# $p$-VARIATION OF THE LOCAL TIMES OF STABLE PROCESSES AND INTERSECTION LOCAL TIME

by

JAY ROSEN[1]

## 1 Introduction

Let $L_t^x$ denote the local time of the symmetric stable process of order $\beta > 1$ in $\Re^1$. $L_t^x$ is known to be jointly continuous (Boylan [1964]). We will study the $p$-variation of $L_t^x$ in $x$, and generalize results concerning Brownian local time of Bouleau and Yor [1981] and Perkins [1982].

Fix $a, b < \infty$ and let $Q(a,b)$ denote the set of partitions $\pi = \{x_0 = a < x_1 \cdots < x_n = b\}$ of $[a,b]$. We use

$$m(\pi) = \sup_i (x_i - x_{i-1})$$

to denote the mesh size of $\pi$.

**Theorem 1.1** *Let $\beta = 1 + \frac{1}{k}$, $k = 1, 2, \ldots$ then*

$$\sum_{x_i \in \pi} (L_t^{x_i} - L_t^{x_{i-1}})^{2k} \longrightarrow \bar{c} \int_a^b (L_t^x)^k \, dx \tag{1.1}$$

*in $L^2$, uniformly both in $t \in [0,T]$ and $\pi \in Q(a,b)$ as $m(\pi) \to 0$.*

*Here*

$$\bar{c} = (2k)!!(4c)^k, \quad c = \int_0^\infty p_t(0) - p_t(1) dt \tag{1.2}$$

*and $p_t(x)$ is the transition density for our stable process.*

[1]Supported in part by NSF DMS 88 022 88, PSC-CUNY Award, and through US-Israel BSF 86-00285

For $k=1$, i.e., Brownian motion, we recover the result of Bouleau and Yor [1981] and Perkins [1982]:

$$\sum_{x_i\in\pi}(L_t^{x_i}-L_t^{x_{i-1}})^2\longrightarrow 4\int_a^b L_t^x dx.$$

This quadratic variation allows one to develop stochastic integrals with respect to the space parameter of Brownian local time, see also Walsh [1983].

We note that the right-hand side of (1.1) is a $k$-fold intersection local time for the self-intersections of our stable process in $[a,b]$.

The methods of this paper only allow us to compute $p$-variations when $p$ is of the form $p=2k$, which limits results of the form (1.1) to $\beta=1+\frac{1}{k}$. In Marcus and Rosen [1990], we obtain analogues of (1.1) for arbitrary $\beta>1$, in the sense of a.s. convergence. The convergence, however, is not uniform in $Q(a,b)$. If we want to obtain results for arbitrary $\beta>1$ by the methods of this paper, we will have to be satisfied with the following:

**Theorem 1.2** *Let $\beta>1$, then*

$$\sum_{x_i\in\pi}\left(\frac{L_t^{x_i}-L_t^{x_{i-1}}}{(x_i-x_{i-1})^\gamma}\right)^{2k}\longrightarrow \bar{c}\int_a^b (L_t^x)^k\,dx \tag{1.3}$$

*in $L^2$, uniformly in both $t\in[0,T]$ and $\pi\in Q(a,b)$ as $m(\pi)\to 0$, where*

$$\gamma=\frac{\beta-1}{2}-\frac{1}{2k}$$

*and $\bar{c}$ is given by (1.2).*

The methods of this paper were a natural outgrowth of our second order limit laws for the local times of stable processes, Rosen [1990].

It is a pleasure to thank M. Yor for drawing my attention to the problem of $p$-variation of stable local times.

## 2 Proofs

*Proof of Theorem 1:* We write, for $\tau\in Q(a,b)$

$$E\left(\left\{\bar{c}\int_a^b (L_t^x)^k\,dx-\sum_{x_i\in\tau}(L_t^{x_i}-L_t^{x_{i-1}})^{2k}\right\}^2\right)$$

$$\begin{aligned}
&= \bar{c}^2 \int_a^b \int_a^b E\left\{(L_t^x)^k (L_t^y)^k\right\} dx dy \\
&- 2\bar{c} \int_a^b \sum_i E\left\{(L_t^{x_i} - L_t^{x_{i-1}})^{2k} (L_t^y)^k\right\} dy \\
&+ \sum_{i,j} E\left\{(L_t^{x_i} - L_t^{x_{i-1}})^{2k} \left(L_t^{x_j} - L_t^{x_{j-1}}\right)^{2k}\right\} \\
&\doteq A - 2B_\epsilon + C_\epsilon, \quad \text{where } \epsilon \doteq m(\tau) \qquad (2.1)
\end{aligned}$$

We will show that as $\epsilon \to 0$, each of $A, B_\epsilon, C_\epsilon$ converges to

$$[(2k)!(2c)^k]^2 \sum_{\tilde{\pi}} \int_a^b dx \int_a^b dy \int_{0 \le t_1 \le \cdots \le t_{2k} \le t} \cdots \int \prod_{i=1}^{2k} p_{\Delta t_i}(\tilde{\pi}_i, \tilde{\pi}_{i-1}) dt_i \qquad (2.2)$$

where the sum runs over all paths $\tilde{\pi} : \{1, \ldots, 2k\} \to \{x, y\}$ which visit $x, y$ an equal number of times (i.e. $k$ times each).

The fact that $A$ equals (2.2) is straightforward, so we turn to $B_\epsilon$. We have

$$\begin{aligned}
&E\left\{(L_t^{x_i} - L_t^{x_{i-1}})^{2k} (L_t^y)^k\right\} \\
&= E\left\{\prod_{\ell=1}^{2k} \int_0^t dL_{s_\ell}^{x_i} - dL_{s_\ell}^{x_{i-1}} \prod_{j=1}^{k} \int_0^t dL_{s_j}^y\right\} \\
&= (2k)!k! \sum_\pi E\left(\int_{0 \le t_1 \le \cdots \le t_{3k} \le t} \cdots \int \prod_{\ell=1}^{3k} d\mathcal{L}_\ell^{\pi_\ell}\right) \qquad (2.3)
\end{aligned}$$

where the sum runs over all paths $\pi : \{1, \ldots, 3k\} \longrightarrow \{x_i, y\}$ which visit $y$ exactly $k$ times, and

$$\begin{aligned}
\mathcal{L}_t^{x_i} &\doteq L_t^{x_i} - L_t^{x_{i-1}} \\
\mathcal{L}_t^y &\doteq L_t^y \qquad (2.4)
\end{aligned}$$

We will say that a path $\pi$ is even if its visits to $x_i$ occur in even runs. A path will be called odd if it is not even.

Assume that $\pi$ is even. Then we can evaluate its contribution to (2.3) by successive application of the Markov property. We use the following observations, where [ ] will be used generically to denote an expression depending only on the path up to the earliest times which are exhibited.

$$\begin{aligned}
&E\left([\quad] \int_{s_{j-2}}^t \left(dL_{s_{j-1}}^{x_i} - dL_{s_{j-1}}^{x_{i-1}}\right) \int_{s_{j-1}}^t dL_{s_j}^{x_i} - dL_{s_j}^{x_{i-1}}\right) \\
&= E\left([\quad] \int_{s_{j-2}}^t \left(dL_{s_{j-1}}^{x_i} + dL_{s_{j-1}}^{x_{i-1}}\right) \int_{s_{j-1}}^t p_{\Delta s_j}(0) - p_{\Delta s_j}(\Delta x_i) ds_j\right) \qquad (2.5)
\end{aligned}$$

$$E\left([\quad]\int_{s_{j-2}}^{t}\left(dL^{x_i}_{s_{j-1}}-dL^{x_{i-1}}_{s_{j-1}}\right)\int_{s_{j-1}}^{t}dL^{x_i}_{s_j}+dL^{x_{i-1}}_{s_j}\right)$$
$$= E\left([\quad]\int_{s_{j-2}}^{t}\left(dL^{x_i}_{s_{j-1}}-dL^{x_{i-1}}_{s_{j-1}}\right)\int_{s_{j-1}}^{t}p_{\Delta s_j}(0)+p_{\Delta s_j}(\Delta x_i)ds_j\right) \tag{2.6}$$

$$E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}dL^{x_i}_{s_j}+dL^{x_{i-1}}_{s_j}\right)$$
$$= E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}p_{\Delta s_j}(y-x_i)+p_{\Delta s_j}(y-x_{i-1})ds_j\right) \tag{2.7}$$

$$E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}dL^{y}_{s_j}\right)$$
$$= E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}p_{\Delta s_j}(0)ds_j\right) \tag{2.8}$$

$$E\left([\quad]\int_{s_{j-2}}^{t}\left(dL^{x_i}_{s_{j-1}}-dL^{x_{i-1}}_{s_{j-1}}\right)\int_{s_{j-1}}^{t}dL^{y}_{s_j}\right)$$
$$= E\left([\quad]\int_{s_{j-2}}^{t}\left(dL^{x_i}_{s_{j-1}}-dL^{x_{i-1}}_{s_{j-1}}\right)\int_{s_{j-1}}^{t}p_{\Delta s_j}(x_i-y)ds_j\right)$$
$$+ E\left([\quad]\int_{s_{j-2}}^{t}dL^{x_{i-1}}_{s_{j-1}}\int_{s_{j-1}}^{t}p_{\Delta s_j}(x_i-y)-p_{\Delta s_j}(x_{i-1}-y)ds_j\right) \tag{2.9}$$

$$E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}dL^{x_i}_{s_j}-dL^{x_{i-1}}_{s_j}\right)$$
$$= E\left([\quad]\int_{s_{j-2}}^{t}dL^{y}_{s_{j-1}}\int_{s_{j-1}}^{t}p_{\Delta s_j}(x_i-y)-p_{\Delta s_j}(x_{i-1}-y)ds_j\right) \tag{2.10}$$

As we see, (2.5), (2.9) and (2.10) give rise to 'difference factors', i.e., factors of the form

$$\int p_s(y)-p_s(y-\Delta x_i)ds \tag{2.11}$$

We will see below in Lemma 1 that such factors give a contribution

$$\bigcirc(\Delta x_i)^{\beta-1}=\bigcirc(\Delta x_i)^{\frac{1}{k}},$$

hence whenever we have $> k$ difference factors, the contribution to $\lim_{\epsilon\to 0} B_\epsilon$ will be zero. We can see by using the above formulae recursively that all terms arising from the evaluation of the expectation associated to an even path $\pi$ have $> k$ difference factors, except for a contribution which can be written as

$$2^k \int\cdots\int_{\sum_{\ell=1}^{2k}\Delta s_\ell+\sum_{j=1}^{k}\Delta t_j\le t}\ \prod_{\ell=1}^{2k}p_{\Delta s_\ell}(\tilde{\pi}_\ell,\ \tilde{\pi}_{\ell-1})\ \prod_{j=1}^{k}p_{\Delta t_j}(0)-p_{\Delta t_j}(\Delta x_i) \tag{2.12}$$

where $\pi$ induces the path $\tilde{\pi}: \{1,\ldots,2k\} \longrightarrow \{x_i, y\}$, (visiting both $x_i$ and $y$ $k$ times) as follows: since visits of $\pi$ to $x_i$ occur in pairs, we simply suppress one visit from each pair. Note that in getting (2.12), we e.g. rewrote the factor

$$\int p_{\Delta s}(y - x_i) + p_{\Delta s}(y - x_{i-1})ds$$

of (2.7) as

$$2\int p_{\Delta s}(y - x_i)ds$$

+ a 'difference factor', and similarly for (2.6) and analogous factors.

We will show below, in Lemma 3, that as $\epsilon \to 0$, the integral in (2.10) summed over $i$ converges to $c^k$ times the integral in (2.2). Furthermore, any given $\tilde{\pi}$ will be induced from precisely one even $\pi$ which will show that the contribution of even paths to $B_\epsilon \longrightarrow$ (2.2).

To see that odd paths $\pi$ give zero contribution in the limit, we use (2.5)–(2.10) recursively to see that every term in the expansion of an odd path $\pi$ has $> k$ 'difference factors'.

We now turn to $C_\epsilon$:

$$\begin{aligned} &E\left\{(L_t^{x_i} - L_t^{x_{i-1}})^{2k}\left(L_t^{x_j} - L_t^{x_{j-1}}\right)^{2k}\right\} \\ &= (2k!)^2 \sum_\pi E\left(\int\cdots\int_{0\le t_1\le\cdots\le t_{4k}\le t} \prod_{\ell=1}^{4k} dL_{t_\ell}^{x_{\pi_\ell}} - dL_{t_\ell}^{x_{\pi_\ell}-1}\right) \end{aligned} \tag{2.13}$$

where the sum runs over all paths $\pi: \{1,\ldots,4k\} \longrightarrow \{i,j\}$ which visit $i,j$ an equal number of times, i.e. $2k$ times each.

We will evaluate

$$E\left(\int\cdots\int_{0\le t_1\le\cdots\le t_{4k}\le t} \prod_{\ell=1}^{4k} dL_{t_\ell}^{x_{\pi_\ell}} - dL_{t_\ell}^{x_{\pi_\ell-1}}\right) \tag{2.14}$$

by using (2.5)–(2.10) together with

$$\begin{aligned} &E\left([\quad]\int_{s_{\ell-2}}^t \left(dL_{s_{\ell-1}}^{x_i} - dL_{s_{\ell-1}}^{x_{i-1}}\right)\int_{s_{\ell-1}}^t dL_{s_\ell}^{x_j} + dL_{s_\ell}^{x_{j-1}}\right) \\ &= E\left([\quad]\int_{s_{\ell-2}}^t \cdot\left(dL_{s_{\ell-1}}^{x_i} - dL_{s_{\ell-1}}^{x_{i-1}}\right)\int_{s_{\ell-1}}^t p_{\Delta s_\ell}(x_i - x_j) + p_{\Delta s_\ell}(x_i - x_{j-1})ds_\ell\right) \\ &+ E\left([\quad]\int_{s_{\ell-2}}^t dL_{s_{\ell-1}}^{x_{i-1}}\int_{s_{\ell-1}}^t \{p_{\Delta s_\ell}(x_i - x_j) - p_{\Delta s_\ell}(x_{i-1} - x_j)\}\right. \\ &\qquad \left. + \{p_{\Delta s_\ell}(x_i - x_{j-1}) - p_{\Delta s_\ell}(x_{i-1} - x_{j-1})\}ds_\ell\right) \end{aligned} \tag{2.15}$$

and

$$E\left([\quad]\int_{s_{\ell-2}}^{t}\left(dL_{s_{\ell-1}}^{x_i}-dL_{s_{\ell-1}}^{x_{i-1}}\right)\int_{s_{\ell-1}}^{t}dL_{s_\ell}^{x_j}-dL_{s_\ell}^{x_{j-1}}\right)$$

$$= E\left([\quad]\int_{s_{\ell-2}}^{t}\left(dL_{s_{\ell-1}}^{x_i}-dL_{s_{\ell-1}}^{x_{i-1}}\right)\int_{s_{\ell-1}}^{t}p_{\Delta s_\ell}(x_j-x_i)-p_{\Delta s_\ell}(x_{j-1}-x_i)ds_\ell\right)$$

$$+ E\left([\quad]\int_{s_{\ell-2}}^{t}dL_{s_{\ell-1}}^{x_{i-1}}\int_{s_{\ell-1}}^{t}\{p_{\Delta s_\ell}(x_j-x_i)-p_{\Delta s_\ell}(x_{j-1}-x_i)\right.$$

$$\left.-p_{\Delta s_\ell}(x_j-x_{i-1})+p_{\Delta s_\ell}(x_{j-1}-x_{i-1})\}ds_\ell\right) \tag{2.16}$$

We now call a path $\pi$ even if both its visits to $i$ and to $j$ occur in even runs. Such a path uniquely induces a path $\tilde{\pi}: \{1,\dots,2k\}\longrightarrow\{i,j\}$ by

$$\tilde{\pi}(\ell) :\doteq \pi(2\ell-1)=\pi(2\ell)$$

We refer to a 'difference factor' of the form (2.11) as an '$x_i$- difference factor', and note that the terms generated by (2.14) will give zero contribution to (2.13), in the limit, if such a term has $k_1$ '$x_i$-difference factors' and $k_2$ '$x_j$-difference factors'—and

$$k\le k_1\wedge k_2,\quad k_1\vee k_2>k.$$

We can see using the above formulae recursively that if $\pi$ is even, the only term giving a non-zero limit will be

$$2^{2k}(2k!)^2\int\cdots\int_{\sum_{\ell=1}^{2k}\Delta t_\ell+\sum_{m=1}^{k}s_m+\sum_{n=1}^{k}\Delta r_n\le t}\prod_{\ell=1}^{2k}p_{\Delta t_\ell}(x_{\tilde{\pi}_i},\ x_{\tilde{\pi}_{i-1}})$$

$$\prod_{m=1}^{k}(p_{\Delta s_m}(0)-p_{\Delta s_m}(\Delta x_i))\prod_{n=1}^{k}p_{\Delta r_n}(0)-p_{\Delta r_n}(\Delta x_j) \tag{2.17}$$

and we show below, in Lemma 3, that this summed over $i,j$ converges to the integral in (2.2).

Finally, we turn to odd paths $\pi$ and show that they contribute 0 in the limit. The only new wrinkle comes from the second term in (2.16), which a-priori generates only one 'difference factor' for the two local time integrals. However, if we fix $\delta>0$, and if

$$|u|\doteq|x_{i-1}-x_{j-1}|\ge\delta\ ,\quad m(\tau)<\frac{\delta}{4},$$

and

$$E\left([\quad]\int_{s_{\ell-2}}^{t}\left(dL^{x_i}_{s_{\ell-1}}-dL^{x_{i-1}}_{s_{\ell-1}}\right)\int_{s_{\ell-1}}^{t}dL^{x_j}_{s_\ell}-dL^{x_{j-1}}_{s_\ell}\right)$$

$$\begin{aligned}&= E\left([\quad]\int_{s_{\ell-2}}^{t}\left(dL^{x_i}_{s_{\ell-1}}-dL^{x_{i-1}}_{s_{\ell-1}}\right)\int_{s_{\ell-1}}^{t}p_{\Delta s_\ell}(x_j-x_i)-p_{\Delta s_\ell}(x_{j-1}-x_i)ds_\ell\right)\\ &+ E\left([\quad]\int_{s_{\ell-2}}^{t}dL^{x_{i-1}}_{s_{\ell-1}}\int_{s_{\ell-1}}^{t}\{p_{\Delta s_\ell}(x_j-x_i)-p_{\Delta s_\ell}(x_{j-1}-x_i)\right.\\ &\quad\left.-p_{\Delta s_\ell}(x_j-x_{i-1})+p_{\Delta s_\ell}(x_{j-1}-x_{i-1})\}ds_\ell\right)\end{aligned}\tag{2.16}$$

We now call a path $\pi$ even if both its visits to $i$ and to $j$ occur in even runs. Such a path uniquely induces a path $\tilde{\pi}:\ \{1,\dots,2k\}\longrightarrow\{i,j\}$ by

$$\tilde{\pi}(\ell):\doteq\pi(2\ell-1)=\pi(2\ell)$$

We refer to a 'difference factor' of the form (2.11) as an '$x_i$- difference factor', and note that the terms generated by (2.14) will give zero contribution to (2.13), in the limit, if such a term has $k_1$ '$x_i$-difference factors' and $k_2$ '$x_j$-difference factors'—and

$$k\le k_1\wedge k_2,\quad k_1\vee k_2>k.$$

We can see using the above formulae recursively that if $\pi$ is even, the only term giving a non-zero limit will be

$$\begin{aligned}2^{2k}(2k!)^2\int\!\!\cdots\!\!\int_{\sum_{\ell=1}^{2k}\Delta t_\ell+\sum_{m=1}^{k}s_m+\sum_{n=1}^{k}\Delta r_n\le t}&\prod_{\ell=1}^{2k}p_{\Delta t_\ell}(x_{\tilde{\pi}_i},\ x_{\tilde{\pi}_{i-1}})\\ \prod_{m=1}^{k}(p_{\Delta s_m}(0)-p_{\Delta s_m}(\Delta x_i))&\prod_{n=1}^{k}p_{\Delta r_n}(0)-p_{\Delta r_n}(\Delta x_j)\end{aligned}\tag{2.17}$$

and we show below, in Lemma 3, that this summed over $i,j$ converges to the integral in (2.2).

Finally, we turn to odd paths $\pi$ and show that they contribute 0 in the limit. The only new wrinkle comes from the second term in (2.16), which a-priori generates only one 'difference factor' for the two local time integrals. However, if we fix $\delta>0$, and if

$$|u|\doteq|x_{i-1}-x_{j-1}|\ge\delta\ ,\quad m(\tau)<\frac{\delta}{4},$$

then we will show below in Lemma 2 that

$$\begin{aligned} &\Big|\int_0^r p_s(u-\Delta x_i+\Delta x_j)-p_s(u-\Delta x_i) \\ &\qquad -p_s(u+\Delta x_j)+p_s(u)\,ds\Big| \\ &\le \frac{c}{\delta^2}\,\Delta x_i\,\Delta x_j, \end{aligned} \tag{2.18}$$

while if $|u|\le\delta$, we can bound (2.18) by breaking it up into pairs—either with $\Delta x_i$ or $\Delta x_j$ as the difference, to get via Lemma 1 a bound

$$c(\Delta x_i)^{\beta-1}\wedge(\Delta x_j)^{\beta-1}. \tag{2.19}$$

The contribution of (2.18) and (2.19) will then be bounded by

$$c\sum_{\substack{i,j\\|u|\le\delta}}\Delta x_i\,\Delta x_j+\frac{c}{\delta^2}\epsilon^\alpha$$

for some $\alpha>0$, $\epsilon=m(\tau)<\frac{\delta}{4}$, and we now take first $\epsilon\to0$ then $\delta\to0$ to see that such terms don't contribute in the limit. This completes the proof of Theorem 1, and that of Theorem 2 is basically the same.

## 3 Lemmas

**Lemma 1**

$$\int_0^T |\,p_t(x)-p_t(y)\,|\;dt\le c\,\Big|\,|x|^{\beta-1}-|y|^{\beta-1}\Big|\le c|x-y|^{\beta-1} \tag{3.1}$$

*and*

$$\int_0^T p_t(0)-p_t(x)dt=c|x|^{\beta-1}+O\left(\frac{|x|^2}{T^{3/\beta-1}}\right) \tag{3.2}$$

*where*

$$c=\int_0^\infty(p_t(0)-p_t(1))dt<\infty \tag{3.3}$$

*Proof:* $p_t(x)$ is monotone in $|x|$, hence if $|x|\le|y|$,

$$\begin{aligned} &\int_0^T |\,p_t(x)-p_t(y)\,|\;dt=\int_0^T p_t(x)-p_t(y)\,dt \\ &=\int_0^T(p_t(0)-p_t(y))-(p_t(0)-p_t(x))\,dt \\ &\le\int_0^\infty(p_t(0)-p_t(y))-(p_t(0)-p_t(x))\,dt \\ &=\int_0^\infty(p_t(0)-p_t(y))\,dt-\int_0^\infty(p_t(0)-p_t(x))\,dt \end{aligned} \tag{3.4}$$

since $p_t(0) - p_t(x) \geq 0$ and we will now show it is integrable in $t$.

For this we use the scaling:

$$p_t(x) = \frac{1}{t^{1/\beta}} p_1\left(\frac{x}{t^{1/\beta}}\right) \tag{3.5}$$

so that

$$\begin{aligned} &\int_0^\infty p_t(0) - p_t(x)\,dt \\ &= \int_0^\infty \left(p_1(0) - p_1\left(\frac{x}{t^{1/\beta}}\right)\right) \frac{dt}{t^{1/\beta}} \\ &= |x|^{\beta-1} \int_0^\infty \left(p_1(0) - p_1\left(\frac{1}{t^{1/\beta}}\right)\right) \frac{dt}{t^{1/\beta}} \end{aligned} \tag{3.6}$$

and the last integral is finite since, for $t$ small we have $|p_1(y)| \leq p_1(0)$ and $\beta > 1$, while for large $t$, we have from symmetry that

$$\left| p_1(0) - p_1\left(\frac{1}{t^{1/\beta}}\right)\right| \leq \frac{\tilde{c}}{t^{2/\beta}}. \tag{3.7}$$

It is now easy to see that (3.3) is the integral on the r.h.s. of (3.6). This proves (3.1).

For (3.2) we write

$$\begin{aligned} &\int_0^T p_t(0) - p_t(x)\,dt \\ &= \int_0^\infty p_t(0) - p_t(x)dt - \int_t^\infty p_t(0) - p_t(x)dt, \end{aligned} \tag{3.8}$$

and use (3.6), together with the bound from (3.5), (3.7)

$$\begin{aligned} &\int_T^\infty p_t(0) - p_t(x)dt \\ &\leq \tilde{c}|x|^{\beta-1} \int_{T/x^\beta}^\infty \frac{1}{t^{3/\beta}}\,dt \\ &= \tilde{c}|x|^{\beta-1} \left(\frac{T}{|x|^\beta}\right)^{1-3/\beta} \\ &= \tilde{c}\frac{|x|^2}{T^{3/\beta-1}} \end{aligned} \tag{3.9}$$

**Lemma 2**

$$\begin{aligned} &\int_0^T |\, p_t(x+a+b) - p_t(x+a) - p_t(x+b) + p_t(x)\,|\; dt \\ &\leq c\frac{|a|\,|b|}{|x|^2}, \end{aligned} \tag{3.10}$$

*for* $|x| \geq 4(a \vee b)$.

*Proof:* We integrate by parts:

$$\begin{aligned}\frac{d^2}{dx^2}p_t(x) &= \frac{d^2}{dx^2}\int e^{ipx}\, e^{-tp^\beta}\,dp \\ &= -\int e^{ipx}\, p^2\, e^{-tp^\beta}\,dp \\ &= \frac{1}{ix}\int e^{ipx}\frac{d}{dp}\left(p^2\, e^{-tp^\beta}\right)dp \\ &= \frac{-1}{x^2}\int e^{ipx}\frac{d^2}{dp^2}\left(p^2\, e^{-tp^\beta}\right)dp \end{aligned} \tag{3.11}$$

so that

$$\begin{aligned}\left|\frac{d^2}{dx^2}p_t(x)\right| &\leq \frac{\tilde{c}}{|x|^2}\int \left|\frac{d^2}{dp^2}\left(p^2\, e^{-tp^\beta}\right)\right|dp \\ &\leq \frac{\tilde{c}}{|x|^2}\int e^{-tp^\beta} + tp^\beta\, e^{-tp^\beta} + \left(tp^\beta\right)^2 e^{-tp^\beta}\,dp \\ &\leq \frac{\tilde{c}}{|x|^2}\,\frac{1}{t^{1/\beta}}\end{aligned}$$

Now, the mean value theorem, our assumption that $|x| \geq 4(a \vee b)$, and the integrability of $\frac{1}{t^{1/\beta}}$ on $[0,T]$ finishes the proof of Lemma 2.

**Lemma 3** *Let* $f \in L^1([0,T]^j)$ *and set*

$$F(s) \doteq \int\cdots\int_{\sum_{i=1}^j t_i \leq s} f(t)\,dt_1 \ldots dt_j$$

*then*

$$\int\cdots\int_{\sum_{i=1}^j r_i + \sum_{\ell=1}^k s_\ell \leq t} f(r_1,\ldots,r_j)\prod_{\ell=1}^k p_{s_\ell}(0) - p_{s_\ell}(x_\ell)\,dr\,ds$$
$$\leq c^k\ |x_1 \ldots x_k|^{\beta-1}\, F(t) \tag{3.12}$$

*and for any* $\delta > 0$*, we have*

$$\int\cdots\int_{\sum_{i=1}^j r_i + \sum_{\ell=1}^k s_\ell \leq t} f(r_1,\ldots,r_j)\prod_{\ell=1}^k p_{s_\ell}(0) - p_{s_\ell}(x_\ell)\,dr\,ds$$
$$= c^k\ |x_1 \ldots x_k|^{\beta-1}\left(F(t) + o(1_\delta) + \bigcirc\left(\sup_\ell \frac{|x_\ell|^{3-\beta}}{\delta^{3/\beta-1}}\right)\right) \tag{3.13}$$

*where* $o(1_\delta)$ *means a term which goes to zero when* $\delta \to 0$.

*Proof:* (3.12) is immediate from Lemma 1. To see (3.13), fix $\delta > 0$, and define

$$\begin{aligned} C &= \{(r,s) \mid \sum_{i=1}^{j} r_i + \sum_{\ell}^{k} s_\ell \le t\} \\ D_\delta &= \{(r,s) \mid s_\ell \le \delta, \quad \text{for all } \ell\} \end{aligned}$$

Note that

$$C \cap (D_\delta^c) \subseteq \left\{(r,s) \;\middle|\; \sum_{i=1}^{j} r_i \le t, \ \text{ and } s_\ell > \delta \text{ for some } \ell \right\}$$

so that

$$\begin{aligned} &\int \cdots \int_{C\cap(D_\delta^c)} f(r) \prod_{\ell=1}^{k} \ p_{s_\ell}(0) - p_{s_\ell}(x_\ell)\, dr ds \\ &\le \ \tilde{c} F(t)\, |x_1 \dots x_k|^{\beta-1} \sup_\ell \frac{|x_\ell|^{3-\beta}}{\delta^{3/\beta-1}} \end{aligned} \tag{3.14}$$

from Lemma 1, and (3.9).

Now set

$$H_\delta = \{(r,s) \ \mid \ \sum_{i=1}^{j} r_i \le t - k\delta\}$$

and note that, by Lemma 1,

$$\begin{aligned} &\int \cdots \int_{C\cap D_\delta \cap(H_\delta^c)} f(r) \prod_{\ell=1}^{k} p_{s_\ell}(0) - p_{s_\ell}(x_\ell)\, dr ds \\ &\le \ \tilde{c}\ |x_1 \dots x_k|^{\beta-1} (F(t) - F(t-k\delta)) \end{aligned} \tag{3.15}$$

Finally, note that

$$C \cap D_\delta \cap H_\delta = \left\{(r,s) \middle| \sum_{i=1}^{j} r_i \le t - k\delta, \text{ and } s_\ell \le \delta, \quad \text{for all } \ell \right\}$$

so that, from Lemma 1,

$$\begin{aligned} &\int \cdots \int_{C\cap D_\delta \cap H_\delta} f(r) \prod_{\ell=1}^{k} \ p_{s_\ell}(0) - p_{s_\ell}(x_\ell)\, dr ds \\ &= \ F(t-k\delta) \prod_{\ell=1}^{k} \int_0^\delta p_s(0) - p_s(x_\ell)\, ds \\ &= \ F(t-k\delta) \left( c^k\, |x_1 \dots x_k|^{\beta-1} + \tilde{c}\, |x_1, \dots x_k|^{\beta-1} \sup_\ell \frac{|x_\ell|^{3-\beta}}{\delta^{3/\beta-1}} \right) \end{aligned} \tag{3.16}$$

(3.14), (3.15) and (3.16) now complete the proof of (3.13).

## References

[1] Bouleau, N. and Yor, M. [1981], "Sur La Variation Quadratique de Temps Locaux de Certaines Semi-Martingales", *C. R. Acad. Sci. Paris, Série I* **292**, pp. 491–492.

[2] Marcus, M. and Rosen, J. [1990], "$p$-Variation of the Local Times of Symmetric Markov Processes Via Gaussian Processes". In preparation.

[3] Perkins, E. [1982], "Local Time is a Semi-Martingale", *Z. Warsch. verw. Geb.* **60**, pp. 79–117.

[4] Rosen, J. [1990], "Second Order Limit Laws for the Local Times of Stable Processes", preprint.

[5] Walsh, J. [1983], "Stochastic Integration with Respect to Local Time", *Seminar on Stochastic Processes 1982*, pp. 237–302, Birkhäuser, Boston.

Jay Rosen
Faculty of Industrial Engineering and Management
Technion—Israel Institute of Technology
Haifa 32000
ISRAEL

Permanent Address:

Department of Mathematics
College of Staten Island, CUNY
Staten Island, New York 10301
U.S.A.