).$$

If $0 < r \leqslant 1$, then

$$\int |f|^2 |\lg(|f|)|^r d\mu \leqslant \|Ff\|_2^2 + \|f\|_2^2\,|\lg(\|f\|_2)|^r, \quad \textit{all } f \in D(F).$$

and if $r = 1$ the inequality $\int |f|^2 \lg(|f|)\, d\mu \leqslant \|Ff\|_2^2$, $f \in D(F)$, $\|f\|_2 = 1$, implies

$$\int |f|^2 \lg(|f|)\, d\mu \leqslant \|Ff\|_2^2 + \|f\|_2^2 \lg(\|f\|_2), \quad \textit{all } f \in D(F).$$

PROOF. Consider first the case $r \geqslant 1$. Take $f \in D(F)$. By assumption

$$\int |f|^2 |\lg(|f|/\|f\|_2)|^r d\mu \leqslant \|Ff\|_2^2.$$

By convexity and monotonicity of x^r, for any $p, q \neq 1, \infty, 1/p + 1/q = 1$, we have

$$\begin{aligned}
\int |f|^2 |\lg(|f|)|^r d\mu &= \int |f|^2 |\lg(|f|/\|f\|_2) + \lg(\|f\|_2)|^r d\mu \\
&= \int |f|^2 \left| \frac{q \lg(|f|/\|f\|_2)}{q} + \frac{p \lg(\|f\|_2)}{p} \right|^r d\mu \\
&\leqslant \int |f|^2 \left(\frac{q|\lg(|f|/\|f\|_2)|}{q} + \frac{|p \lg(\|f\|_2)|}{p} \right)^r d\mu \\
&\leqslant q^{r-1} \int |f|^2 |\lg(|f|/\|f\|_2)|^r d\mu + p^{r-1} \|f\|_2^2 \, |\lg(\|f\|_2)|^r \\
&\leqslant q^{r-1} \|Ff\|_2^2 + p^{r-1} \|f\|_2^2 \, |\lg(\|f\|_2)|^r.
\end{aligned}$$

The assertion for $0 < r \leqslant 1$ follows similarly using the monotonicity and subadditivity of x^r. The assertion for $r = 1$ is trivial.

Finally, (8) follows from (7) for f normalized by the spectral theorem and Holder's inequality.

3. Higher order inequalities.

THEOREM 3. *Let $r > 0$ be such that $|\phi(x)|^r \leqslant a(\nabla\phi \cdot \nabla\phi - \Delta\phi + b)$; then for all $k \in \mathrm{N}$*

$$\begin{aligned}
&\int |f|^2 |\lg(|f|)|^{rk} \exp(-2\phi) d^n x \\
&\leqslant c \left\{ \sum_{|\alpha|=0}^{k} \|D^\alpha f\|_{2,\phi}^2 + \|f\|_{2,\phi}^2 |\lg(\|f\|_{2,\phi})|^{rk} \right\},
\end{aligned} \tag{19}$$

$f \in L_2^{(k)}(\mathrm{R}^n, \phi)$.

PROOF. Let us prove (19) by induction on k. The case $k = 1$ is our first order inequality (6). Assume we have proven (19) for $k = 1, \ldots, m$. Let us show that

$$\int |f|^2 |\lg(|f|)|^{r(m+1)} \exp(-2\phi) d^n x \leqslant c \left(1 + \sum_{|\alpha|=0}^{m+1} \|D^\alpha f\|_{2,\phi} \right)^5, \quad f \in L_2^{(m+1)}(\mathrm{R}^n, \phi). \tag{20}$$

Then, by homogeneity, and our usual use of monotonicity, convexity and subadditivity

$$\begin{aligned}
&\int |f|^2 |\lg(|f|)|^{r(m+1)} \exp(-2\phi) d^n x \\
&\leqslant c \left(\sum_{|\alpha|=0}^{m+1} \|D^\alpha f\|_{2,\phi}^2 + \|f\|_{2,\varphi}^2 \cdot \left| \lg \left(\sum_{|\alpha|=0}^{m+1} \|D^\alpha f\|_{2,\phi} \right) \right|^{r(m+1)} \right).
\end{aligned} \tag{21}$$

Then, since $(\lg(x))^{r(m+1)} \leqslant bx$, $x \geqslant 1$, for some b, (21) yields (19) for the special case $\|f\|_{2,\phi} = 1$. The general case now follows by Lemma 2.

It suffices by continuity to prove (20) for $f \in C_0^\infty(\mathbf{R}^n)$. We have

$$\begin{aligned} \int f^2 |\lg(f^2)|^{r(m+1)} \exp(-2\phi) d^n x & \\ &\leqslant \int (f^2+4)(\lg(f^2+4))^{r(m+1)} \exp(-2\phi) d^n x \\ &\leqslant \int (f^2+4)(\lg(f^2+4))^{rm} \\ &\quad \cdot (\lg[(f^2+4)(\lg(f^2+4))^{rm}])^r \exp(-2\phi)\, d^n x. \end{aligned} \tag{22}$$

If we set $g = (f^2+4)^{1/2}(\lg(f^2+4))^{rm/2}$ we can write (22) as

$$\begin{aligned} \int f^2 |\lg(f^2)|^{r(m+1)} \exp(-2\phi)\, d^n x &\leqslant \int g^2 (\lg(g^2))^r \exp(-2\phi) d^n x \\ &\leqslant c \left\{ \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{2,\phi}^2 + \|g\|_{2,\phi}^2 |\lg(\|g\|_{2,\phi})|^r + \|g\|_{2,\phi}^2 \right\} \end{aligned} \tag{23}$$

where the last line follows from our first order inequality (6).

Now

$$\begin{aligned} \frac{\partial g}{\partial x_i} &= \frac{\partial}{\partial x_i} ((f^2+4)^{1/2} (\lg(f^2+4))^{rm/2}) \\ &= \frac{\partial f}{\partial x_i} \frac{f}{(f^2+4)^{1/2}} ((\lg(f^2+4))^{rm/2} + rm(\lg(f^2+4))^{rm/2-1}). \end{aligned}$$

Therefore

$$\left\| \frac{\partial g}{\partial x_i} \right\|_{2,\phi}^2 \leqslant c \int \left| \frac{\partial f}{\partial x_i} \right|^2 (\lg(f^2+4))^{rm} \exp(-2\phi) d^n x.$$

Now, Young's inequality [6], [7] states that

$$\int |U|\,|V|\, d\mu \leqslant c \left\{ 1 + \int |U|\,|\lg(|U|)|^{rm} d\mu + \int \exp(|V|^{1/rm}) d\mu \right\}$$

so that, by our induction hypothesis (19),

$$\begin{aligned} \left\| \frac{\partial g}{\partial x_i} \right\|_{2,\phi}^2 &\leqslant c \left\{ 1 + \int \left| \frac{\partial f}{\partial x_i} \right|^2 \left| \lg\left(\left| \frac{\partial f}{\partial x_i} \right|^2 \right)^{rm} \right. \right. \exp(-2\phi) d^n x \\ &\qquad \left. + \int \exp(\lg(f^2+4)) \exp(-2\phi) d^n x \right\} \\ &\leqslant c \left\{ 1 + \sum_{|\alpha|=1}^{m+1} \|D^\alpha f\|_{2,\phi}^2 + \left\| \frac{\partial f}{\partial x_i} \right\|_{2,\phi}^2 \cdot \left| \lg\left(\left\| \frac{\partial f}{\partial x_i} \right\|_{2,\phi} \right) \right|^{rm} + \|f\|_{2,\phi}^2 \right\} \\ &\leqslant c \left(1 + \sum_{|\alpha|=0}^{m+1} \|D^\alpha f\|_{2,\phi}^2 \right)^2. \end{aligned} \tag{24}$$

Similarly we see that

$$\|g\|_{2,\phi}^2 = \int (f^2 + 4)\,(\lg(f^2 + 4))^{rm} \exp(-2\phi) d^n x$$

$$\leq c\left(1 + \sum_{|\alpha|=0}^{m} \|D^\alpha f\|_{2,\phi}^2\right)^2. \tag{25}$$

(23), (24) and (25) now prove (20), completing our proof of Theorem 3.

4. **Supercontractivity.**

THEOREM 4. *Let $r > 1$ be such that $|\phi(x)|^r \leq a(\nabla\phi \cdot \nabla\phi - \Delta\phi + b)$; then $e^{-t\nabla^* \cdot \nabla}$ is a bounded map from $L_q(\mathbf{R}^n, \phi)$ to $L_p(\mathbf{R}^n, \phi)$ for any $q, p \neq 1, \infty$, for all $t > 0$.*

PROOF. To prove our theorem we appeal to a result due to L. Gross [1], in a generalized form of J.-P. Eckmann [5].

"Let μ be a probability measure on $\mathbf{R}^n$ and let G be a selfadjoint operator on $L_2(d\mu)$. Suppose that the set C_B^2 of twice continuously differentiable functions with bounded first and second derivatives is a core for G and that $\int \bar{f}(Gg)d\mu = \int \nabla\bar{f} \cdot \nabla g d\mu$, $f, g \in C_B^2$. If there exist constants $0 < u$ and $v < \infty$ such that

$$\int |f|^2 \lg(|f|) d\mu \leq u(f, Gf) + v\|f\|_2^2 + \|f\|_2^2 \lg(\|f\|_2),$$

then $\|e^{-tG}\|_{q, 1+(q-1)e^{2t/u}, d\mu} \leq e^{tv}$."

Now Theorem 1 tells us that, with

$$d\mu = \exp(-2\phi)d^n x \Big/ \int \exp(-2\phi) d^n x,$$

$$\int |f|^2 \lg(|f|)\, d\mu \leq c(f, (\nabla^* \cdot \nabla + 1)^{1/r} f) + \|f\|_2^2 \cdot \lg(\|f\|_2).$$

By the spectral theorem this implies that for any $\epsilon > 0$ there exists a $c(\epsilon)$ such that

$$\int |f|^2 \lg(|f|) d\mu \leq \epsilon(f, \nabla^* \cdot \nabla f) + c(\epsilon)\|f\|_2^2 + \|f\|_2^2 \cdot \lg(\|f\|_2).$$

The general assertion of our theorem follows from the result quoted.

5. **Applications.**

THEOREM 5. *If $\phi \sim a|x|^{1+s}$, $s > 0$, $a > 0$, and $D^\alpha\phi \sim aD^\alpha|x|^{1+s}$, $|\alpha| = 1, 2$, as $|x| \to \infty$, then for any $k \in \mathbf{Z}^+$*

$$\int |f|^2 |\lg(|f|)|^{2sk/(s+1)} \exp(-2\phi) d^n x$$

$$\leq c\left\{\sum_{|\alpha|=0}^{k} \|D^\alpha f\|_{2,\phi}^2 + \|f\|_{2,\phi}^2 |\lg(\|f\|_{2,\phi})|^{2sk/(s+1)}\right\}, \tag{26}$$

$f \in L_2^{(k)}(\mathbf{R}^n, \phi)$. *If* $s > 1$, *then*

$$\int |f|^2 \lg(|f|) \exp(-2\phi) d^n x \leqslant c \left\{ \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{2,\phi}^{(s+1)/s} + 1 \right\}, \tag{27}$$

$$f \in L_2^{(1)}(\mathbf{R}^n, \phi), \ \|f\|_{2,\phi} = 1,$$

and

$$e^{-t\nabla^* \cdot \nabla} : L_q(\mathbf{R}^n, \phi) \longrightarrow L_p(\mathbf{R}^n, \phi), \qquad p, q \neq 1, \infty, \tag{28}$$

is bounded for all $t > 0$. *Furthermore, for any* $s > 0$, *if*

$$D^\alpha \phi \sim a D^\alpha |x|^{1+s}, \qquad |\alpha| = 0, 1, \ldots, k, \tag{29}$$

then for any $m \in \mathbf{Z}^+$, *there are* $f \in L_2^{(k)}(\mathbf{R}^n, \phi)$ *with*

$$\int |f|^2 |\lg(|f|)|^{2sk/(s+1)} \overbrace{\lg^+(\cdots \lg^+(|f|) \cdots)}^{m} \exp(-2\phi) d^n x = \infty.$$

PROOF. (26), (27), and (28) follow from Theorems 1, 3 and 4 once we have verified $|\phi(x)|^{2s/(s+1)} \leqslant a(\nabla\phi \cdot \nabla\phi - \Delta\phi + b)$, but by our hypothesis both $|\phi(x)|^{2s/(s+1)}$ and $\nabla\phi \cdot \nabla\phi - \Delta\phi$ are $\sim O(|x|^{2s})$.

To prove the second part of our theorem, consider a function f such that

$$f(x) = \frac{\exp(\phi(x))|x|^{-(n-1)/2}}{|x|^{sk}(|x| \lg(|x|) \cdots \lg_{m-2}(|x|) \, [\lg_{m-1}(|x|)]^2)^{1/2}}$$

for $|x| \geqslant x_0$ large, $f(x) = 0$ for $|x| \leqslant x_0 - 1$ and $f(x) \in C^\infty(\mathbf{R}^n)$, where we have used the notation $\lg_j(x)$ to mean that $\lg(\cdots \lg(x) \cdots)$ occurs j times.

To see that $f \in L_2^{(k)}(\mathbf{R}^n, \phi)$ compute for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = k$,

$$D^\alpha f = \frac{\Pi_{i=1}^n (\partial\phi(x)/\partial x_i)^{\alpha_i} \exp(\phi(x))|x|^{-(n-1)/2}}{|x|^{sk}(|x| \lg(|x|) \cdots \lg_{m-2}(|x|) [\lg_{m-1}(|x|)]^2)^{1/2}}$$

$+$ terms smaller at ∞ using our assumption (29), and in fact, by (29),

$$\|D^\alpha f\|_{2,\phi}^2 \leqslant c_0 + c \int_{x_0}^\infty \frac{dx}{x \lg(x) \cdots \lg_{m-2}(x) [\lg_{m-1}(x)]^2}$$

$$= c_0 + c \int_{x_0}^\infty \frac{d}{dx} \left(\frac{-1}{\lg_{m-1}(x)} \right) dx = c_0 + c(\lg_{m-1}(x_0))^{-1} < \infty.$$

On the other hand, since $|\phi(x)| \sim O(|x|^{s+1})$

$$\int |f|^2 |\lg(f)|^{2sk/(s+1)} \lg_m^+(f) \exp(-2\phi) d^n x$$

$$\geqslant c_0 \int_{|x| \geqslant x_0} \frac{|\phi(x)|^{2sk/(s+1)} \lg_{m-1}(\phi(x)) |x|^{-(n-1)} d^n x}{|x|^{2sk} |x| \lg(|x|) \cdots \lg_{m-2}(|x|) [\lg_{m-1}(|x|)]^2}$$

$$\geqslant c_0 \int_{x_0}^{\infty} \frac{1}{x \lg(x) \cdots \lg_{m-1}(x)} dx$$

$$= c_0 \int_{x_0}^{\infty} \frac{d}{dx} (\lg_m(x)) = \infty.$$

REMARK. Let $P(x) = \Sigma_{i=0}^{2p} a_i x^i$ with $a_{2p} > 0$, and consider the anharmonic oscillator H in $L_2(\mathbf{R}^1, dx)$

$$H = -d^2/dx^2 + P(x).$$

The normalized groundstate $\Omega(x)$ is strictly positive and can be written as $\Omega(x) = \exp(-\phi)$, for ϕ satisfying all the requirements of Theorem 5 with $s = p$ [7], [11], [12].

For extensions to anharmonic oscillators in $L_2(\mathbf{R}^n, d^n x)$ see [13].

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