# Self-Intersections of Stable Processes in the Plane: Local Times and Limit Theorems

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## 1. Introduction

$X_t$ will denote the symmetric stable process of index $\beta>1$ in $\mathbb{R}^2$, with transition density $p_t(x)$ and $\lambda$–potential

$$G_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x)\, dt.$$

We recall that

$$(1.1) \qquad G_0(x) = \frac{\Gamma(\frac{2-\beta}{2})}{\Gamma(\beta/2)} \cdot \frac{1}{2^\beta \pi} \cdot \frac{1}{|x|^{2-\beta}}$$

To study the k–fold self–intersections of X we will attempt to give meaning to the formal expression

$$(1.2) \qquad \int_{0\le t_1\le \cdots \le t_k \le t} \cdots \int \delta(X_{t_2} - X_{t_1}) \cdots \delta(X_{t_k} - X_{t_{k-1}})$$

Let $f\geq 0$ be a continuous function supported in the unit disc, and set

$$f_\epsilon(x) = \frac{1}{\epsilon^2} f(x/\epsilon)$$

If we think of $f_\epsilon$ as an approximate $\delta$ function, we are led to consider

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$$(1.3)\quad \alpha_{k,\epsilon}(t) = \underset{0\le t_1\le\ldots\le t_k\le t}{\int\cdots\int} dt_1 \prod_{i=2}^{k} f_\epsilon(X_{t_i} - X_{t_{i-1}})dt_i$$

as an approximation to (1.2).

As $\epsilon \to 0$, $\alpha_{k,\epsilon}(t)$ will diverge (due to the contributions near the 'diagonals' $\{t_i=t_j\}$). To get a non-trivial limit we must 'renormalize', which in our case means subtracting from $\alpha_{k,\epsilon}(t)$ terms involving lower order intersections. Thus, we define the approximate renormalized self-intersection local time,

$$(1.4)\qquad \gamma_{k,\epsilon}(t) = \sum_{j=1}^{k} (-h_\epsilon)^{k-j} \binom{k-1}{j-1} \alpha_{j,\epsilon}(t)$$

$$= \underset{0\le t_1\le\ldots\le t_k t}{\int\cdots\int} dt_1 \prod_{i=2}^{k} \left[f_\epsilon(X_{t_i} - X_{t_{i-1}})dt_i - h_\epsilon\ \delta_{t_{i-1}}(dt_i)\right]$$

where

$$(1.5)\qquad h_\epsilon = \int f_\epsilon(x) G_0(x) d^2x = \frac{1}{\epsilon^{2-\beta}} \int G_0(x)\ f(x)\ d^2x.$$

Note that $\gamma_{1,\epsilon}(t) = t$.

Following Dynkin [1988B], to reduce our anlaysis to managable proportions, rather than study $\gamma_{k,\epsilon}(t)$ for fixed t, we study $\gamma_{k,\epsilon}(\zeta)$ where $\zeta$ is an independent exponential random variable

$$(1.6)\qquad P(\zeta>t) = e^{-\lambda t}$$

We will find that $\gamma_{k,\epsilon}(\zeta)$ converges, as $\epsilon \to 0$, if and only if $\beta$ is sufficiently large. We recall that X has k-fold self-intersections if and only if $k(2-\beta)<2$.

**Theorem 1:** If $(2k-1)(2-\beta)<2$, then $\gamma_{k,\epsilon}(\zeta)$ converges in $L^2$ to a non-trivial random variable denoted by $\gamma_k(\zeta)$. Moreover, we have

$$\|\gamma_{k,\epsilon}(\zeta) - \gamma_k(\zeta)\|_2 \le c\ \epsilon^{\alpha/2}$$

where
$$\alpha = 2-(2k-1)(2-\beta)>0$$

Aside from the intrinsic interest of $\gamma_k(\zeta)$ as a measure of k-fold intersections, we hope to show in future work that $\gamma_k(\zeta)$ arises naturally in the asymptotic expansion for the area of the 'stable sausage'

$$S_\epsilon = \left\{x \in \mathbb{R}^2 \,\middle|\, \inf_{0\le s<\zeta} \|X_s - x\| \le \epsilon\right\}$$

generalizing the work of LeGall [1988] for Brownian motion. We also note our previous work involving a different form of renormalization, Rosen [1986]. The simplifications arising from the present form of renormalization will be most helpful in what follows.

When the condition of Theorm 1 is not satisfied, $\gamma_{k,\epsilon}(\zeta)$ will not converge in $L^2$. Instead, appropriately normalized, we get a central limit type theorem involving L, a random variable with density $\frac{1}{2}e^{-|x|}$, [known as Laplace's first law].

**Theorem 2:** If $(2k-1)(2-\beta) = 2$ then

$$\frac{\gamma_{k,\epsilon}(\zeta)}{\sqrt{\lg(1/\epsilon)}} \xrightarrow{\text{(dist.)}} \left[\sqrt{\frac{c(\beta,k)}{\lambda}}\ \right] L$$

where

$$c(\beta,k) = 2\pi \left[\frac{\Gamma\left[\frac{2-\beta}{2}\right]}{\Gamma(\beta/2)} \cdot \frac{1}{2^\beta \pi}\right]^{2k-1}.$$

**Remark:** (i) compare (1.1).

(ii). If $B_t$ denotes a real Brownian motion then $B_\zeta$ and $\frac{1}{\sqrt{2\lambda}} L$ have the same law. This provides a conceptual link between Theorem 2 and Rosen [1988], Yor [1985].

**Theorem 3:** If $(2k-1)(2-\beta)>2$ but $(2(k-1)-1)(2-\beta)<2$ then

$$\epsilon^{\alpha/2}\gamma_{k,\epsilon}(\zeta) \xrightarrow{(dist.)} \left[\sqrt{\frac{c(\beta,k)}{\lambda}}\,\right] L$$

where $\alpha = (2k-1)(2-\beta)-2>0$ and $c(\beta,k)$ is an explicit constant.

**Remark:** In the proof of Theorem 3, we will find that

$$c(\beta,k) = \lim_{\epsilon\to 0} \frac{\lambda\epsilon^{\alpha}}{2} E(\gamma^2_{k,\epsilon}(\zeta)),$$

and we will give an explicit formula for $c(\beta,k)$.

For more information on self-intersection local times see the survey of Dynkin [1988A] and the references therein.

## 2. Preliminaries

We have formulated our theorems in terms of $\gamma_{k,\epsilon}(t)$, an expression which does not involve $\lambda$, the parameter of the exponential time $\zeta$. In our proofs, it will be more convenient to work with

$$(2.1)\qquad \Gamma_{k,\epsilon}(t) = \sum_{j=1}^{k}(-H_\epsilon)^{k-j}\begin{bmatrix}k-1\\ j-1\end{bmatrix}\alpha_{k,\epsilon}(t)$$

$$= \int\cdots\int_{0\le t_1\le\ldots\le t_k\le t} dt_1 \prod_{j=1}^{k}\left[f_\epsilon(X_{t_i}-X_{t_{i-1}})\, dt_i - H_\epsilon\delta_{t_{i-1}}(dt_i)\right]$$

which differs from $\gamma_{k,\epsilon}(t)$, (1.4) in that $h_\epsilon=\int f_\epsilon(x)G_0(x)dx$ is replaced by

$$(2.2)\qquad H_\epsilon=\int f_\epsilon(x)G_\lambda(x)\ dx$$

It is easily checked that

$$(2.3)\qquad \gamma_{k,\epsilon}(t) = \sum_{j=1}^{k}\left[-(h_\epsilon - H_\epsilon)\right]^{k-j}\binom{k-1}{j-1}\ \Gamma_{j,\epsilon}(t).$$

This expression will allow us to derive results about the $\gamma$'s from results on the $\Gamma$'s.

The main point of this section is to derive a useful expression for

$$(2.4)\qquad E\Big[\prod_{j=1}^{n}\ \Gamma_{k,\epsilon_j}(\zeta)\Big]$$

$$=E\Big[\ \int\cdots\int\prod_{j=1}^{n}dt_1^j\ \prod_{i=2}^{k}\Big[f_{\epsilon_j}\Big(X_{t_i^j}-X_{t_{i-1}^j}\Big)dt_i^j-H_{\epsilon_j}\delta_{t_{i-1}^j}(dt_i^j)\Big]\Big]$$

$$=\sum_D I(D)$$

where

$$(2.5)\qquad I(D)$$

$$=E\Big[\int\underset{D}{\cdots}\int\prod_{j=1}^{n}dt_1^j\prod_{i=2}^{k}\Big[f_{\epsilon_j}\Big(X_{t_i^j}-X_{t_{i-1}^j}\Big)dt_i^j-H_\epsilon\delta_{t_{i-1}^j}(dt_i^j)\Big]\Big]$$

and D runs over the set of orderings of the nk+1 points $0,t_i^j$; $1\le i\le k$; $1\le j\le n$; such that $0\le t_1^j\le t_2^j\le\cdots\le t_k^j$ for all j.

Fix D. We call a set S of t's **elementary**, relative to D, if

$$(2.6)\qquad S=\left\{t_i^j,\ t_{i+1}^j,\cdots,\ t_{i+\ell}^j,\ t_{\bar i}^{\bar j}\right\}$$

and S satisfies

a) $t_{i+\ell}^j\le t_{\bar i}^{\bar j}$

b) no other t's come between $t_i^j$ and $t_{\bar i}^{\bar j}$ in D, (except $t_{i+m}^j$, $2\le m\le\ell$)

c) S is maximal in the sense that the t preceeding $t_i^j$ in D is not $t_{i-1}^j$.

With such an elementary sequence S, (2.6), we associate a function $H_S(Y)$ of nk variables

$$Y = \left\{Y_i^j;\ 1\leq i\leq k;\ 1\leq j\leq n\right\}$$

by the formula

$$(2.7)\quad H_S(Y) = G_\lambda(y_{i+1}^j)\cdots G_\lambda(y_{i+\ell}^j)\Delta^\ell_{y_{i+1}^j,\ldots,y_{i+\ell}^j} G_\lambda(Y_i^j - Y_{\bar{i}}^{\bar{j}})$$

Here $y_{i+1}^j \doteq Y_{i+1}^j - Y_i^j$, etc.

$$\Delta^\ell_{a_1,\ldots,a_\ell} F = \Delta_{a_1}\Delta_{a_2}\cdots\Delta_{a_\ell} F$$

and

$$\Delta_a F(x) = F(x+a) - F(x)$$

In particular, if $S = \left\{t_i^j,\ t_{\bar{i}}^{\bar{j}}\right\}$ has only two elements, then the above reduces to

$$(2.8)\qquad H_S(Y) = G_\lambda(Y_{\bar{i}}^{\bar{j}} - Y_i^j)$$

Let $\epsilon(D)$ denote the elementary sequences in D. Our formula for I(D) is

$$(2.9)\qquad I(D) = \int\cdots\int\left[\prod_{\substack{i=2\\ \forall j}}^{k} f_{\epsilon_j}(y_i^j)\right]\prod_{S\in\mathcal{E}(D)} H_S(Y)\,dY$$

as is easily checked, using (2.2).

The following lemma, proven in Section 7 is basic.

**Lemma 1:** Let $\beta > 1$, then

$$(2.10)\qquad 0 \leq G_\lambda(z) \leq c\left[G_0(z)\wedge\frac{1}{|z|^3}\right].$$

If $|z| \geq 2\ell\epsilon$, then

$$(2.11)\qquad \sup_{|a_i|\leq\epsilon}\left|\Delta^\ell_{a_1,\ldots,a_\ell} G_\lambda(z)\right| \leq c\left[\frac{\epsilon}{|z|}\right]^\ell G_0(z)\ \mathbb{R}(z)$$

$$(2.12)\qquad \sup_{|a_i| \leq \epsilon} \left[\prod_{i=1}^{\ell} G_\lambda(a_i)\right] \left|\Delta^{\ell}_{a_1,\dots,a_\ell} G_\lambda(z)\right| \leq c\, G_0^{1+\ell}(z) \left[\frac{\epsilon}{|z|}\right]^{(\beta-1)\ell} R(z)$$

where $R(z)$ is a bounded monotone decreasing integrable function. (In fact we can take $R(z) = \frac{1}{1+z^{1+\beta}}$).

If $S \in \epsilon(D)$ has the form (2.6), we say that S has length $\ell$, and write $\ell(S) = \ell$. For this S, (2.7) and lemma 1 mean that

$$(2.13)\qquad |H_S(Y)| \leq c\, G_0^{\ell(S)+1}(Z) \left[\frac{\epsilon_j}{|Z|}\right]^{(\beta-1)\ell(S)} R(z)$$

whenever $Z = Y^{\bar{j}}_{\bar{i}} - Y^{j}_{i}$ satisfies $|Z| \geq 2\ell\epsilon_j$.

## 3. Proof of Theorem 1

From now on, $\lambda$ is fixed and $G(x)$ without a subscript will refer to $G_\lambda(x)$. Similarly, we write $\gamma_{k,\epsilon}$ for $\gamma_{k,\epsilon}(\zeta)$, etc.

We first show that to prove theorem 1, it suffices to prove the following analogue for $\Gamma$.

**Proposition 1:** If $(2-\beta)(2k-1)<2$, then $\Gamma_{k,\epsilon}$ converges in $L^2$ to a non-trivial random variable denoted by $\Gamma_k$. Moreover, we have

$$(3.1)\qquad \|\Gamma_{k,\epsilon} - \Gamma_k\|_2 \leq c\, \epsilon^{\alpha/2}$$

where $\alpha = 2-(2k-1)(2-\beta) > 0$.

To see that proposition 1 implies theorem 1, define

$$(3.2)\qquad H(x) = G_0(x) - G_\lambda(x) = \int_0^\infty (1-e^{-\lambda t}) p_t(x)\,dt = \frac{1}{(2\pi)^2} \int e^{ipx} \frac{\lambda}{p^\beta(\lambda+p^\beta)}\, d^2p.$$

Since $\beta > 1$, $H(x)$ is continuous, bounded and

$$(3.3)\qquad |h_\epsilon - H_\epsilon - H(o)| = |\int f_\epsilon(x)\ [H(x) - H(o)]dx$$

$$= |\ \frac{1}{(2\pi)^2}\int f_\epsilon(x)\left[\int (e^{ipx}-1)\ \frac{\lambda}{p^\beta(\lambda+p^\beta)}\ d^2p\right]dx$$

$$\leq c\left[\int f_\epsilon(x)\ \ |x|^\delta\ dx\right]\left[\int \frac{p^\delta}{p^\beta(\lambda+p^\beta)}\ d^2p\right]$$

$$\leq c\ \ \epsilon^\delta\int \frac{p^\delta}{p^\beta(\lambda+p^\beta)}\ d^2p$$

for any $0 \leq \delta \leq 1$.

Thus,

$$(3.4)\qquad |h_\epsilon - H_\epsilon - H(o)| \leq \begin{cases} c\epsilon, & \text{if } \beta > 3/2 \\ c\epsilon^{2\beta-2-\bar\delta}, & \text{if } \beta \leq 3/2 \end{cases}$$

for any $\bar\delta > 0$.

We write

$$2\beta - 2 - \bar\delta$$
$$= 2-2(2-\beta) - \bar\delta$$
$$= \frac{1}{2}(2-(2k-1)(2-\beta)) + 1 - \bar\delta + (k - \frac{5}{2})(2-\beta)$$
$$> \frac{1}{2}(2-(2k-1)(2-\beta))$$

since $k \geq 2$, and $\bar\delta > 0$ can be chosen small.

Since, obviously

$1 > \frac{1}{2}(2-(2k-1)(2-\beta))$, (3.4) gives

$$(3.5)\qquad |h_\epsilon - H_\epsilon - H(o)| \leq c\ \epsilon^{(2-(2k-1)(2-\beta))/2}$$

so that (2.3) and proposition 1 now imply Theorem 1, with

$$(3.6)\qquad \gamma_k \doteq \sum_{j=1}^{k} (-H(o))^{k-j}\begin{bmatrix}k-1\\ j-1\end{bmatrix}\Gamma_j$$

Proposition 1 will follow from

**Proposition 2:** If $(2-\beta)(2k-1) < 2$, then for

$$0 < \epsilon \le \bar{\epsilon} \le 2\epsilon < 1$$

we have

(3.7) $$\|\Gamma_{k,\epsilon} - \Gamma_{k,\bar{\epsilon}}\|_2 \le c\ \bar{\epsilon}^{\alpha/2}$$

where $\alpha = 2 - (2k-1)(2-\beta)$

For, assume proposition 2. Given any $0 < \epsilon < \bar{\epsilon} < 1$. choose $n \ge 0$ such that

$$\frac{\bar{\epsilon}}{2^{n+1}} < \epsilon \le \frac{\bar{\epsilon}}{2^n}.$$ Then by (3.7),

(3.8) $$\begin{aligned}\|\Gamma_{k,\epsilon} - \Gamma_{k,\bar{\epsilon}}\|_2 &\le \sum_{i=0}^{n-1} \|\Gamma_{k,\bar{\epsilon}/2^i} - \Gamma_{k,\bar{\epsilon}/2^{i+1}}\|_2 \\ &\quad + \|\Gamma_{k,\bar{\epsilon}/2^n} - \Gamma_{k,\epsilon}\|_2 \\ &\le c\ \bar{\epsilon}^{\alpha/2} \sum_{i=0}^{n} \frac{1}{(2^{\alpha/2})^i} \\ &\le c\ \bar{\epsilon}^{\alpha/2}.\end{aligned}$$

This shows the $L^2$ convergence of $\Gamma_{k,\epsilon}$, and also establishes (3.1)

**Proof of Proposition 2:** According to section 2

(3.9) $$\|\Gamma_{k,\epsilon} - \Gamma_{k,\bar{\epsilon}}\|_2^2$$

$$= E((\Gamma_{k,\epsilon} - \Gamma_{k,\bar{\epsilon}})^2) = \sum_D I(D)$$

$$= \sum_D \int \cdots \int F_{\epsilon,\bar{\epsilon}}(y^1.) F_{\epsilon,\bar{\epsilon}}(y^2.) \prod_{S \in \mathcal{E}(D)} H_S(Y)\, dY$$

where

(3.10) $$F_{\epsilon,\bar{\epsilon}}(y.) = \prod_{i=2}^{k} f_\epsilon(y_i) - \prod_{i=2}^{k} f_{\bar{\epsilon}}(y_i)$$

Fix D.

The ordering D, in a natural way, induces an ordering on $Y^1_{\cdot}$, $Y^2_{\cdot}$. Thus, if $t^m_i \leq t^{\bar{m}}_{\bar{i}}$, we will say that $Y^m_i$ comes before $Y^{\bar{m}}_{\bar{i}}$. This induces an order on $\mathcal{E}(D)$. We may assume that the first element in D is $t^1_1$, hence our first element of $\mathcal{E}(D)$ is $\{o, t^1_1\}$ giving rise to the factor $G(Y^1_1)$. Let $S = \left\{t^1_1, \ldots, t^1_l, t^2_1\right\}$ be the next element in $\mathcal{E}(D)$. Let $Z = Y^2_1 - Y^1_1$.

We first show that the contribution to I(D) from the region $\left\{|Z| \leq 4k\bar{\epsilon}\right\}$ is $O(\epsilon^\alpha)$.

To see this, we first integrate the Y's in reverse order; we start with the last Y and integrate successively until we reach $Y^2_1$ using the bound

(3.11)
$$\int f_\epsilon(x-a)G(x)\, dx$$

$$= c\int e^{ipa}\, \hat{f}(\epsilon\, p)\, \frac{1}{\lambda+p^\beta}\, d^2p$$

$$\leq c\int |\hat{f}(\epsilon\, p)|\, \frac{1}{p^\beta}\, d^2p$$

$$\leq \frac{c}{\epsilon^{(2-\beta)}}.$$

For the $Y^2_1$ integral we use

(3.12)
$$\int_{|Z|\leq 4k\bar{\epsilon}} G(Y^2_1 - Y^1_m)\, dY^2_1$$

$$\leq \int_{|Z|\leq 6k\bar{\epsilon}} G(Z)\, d^2Z$$

$$\leq \int_{|Z|\leq 6k\bar{\epsilon}} G_o(Z)\, d^2Z$$

$$\leq c \frac{\bar{\epsilon}^2}{\bar{\epsilon}^{(2-\beta)}}$$

The remaining $Y^1_\ell, Y^1_{\ell-1}, \ldots, Y^1_2$ integrals are handled using (3.11), and finally $\int G(Y^1_1)dY^1_1 = \frac{1}{\lambda}$.

Since there were $\leq 2k$ G factors in (3.9), we find that the contribution from the region $\left\{|Z| \leq 4k\bar{\epsilon}\right\}$ is

$$0\left[\frac{\epsilon^2}{\epsilon^{(2-\beta)(2k-1)}}\right] = 0(\epsilon^\alpha)$$

Thus, for the remainder of our proof we can assume that $|Z| \geq 4K\bar{\epsilon}$. In view of (2.13), we can bound the integral $I(D)$ over $|Z| \geq 4k\bar{\epsilon}$ by

$$(3.13) \qquad \int_{|Z| \geq 4k\bar{\epsilon}} G_o^{2K-1}(Z)\left[\frac{\bar{\epsilon}}{|Z|}\right]^{(\beta-1)\ell(D)} R(Z)dZ$$

where $\ell(D) = \sum_{S \in \mathcal{E}(D)} \ell(S)$.

If $\ell(D) \geq 2$, we can bound (3.13) by replacing $(\beta-1)\ell(D)$ with

$$2(\beta-1) = 2 - 2(2-\beta),$$

giving

$$(3.14) \qquad \int_{|Z| \geq 4k\bar{\epsilon}} \bar{\epsilon}^{2-2(2-\beta)} \frac{1}{|Z|^{2+(2k-3)(2-\beta)}} dZ = 0(\bar{\epsilon}^\alpha)$$

since $k \geq 2$.

We can thus assume that $\ell(D) \leq 1$. If $\ell(D) = o$, D must be the ordering

$$(3.15) \qquad D_* = t^1_1 \leq t^2_1 \leq t^1_2 \leq t^2_2 \leq t^1_3 \leq \ldots \leq t^2_k$$

and then

$$(3.16) \quad I(D_*) = \int F_{\epsilon,\bar{\epsilon}}(y^1)F_{\epsilon,\bar{\epsilon}}(y^2)\prod_{i=1}^{k} G(Y^1_i - Y^2_{i-1})G(Y^2_i - Y^1_i)dY$$

We note that

$$G(Z+a+b) = G(Z)+\Delta_a G(Z) + \Delta_b G(Z) + \Delta^2_{a,b} G(Z)$$

and we use this to expand $G(Y^2_i - Y^1_\ell)$, with $Z = Y^2_1 - Y^1_1$ as before

and
$$a = Y^1_1 - Y^1_\ell = -\sum_{j=2}^{l} y^1_j$$
$$b = Y^2_i - Y^2_1 = \sum_{j=2}^{i} y^2_j$$

We can thus write the product in I(D) as a sum of monomials in $G(Z)$, $\Delta_a G(Z)$ and $\Delta^2_{a,b} G(Z)$. If any monomial contains either a $\Delta^2 G$ factor, or 2 $\Delta G$ factors then we can use (2.11), in a manner similar to (3.13), (3.14) to show that the integral over $|Z| \geq 4k\bar{\epsilon}$ is $O(\bar{\epsilon}^\alpha)$.

But, because of the factor $F_{\epsilon,\bar{\epsilon}}(y^1_\cdot)\ F_{\epsilon,\bar{\epsilon}}(y^2_\cdot)$ in (3.16), it is clear that the integral will vanish if our monomial is of the form $G^{2k-1}(Z)$ or $G^{2k-1}(Z)\Delta_a G(Z)$.

A similar analysis applies to the case of $\ell(D) = 1$, completing the proof of proposition 2, hence of Theorem 1.

## 4. The second moment

In this section we calculate the asymptotics of $E(\Gamma^2_{k,\epsilon})$ as $\epsilon \to 0$. If $(2k-1)(2-\beta) < 2$, then the last section shows that

(4.1) $$E(\Gamma^2_{k,\epsilon}) \longrightarrow \frac{2}{\lambda} \int G^{2k-1}(z)\ d^2 z.$$

Consider now the case $(2k-1)(2-\beta) = 2$, so that $\alpha = 0$. It is easily checked that all estimates of the previous section which were $O(\epsilon^\alpha)$, also hold in this case, i.e. are

$O(1)$, leading to

$$(4.2) \qquad E(\Gamma^2_{k,\epsilon}) = \frac{2}{\lambda} \int_{|Z|\geq 4k\epsilon} G^{2k-1}(z)d^2z + O(1)$$

$$= \frac{2}{\lambda} \int_{4k\epsilon\leq|z|\leq 1} G^{2k-1}(z)d^2z + O(1)$$

since $G(z)$ is bounded and integrable for $|z|\geq 1$.

As in (3.2), we write

$$(4.3) \qquad G(z) = G_0(z) - H(z)$$

with $H$ bounded, and we find immediately that (using (1.1))

$$(4.4) \qquad E(\Gamma^2_{k,\epsilon}) = \frac{2}{\lambda} \int_{4k\epsilon\leq|Z|\leq 1} G_0^{2k-1}(z)d^2z + O(1)$$

$$= 2\,\frac{c(\beta,k)}{\lambda}\,\lg(1/\epsilon) + O(1)$$

where $c(\beta,k) = 2\pi\left[\frac{\Gamma\left[\frac{2-\beta}{2}\right]}{\Gamma(\beta/2)}\,\frac{1}{2^\beta\pi}\right]^{2k-1}$ as in Theorem 2.

We next consider the case where $(2k-1)(2-\beta) > 2$. Here we will see that all orderings $D$ will contribute a term of order $\frac{1}{\epsilon^\alpha}$ (where now $\alpha = (2k-1)(2-\beta)-2>0$), plus terms of lower order.

Consider a fixed ordering $D$ as before, and

$$(4.5) \qquad I(D) = \int\cdots\int F_\epsilon(y^1_\cdot)F_\epsilon(y^2_\cdot) \prod_{S\epsilon\in(D)} H_S(Y)\,dY$$

with

$$(4.6) \qquad F_\epsilon(y_\cdot) = \prod_{i=2}^{k} f_\epsilon(y_i).$$

Assume for definiteness, as in section 3, that the first element in $\in(D)$ is $\left\{o,t^1_1\right\}$, so that we have a factor $G(Y^1_1)$ in (4.5). We change variables

$Y^1_\cdot, Y^2_\cdot \longrightarrow X_i$, $i = 1,\dots,2r$ where $X_i$ is the argument of the i'th G factor in I(D). More precisely, if the i'th interval in $D = \left\{0<t^1_1<\dots\right\}$ is $t^m_j < t^{\bar{m}}_{\bar{j}}$, then $X_i = Y^{\bar{m}}_{\bar{j}} - Y^m_j$. We integrate out $dX_1 = dY^1_1$ and write

$$(4.7)\quad I(D) = \frac{1}{\lambda}\int\cdots\int F_\epsilon(y^1_\cdot)F_\epsilon(y^2_\cdot)\prod_{S\in\bar{\mathcal{E}}(D)} H_S(Y)\ dX_2\dots dX_{2r}$$

where $\bar{\mathcal{E}}(D)$ is obtained from $\mathcal{E}(D)$ by removing the first sequence, $\left\{o, t^1_1\right\}$.

We write $G(z) = G_o(z) - H(z)$ as in (4.3), and use this to rewrite (4.7) as the sum of many terms. One term is

$$(4.8)\quad \frac{1}{\lambda}\int F_\epsilon(y^1_\cdot)F_\epsilon(y^2_\cdot)\prod_{S\in\bar{\mathcal{E}}(D)} H^o_S(Y)\ dX_2\cdots dX_{2k}$$

where $H^o_s$ is defined by replacing each G in $H_S$ with $G_o$. The other terms arising from (4.7) differ from (4.8) in that at least one G has been replaced by H. We first deal with (4.8), which will turn out to be the dominant term.

We scale in (4.8), and obtain

$$(4.9)\quad \frac{1}{\epsilon^\alpha}\frac{1}{\lambda}\int F(y^1_\cdot)F(y^2_\cdot)\prod_{S\in\bar{\mathcal{E}}(D)} H^o_S(Y)dX_2\cdots dX_{2k}$$

where now

$$F(y) = \prod_{i=2}^{k} f(y_i).$$

Let us show that the integral in (4.9) converges. If the first sequence in $\bar{\mathcal{E}}(D)$ is $\left\{t^1_1, t^1_2, \dots t^1_\ell, t^2_1\right\}$, set $Z = X_{\ell+1} = Y^2_1 - Y^1_\ell$. If $|Z| \geq 4k$, then by the $H^o_S$ analogue of (2.13) we can bound our integral by

$$c\int_{|Z|\geq 4k} G^{2k-1}_o(z)\ dz < \infty.$$

If, on the other hand, $|Z| \leq 4k$. then all $|X_i| \leq 8k$, and using $\int_{|X| \le c} G_0(x)\, dx < \infty$ we can bound our integral by integrating in reverse order $dX_{2k}, \ldots, d\ell$.

Next, consider a term arising from the expansion of (4.7), in which at least one of the $G_0$ factors of (4.8) has been replaced by $H(\cdot)$.

If $|Z| \le 4k\epsilon$, we first bound any $H(\cdot)$ factor by a constant, and then scale. We obtain an integral, which can be bounded as above (since now $|Z| \le 4k$) multiplied by $\frac{1}{\epsilon^{\bar{\alpha}}}$ with $\bar{\alpha} < \alpha$.

If $|Z| \ge 4k\epsilon$, then by (7.10) and (7.12) we find that for any $\ell$, including $\ell = 0$, and $|a_i| \le \epsilon$,

$$(4.10)\quad |\Delta^{\ell}_{a_1,\ldots,a_\ell} H(x)| \le c\left[\frac{|a_1|\cdot\ldots|a_\ell|}{x^{\ell}} \cdot \frac{1}{x^{2-\beta}}\right] \wedge 1$$

$$\le c\left[\frac{(|a_1|\cdot\ldots|a_\ell|)^{2-\beta}}{x^{(2-\beta)(\ell+1)}}\right]^{\delta} \qquad ,|X| \ge 2\ell\epsilon$$

for any $0 \le \delta \le 1$. Scaling with these bounds, gives a factor $\frac{1}{\epsilon^{\bar{\alpha}}}$ with $\bar{\alpha} < \alpha$ if $\delta < 1$, and an integral which can be bounded as long as $\delta$ is chosen close enough to 1 so that

$$(2k-1)(2-\beta)\,\delta > 2.$$

Thus we finally have

$$(4.11)\quad E\left[\Gamma^2_{k,\epsilon}\right] = \frac{1}{\epsilon^{\alpha}}$$

$$\frac{1}{\lambda}\sum_{D}\int\cdots\int F(y^1_{\cdot})F(y^2_{\cdot}) \prod_{S\in\bar{E}(D)} H^0_S(Y)\,dX_2\cdots dX_{2k}$$

$$+o\left[\frac{1}{\epsilon^{\alpha}}\right]$$

## 5. Proof of Theorem 2

We proceed by the method of moments. Since

$$(5.1)\qquad \begin{cases} E(L^{2n}) = (2n)! \\ E(L^{2n+1}) = 0 \end{cases}$$

it suffices to show that

$$(5.2)\qquad \begin{cases} E\left[\dfrac{\Gamma_{k,\epsilon}}{\sqrt{\lg(1/\epsilon)}}\right]^{2n} \longrightarrow (2n)! \left[\dfrac{c(\beta,k)}{\lambda}\right]^{n} \\ E\left[\dfrac{\Gamma_{k,\epsilon}}{\sqrt{\lg(1/\epsilon)}}\right]^{2n+1} \longrightarrow 0 \end{cases}$$

in order to get

$$\frac{\Gamma_{k,\epsilon}}{\sqrt{\lg(1/\epsilon)}} \xrightarrow{\text{(dist)}} \left[\sqrt{\frac{c(\beta,k)}{\lambda}}\right] L,$$

which then implies Theorem 2, by (2.3) and Theorem 1.

We recall from section 2 that

$$(5.3)\quad E(\Gamma^{m}_{k,\epsilon}) = \sum_{D} \int \cdots \int \left[\prod_{j=1}^{m} F_{\epsilon}(y^{j}_{\cdot})\right] \prod_{S\in\mathfrak{E}(D)} H_S(Y)\, dY$$

where D runs over all orderings of

$\left\{o, t^{j}_{i},\ j=1,\dots,m; i=1\dots,k\right\}$

Let

$$(5.4)\qquad U(D) = \bigcup_{j=1}^{m} \left[t^{j}_{1},\ t^{j}_{k}\right]$$

U(D) naturally decomposes into the union of its components; $U^1$, $U^2,\dots,U^j$. If, say,

$$U^{i} = \bigcup_{\ell=1}^{p} \left[t^{j_\ell}_{1},\ t^{j_\ell}_{k}\right]$$

then we say that $U^i$ has height p, and denote by $D^i$ the ordering induced on

$$\left\{o,\ t^{j_\ell}_{n},\quad \ell = 1,\dots,p;\quad n = 1,\dots,k\right\}$$

by D. By translation invariance we find that

(5.5) $$I(D) = \prod_{i=1}^{j} I(D^i)$$

It is clear from this that if any component of U(D) has height 1, then I(D) = 0. Furthermore, from section 4 we know that if $D^i$ has height 2, then

$$I(D^i) = \begin{cases} \frac{c(\beta,k)}{\lambda} \lg(1/\epsilon) + O(1), \text{ if } D = D_*, D_{**} \\ O(1) \quad \text{otherwise} \end{cases}$$

where $D_*$ is given by (3.15), and $D_{**}$ is obtained from $D_*$ by permuting $t^1_\cdot$ with $t^2_\cdot$.

If m = 2n, and U(D) has n components of height 2, then the above allows us to compute I(D), and since there are (2n)! ways to permute the $t^j$'s, we see that the contribution to (5.2) from orderings D with n components of height 2 is

$$(2n)! \left[\frac{c(\beta,k)}{\lambda}\right]^n (\lg 1/\epsilon)^n + O(\lg(1/\epsilon))^{n-1}$$

To complete the proof of (5.2) it suffices to show that if U(D) is connected and of height n > 2, then

(5.6) $$I(D) = o(\lg(1/\epsilon))^{n/2}$$

We will develop a three step procedure to prove (5.6).

We will refer to $Y^1_\cdot, Y^2_\cdot, \ldots, Y^n_\cdot$ as n letters, and to $Y^i_j$ as the j'th component of the letter $Y^i_\cdot$. If $S \epsilon \mathcal{E}(D)$ is of the form (2.6), i.e.,

(5.7) $$S = \left\{t^j_i, \ldots, t^j_{\ell+i}, t^{\bar{j}}_{\bar{i}}\right\}$$

and if $\ell > o$, then $H_S(Y)$, see (2.7), contains factors $G(y^j_{i+1}) \ldots G(y^j_{i+\ell})$, and we say that the letter $Y^j_\cdot$ has $\ell$ isolated G factors. This terminology refers to the fact that in these factors $Y^j_\cdot$ appears alone, without any other letter. Let

$$I = \left\{ i \,|\, Y^i_{\cdot} \text{ has isolated G factors} \right\}.$$

It is the presence of isolated G factors which complicates the proof of (5.6), and necessitates the three step procedure which we soon describe.

For each $S \in \mathcal{E}(D)$ of the form (5.7), (even if $\ell = o$) we write

$$(5.8) \quad H_S(Y) = H_S(Y) \left[ 1_{\left\{ |y_1^j - y_1^{\bar{j}}| \le 4n\epsilon \right\}} + 1_{\left\{ |Y_1^j - Y_1^{\bar{j}}| > 4n\epsilon \right\}} \right]$$

and expand the product in (5.3) into a sum of many terms. We work with one fixed term. We then say that $Y^j_{\cdot}$ and $Y^{\bar{j}}_{\cdot}$ are G–close or G–separated depending on whether the first or second characteristic function in (5.8) appears in our integral. If $Y^j, Y^{\bar{j}}$ never appear together in any $H_S(Y)$, then they are neither G–close nor G–separated. (This determination of G–close, etc. is fixed at the onset, and is not amended during the proof.)

For ease of reference we spell out two simple lemmas.)

<u>**Lemma 2:**</u> Let $g_i(Z) \ge 0$ be monotone decreasing in $|Z|$. If

$$(5.9) \qquad \int_{|Z| \ge \epsilon} \prod_{i=1}^{p} g_i(Z)\, d^n Z \le M(\epsilon).$$

then for any $a_1, \ldots, a_p$

$$(5.10) \qquad \int_{\{|Z-a_i| \ge \epsilon, \forall_i\}} \prod_{i=1}^{p} g_i(Z-a_i)\, d^n Z \le pM(\epsilon).$$

**Proof:** The integral in (5.10) is bounded by

$$\sum_{j=1}^{p} \int_{\left\{\begin{array}{l}|Z-a_i|\geq\epsilon,\forall_i \\ |Z-a_i|\geq|Z-a_j|,\forall_i\end{array}\right\}} \prod_{i=1}^{p} g_i(Z-a_i)d^nZ$$

$$\leq \sum_{j=1}^{p} \int_{|Z-a_j|\geq\epsilon} \prod_{i=1}^{p} g_i(Z-a_j)d^nZ \leq p\ M(\epsilon)$$

by (5.9). []

**Lemma 3**:

$$(5.11) \quad \int_{|Y_1|\leq\epsilon} F_\epsilon(y.) \prod_{i=1}^{\ell} G_o(Y_{j_i}-a_i)dY_1\ldots dY_k \leq c\epsilon^{2-\ell(2-\beta)}$$

**Proof:** See the discussion about (3.11), (3.12). []

If S is of the form (5.7), and if $Y^j_., Y^{\bar{j}}_.$ are G–separated we recall the bound of (2.13):

$$(5.12) \qquad |H_S(Y)| \leq c\ G_o^{\ell(s)+1}(Z)\left[\frac{\epsilon}{|Z|}\right]^{\delta} R(Z)$$

where $Z = Y^j_1 - Y^{\bar{j}}_1$, and $0 \leq \delta \leq (\beta-1)\ell(s)$ is at our disposal.

Let

$$(5.13) \quad I_0 = \left\{i\epsilon I | Y^i \text{ is not G–close to any } Y^j,\ j\epsilon I\right\}$$

$$(5.14) \qquad I_1 = I-I_0$$

We briefly outline our three steps, and then return to spell out the details. We integrate out one letter at a time, in a manner which allows us to keep track of potential problems.

<u>Step 1:</u> We integrate out $Y^i$, $i\epsilon I_0$ using (5.12) when applicable.

<u>Step 2:</u> We integrate out the letters from $I_1$, using (5.11) whenever possible.

<u>Step 3:</u> We integrate the letters from $I^c$, i.e. letters without isolated G–factors. This is the most straightforward case.

Before spelling out the details, we can immediately recognize a potential problem. After integrating several letters, we may, inadevertently, have integrated out all G–factors containing some other letter, not yet integrated. Its integral might then diverge. To remedy this, before integrating each letter we carry out the following.

<u>Preservation Step:</u> Before integrating Y., we search for any two letters, say X.,Z. with components which are separated only by components of Y. Thus we may have factors of the form

(5.15)

$$G\left[X-Y_i\right]G\left[Y_{i+1}-Y_i\right]\cdots G\left[Y_{i+\ell}-Y_{i+\ell-1}\right]\Delta^{\ell}_{y_{i+1},\ldots,y_{i+\ell}}G(Y-Z)$$

(if (5.12) is not applied) or (if (5.12) is applied) of the form

$$(5.16)\quad G\left[X - Y_i\right]\; G_0^{\ell(s)+1}\left[Y_1 - Z_1\right]\left[\frac{\epsilon}{|Y_1 - Z_1|}\right]^{\delta}\mathbb{R}\left[Y_1 - Z_i\right]$$

(We include the case X. = 0, i = 1).

In the case of (5.15), we write out $\Delta^{\ell}G$ as a sum of many terms, focus on one of them, say

$$G\left[Y_i + y_{j_1} + y_{j_2} + \ldots + y_{j_p} - Z\right]$$

From (5.15) we select the factors

$$(5.17)\quad G\left[X-Y_i\right]G\left[y_{j_1}\right]\ldots G\left[y_{j_p}\right]G\left[Y_i + y_{j_1} + \ldots + y_{j_p}-Z\right]$$

Now

$$(5.18)\quad |X-Z| \le \left|\left[X-Y_i\right]-y_{j_1}-y_{j_2}\ldots-y_{j_p}+\left[Y_i+y_{j_1}+\ldots+y_{j_p}-Z\right]\right|$$

$$\le |X-Y_i|+|y_{j_1}|+\ldots+|y_{j_p}|+|Y_i+y_{j_1}+\ldots+y_p-Z|$$

Hence $|X-Z|$ is less than (p+2) times the maximum of the terms on the right hand side of (5.18). Hence one of the factors in (5.17) can be bounded by a constant times $G(X-Z)$.

If we have the form (5.16), then necessarily $|Y_1-Z_1| \ge 4n\epsilon$. If $|X-Z_1| \le 4n\epsilon$, then we can bound

$$W(Y_1-Z_1) \doteq G_o(Y_1-Z_1)\left[\frac{\epsilon}{|Y_1-Z_1|}\right]^{\delta}R(Y_1-Z_1) \text{ by } W(X-Z_1).$$

Note that $W(\cdot)$ is integrable. If $|X - Z_1| \ge 4n\epsilon$, then we use

$$(5.19)\quad |X - Z_1| \le |X - Y_i| + |Y_i - Y_1| + |Y_1 - Z_1|$$

$$\le |X - Y_i| + k\epsilon + |Y_1 - Z|$$

so that

$$(5.20)\qquad |X - Z_1| \le 2(|X- Y_i| + |Y_1 - Z_1|)$$

so that as before we can replace either the first factor in (5.16) by $G(X - Z_1)$, or a factor $W(Y_1 - Z_1)$ by $W(X - Z_1)$.

Note that this step actually lowers the number of G–factors involving Y. prior to integrating Y.. After integrating Y., we find that we have not increased the number of G–factors involved with X., (or Z).

One way to think of this preservation step, is to suppress all Y.'s, and 'link up' with G or W the remaining letters which are now adjacent. (The case $X. = 0$ is included). The upshot is that we never lose any letters prior to their integration.

We finally remark that in (5.15), (5.16) we took our first factor to be $G(X. - Y_i)$. If this factor is actually $W(X. - Y_i)$ the same analysis pertains.

We now give the details of our three steps.

<u>Step 1:</u> We apply the bound (5.12) whenever S is of the form (5.7), with $j \in I_0$ having isolated G–intervals (i.e. $\ell(S) \neq 0$) and $|Y_1^j - Y_1^{\bar{j}}| \geq 4\epsilon n$. This is the only place we will apply (5.12). Note that (5.12) does not increase the total number of G–factors in our integral (we count both $G_\lambda$ and $G_0$), but may increase the number of G factors containing $Y^i_\cdot$. Let $N_i$ denote this latter quantity. I claim that

$$\sum_{i \in I_0} N_i \leq 2k|I_0|. \tag{5.12}$$

To see this, let $\ell(i)$ denote the number of isolated G–factors containing $Y^i$ in the original integral, i.e., prior to applying the bound (5.12). At that stage $Y^i_\cdot$ could not have appeared in more than $2k-\ell(i)$ G–factors. The effect of (5.12) is to replace certain of the $\ell(i)$ isolated G–factors each of which had contributed 1 to $N_i$ and zero to any $N_j$, $j \neq i$, by G–factors which contribute 1 to $N_i$ and, at most, 1 to one other $N_j$. This proves (5.12)

If some $N_i \leq 2k-1$ then as in section 4 the $dY^i$ integral is bounded. For, since $i \in I_0$, $Y^i$ has isolated G–factors – hence, either it is close to some other letter, in which case lemma 3 shows the integral to be O(1), or else we will have applied (5.12), in which case lemma 2, with $\delta > o$ small, will show our integral to be O(1) as seen in section 4. (But remember, we always apply the preservation step prior to integrating !).

We proceed in this manner integrating all $Y^i$ with $N_i \leq 2k-1$, (after each integration we update the remaining $N_j$'s).

If all remaining $N_i \geq 2k$, then since (5.21) still holds, showing that now all $N_i = 2k$. The analysis of (5.21), in fact, shows that in such a case isolated G–factors containing such $Y^i$ must be contained in factors $H_S(Y)$ containing a remaining $Y^j$, $j \in I_o$ and to which (5.12) has been applied; in particular, $|Y_1^i - Y_1^j| \geq 4n$. In such a case we check that $Y^i_\cdot, Y^j_\cdot$ cannot be contained together in all 2k factors, hence $Y^i$ must be contained in at least one factor with another letter, say $Y^{\bar{j}}$. If the preservation step does not directly reduce the number of G–factors containing $Y^i$, then, since $|Y_1^j - Y_1^i| \geq 4n\epsilon$, we can still bound one factor by $W(Y^j_\cdot - Y^{\bar{j}}_\cdot)$, by using the same approach as in the preservation step, arguing separately for $|Y_1^j - Y_1^{\bar{j}}| \leq 4n\epsilon$ or $> 4n\epsilon$.

In this manner we integrate out all letters $Y^i$, $i \in I_o$.

<u>Step 2:</u> $I_1$ is naturally partitioned into equivalence classes $Q_1, \ldots, Q_q$, where $i \sim j$ if we can find a sequence

$$i = i_1,\ i_2,\ i_3, \ldots,\ i_\ell = j$$

with $Y^{i_p}$ G–close to $Y^{i_{p+1}}$.

Consider $Q_1$. Choose a $j \in Q_1$ such that $\ell(j) \leq \ell(i)$, $\forall i \in Q_1$. All $Y^i$, $i \in Q_1$, are <u>close</u> to $Y^j$ in the sense that $|Y_1^i - Y_1^j| \leq 4n^2\epsilon$. We then use lemma 3 to integrate, in any order, all $Y_\cdot^i$,
$i \in Q_1$, $i \neq j$. Since $Q_1 \subsetneq I$, we have $\ell(i) \geq 1$ so that the contribution from the $dY_\cdot^i$ integral is at most

$$(5.22)\quad 0\left[\epsilon^{2 - (2k-\ell(i))(2-\beta)}\right] = 0\left[\epsilon^{(\ell(i)-1)(2-\beta)}\right]$$

The $dY_\cdot^j$ integral, which is done last, is at most

$$(5.23)\qquad 0\left[\epsilon^{-\ell(j)(2-\beta)}\right],$$

from the $\ell(j) \geq 1$ isolated G–factors.

Combining (5.22) and (5.23) with $\ell(j) \geq \ell(i) \geq 1$, we see that the total contribution from $Q_1$ is $0(1)$ unless either $\ell(i) = 1$, $\forall\, i \in Q_1$ or if some $\ell(i) > 1$, then necessarily $Q_1 = \{i,j\}$ and $\ell(i) = \ell(j)$. In the former case we can also integrate out all $i \neq j$ except for one – so in both cases we can reduce ourselves to $Q_1 = \{i, j\}$, $\ell(i) = \ell(j) \geq 1$. We call such a pair a twin. $Y^i$, $Y^j$ are close to each other, and we can assume they are close to no remaining letter (otherwise (5.23) can be improved to (5.22)). We leave such twins to step three.

We handle $Q_2, \ldots, Q_q$ similarly.

<u>Step 3:</u> We begin with the remaining letter, say $Y^i$, which appears at the extreme right. Because of this, $Y^i$ appears in $\leq$ 2k–1 G–factors. If $Y^i$ were part of a twin, then it has at most $2k - \ell(i) - 1$ G–factors, as opposed to the $2k - \ell(i)$ assumed for (5.22). This controls the twin.

If $Y^i$ is not part of a twin, then $i \in I^c$. If $Y^i$

appears in 2k–1 G–factors with $Y^j$, then the analysis of section 4, shows that the $dY^i$ $dY^j$ integral is at most $O(\lg(1/\epsilon))$.

It $Y^i$ appears with 2 letters, we already know how to reduce the number of G–factors, so that the $dY^i$ integral is bounded. We proceed in this manner until all letters are integrated.

This analysis shows that (5.6) holds unless $I = \phi$, and the rightmost letter has all G–factors in common with one other letter – but then these two letters form a component, contradicting the assumption that U(D) is connected of height > 2. This completes the proof of theorem 2.

## 6. Proof of Theorem 3

Taking over the notation of section 5, it suffices to show that if U(D) is connected and of height n > 2, then

(6.1) $$I(D) = o(\epsilon^{-\alpha})^{n/2}$$

where $\alpha = (2k-1)(2-\beta)-2$.

The situation here is more complicated than that of Theorem 2, since typically our integrals diverge and we must control the divergence. We make two major modifications. In (5.12) we now take $\delta = 0$, and in applying the preservation step, or any other time we bound a factor such as G or W with factors not involving X in order to reduce the number of factors involving X to $\leq$ 2k–2, we only bound $G^\gamma$, $W^\gamma$ where $\gamma$ is close to, but not equal to, one. This will not significantly affect the order of our X. integral – but when we come to integrate the other letters, a situation which would have led to $O(\epsilon^{-\alpha})$ with $\gamma = 1$ will now lead to $o(\epsilon^{-\alpha})$. These modifications will be taken for granted in what follows.

As in the last section, we will find that we can associate a factor $O(\epsilon^{-\alpha/2})$ with each letter, while at least one letter will be associated with $o(\epsilon^{-\alpha/2})$. By the remarks in the previous paragraph, and as detailed in the sequel, this will occur if any factors associated with our letter were obtained through a preservation like step.

We will assume that $(2k-2)(2-\beta) > 2$. The other cases are similar, but simpler.

<u>Step 1:</u> As in (5.21), we have

$$\sum_{i\in I_o} N_i \le 2k\ |I_o| \tag{6.2}$$

where $N_i$ are the number of G–factors involving $Y^i$, after application of (5.12).

If $N_i < 2k-1$ for any i, the $dY^i$ integral is $O\left[\epsilon^{-[(2k-2)(2-\beta)-2]}\right] = o\left[\epsilon^{-\alpha/2}\right]$, since our assumption

$$(2k-3)(2-\beta) < 2 \text{ implies } (2k-2)(2-\beta)-2 < (2-\beta).$$

Now assume $N_i = 2k-1$. If $Y^i$ is linked to at least two other letters, then as in section 5, we can reduce the number of factors involving $Y^i$, and now the $dY^i$ integral is $o(\epsilon^{-\alpha/2})$. If $Y^i$ is linked to only one other letter, say $Y^j$, then $N_i = 2k-1$ is possible only if all $Y^i, Y^j$'s are contiguous. (We note for later that $Y^j$ can be in $I^c$ or $I_o$ but not in $I - I_o$). The $dY^i$ integral is $O(\epsilon^{-\alpha})$, while the $dY^j$ integral will be bounded.

We can assume that all remaining $N_i \geq 2k$, so that by (6.2), we actually have $N_i = 2k$. We recall that this can occur only if (5.12) is applied with pairs in $I_o$. We leave this for the next step.

**Step 2:** We begin integrating from the right. Let X denote the rightmost remaining letter.

If $X \in I^c$, it has no isolated factors, and being rightmost can appear in at most 2k–1 G–factors (the extra factors arising from (5.12) have either been integrated away, or involve only letters from $I_o$). If there were actually < 2k–1 G–factors, then the dX integral would be $o(\epsilon^{-\alpha/2})$. If X is linked to two distinct letters, we can reduce the number of factors as before, while if all 2k–1 links are to the same letter, say Y, then Y is necessarily in $I^c$, and the dX integral is $O(\epsilon^{-\alpha})$, with the dY integral bounded.

If, as we integrate, we find the rightmost letter $X = Y^i \in I_o$, we can check that $N_i = 2k$ is no longer possible, and we return to the analysis of step 1.

Let us now suppose that the remaining rightmost letter

$$X \in I - I_o.$$

Then $X \in Q_i$ for some i, say i = 1. Assume first that X is within $4k^2\epsilon$ of some letter in $Q_1^c$ (we include o), then automatically an analogous statement holds for all letters in $Q_1$. Before applying this we consider all $Q_1$ as one letter and apply the preservation step to $Q_1^c$. This way, we do not attempt to preserve letters of $Q_1$ itself. By the definition of $Q_1$, each letter has at least one isolated

G–factor, hence $\leq$ 2k–1 G–factors, while X, being rightmost, must have $\leq$ 2k–2. We begin by integrating dX, giving $o(\epsilon^{-\alpha/2})$. Again, by the definition of $Q_1$, X had a G–factor in common with at least one other letter of $Q_1$, hence that letter now has $\leq$ 2k–2 G–factors and we can integrate it, again giving a contribution $o(\epsilon^{-\alpha/2})$. At any stage in our successive integration of the letters of $Q_1$, it must be that some remaining letter has had on G–factor removed – since $Q_1$ was defined by an equivalence relation. This gives a contribution $o(\epsilon^{-\alpha/2})$ for each letter of $Q_1$.

Assume now that $X \epsilon Q_1$ is not within $4k^2\epsilon$ of any letter in $Q^c$, so that in fact no letter of $Q_1$ is within $4k\epsilon$ of any letters of $Q_1^c$. If $|Q_1| \geq 3$, we integrate dX. We can use lemma 3 since X is close to the remaining letters of $Q_1$. Being the rightmost letter, its contribution is $o(\epsilon^{-\alpha/2})$. Prior to the dX integration we preserve all other letters, including $Q_1 - X$. Because of this, it is now possible that the remaining letters in $Q_1$ no longer form an equivalence class, but it will always be true that they are within $4k\epsilon$ of each other and of no letters in $Q_1^c$.

We continue in this fashion and can assume that X is in (an updated) $Q_1$, with $Q_1 = \{X,Y\}$. If $\ell(Y) \leq \ell(X)$, we do the dX integral using lemma 3 for a contribution $0\left[\epsilon^{2-(2k-\ell(X)-1)(2-\beta)}\right]$. When we reach Y, we have $\ell(Y)$ isolated G–factors contributing $0\left[\epsilon^{-\ell(Y)(2-\beta)}\right]$, and $\leq 2k - 2\ell(Y) - 1$ G–factors which give a convergent integral by lemma 2. Thus, the total contribution is $O(\epsilon^{-\alpha})$ if $\ell(Y) = \ell(X)$, and $o(\epsilon^{-\alpha})$ if in fact $\ell(Y) < \ell(X)$.

If, on the other hand $\ell(X) < \ell(Y)$, we first do the dY integral using lemma 3. Y has at most $2k-\ell(Y)$ G–factors. If in fact this is $\leq 2k-\ell(Y)-1 \leq 2k-\ell(X)-2$ then the dY integral is

$$O\left[\epsilon^{2-[2k-\ell(X)-2](2-\beta)}\right] = o(\epsilon^{-\alpha/2})\ O(\epsilon^{\ell(X)(2-\beta)})$$

and the dX integral is $O\left[\epsilon^{-\ell(X)(2-\beta)}\right]$ as above.

Otherwise, we preserve $Q_1^c$, then if Y still has $2k-\ell(Y)$ G–factors, we first assume that at least one of these G–factors links Y with some $Z \neq X$. We bound $G(Y-Z) \leq c\ G(X-Z)$, and after the dY integral there remain $\ell(X)$ isolated G–factors for X and $\leq 2k - 2\ell(X) \leq 2k - 2$ G–factors linking X with other letters. Thus the dX integral is bounded by $O\left[\epsilon^{-\ell(X)(2-\beta)}\right]o(\epsilon^{-\alpha/2})$ and altogether the dX dY integral is $o(\epsilon^{-\alpha})$.

If none of the $2k-\ell(Y)$ G–factors involving Y, involve any letters $Z \neq X$, then all non–isolated G–factors must link X and Y, in particular those factors to the immediate right and left. Since X occurs on the immediate left of Y, we needn't bother preserving it from the Y integration; which is

$$O\left[\epsilon^{2-(2k-\ell(Y))(2-\beta)}\right] = O\left[\epsilon^{2-(2k-1)(2-\beta)}\right]\ O\left[\epsilon^{(\ell(X)(2-\beta)}\right]$$
$$= O(\epsilon^{-\alpha})\ O\left[\epsilon^{\ell(X)(2-\beta)}\right]$$

and the contribution from dXdY is $O(\epsilon^{-\alpha})$.

In this manner we see that $I(D) = O(\epsilon^{-\alpha/2})^n$.

__Step 3:__ we must now show that in fact

$$I(D) = o(\epsilon^{-\alpha/2})^n \tag{6.3}$$

Let us agree to call two letters X,Y totally paired if there are no other letters between them. From the above analysis, we know that (6.3) holds unless D is such that all letters X fall into one of the following three types.

1) $X \in I^c$, and X is totally paired.

2) $X \in I_o$, and X totally paired. We recall that it cannot be paired with a letter from $I - I_o$.

3a) $X \in I - I_o$, and $X \in Q_i$, $|Q_i| = 2$. If, say $Q_1 = \{X,Y\}$, then necessarily X,Y are G–close, hence have at least one common G–factor, and by the above we know that $\ell(X) = \ell(Y)$ and X,Y are far (i.e. not within $4k\epsilon$) from $Q_1^c$.

3b) $Q_i = \{X,Y\}$ with X,Y totally paired.

Consider now Xhe very first letter on the right, X. X cannot be totally paired, since that would mean we have a component of height 2, contrary to our assumption that U(D) is connected of height $\geq 3$. Thus X is of type 3a, say $X \in Q_1 = \{X,Y\}$.

Once again, $Q_1$ cannot be totally paired, hence, proceeding from the right there is a first letter, call it Z interrupting X,Y. Following Z there may be other letters from $Q_1^c$ – we let W be the last of these prior to the next X or Y. (Of course, we can have $Z \equiv W$).

We begin by trying to preserve this W from $Q_1$. If this step removes a G–factor involving X or Y we break up the analysis into three cases.

a) If the removed G–factor contained X, then X now has $\leq 2k-\ell(X)-2$ G–factors, leading to an $o(\epsilon^{-\alpha/2})$ contribution as in step 2.

b) If the removed G–factor linked Y, but Z links X, then bound $G(X-Z) \leq c\ G(Y-Z)$. Now preserve $Q_1^c$ from $Q_1$. Once again X has $\leq 2k-\ell(X)-2$ factors, and while apriori Y has gained an extra G-factor, this gain is compensated by the loss of the G-factors which X,Y have in common. Note: we didn't have to preserve Y from the dX integration, because we have the factor $G(Y-Z)$.

c) If both the removed G-factor and Z link to Y, then bound $G(Z-Y) \leq c\ G(Z-X)$. Preserve $Q_1^c$ from $Q_1$, and do the dY integral first, since Y now has $\leq 2k-\ell(Y)-2$ factors. (In fact, the gain of $G(Z-X)$ is compensated by the loss of a factor in common with Y). In any event the X,Y integral is $o(\epsilon^{-\alpha})$.

We can thus assume that our attempt to preserve the above W didn't remove any G-factors from X or Y. This can only happen if there is another W linked to X or Y to the left. We use step 2 to bound the X,Y integral by $O(\epsilon^{-\alpha})$, and now show that our resultant removal of two G-factors involving W will yield a proof of (6.3).

If W is of type 1), 2) or 3b) this is obvious, since they require total pairing without any loss of G-factors. Thus, W is of type 3a, hence part of a pair $Q_2 = \{U,W\}$. If W is to the right of U, the analysis of step 2 gives the desired result. Even of W is to the left of U, W has at most $2k-\ell(W)-2$ G-factors so that the dW integral is

$$O\left[\epsilon^{2-[2k-\ell(W)-2](2-\beta)}\right] = o(\epsilon^{-\alpha/2})\ O\left[\epsilon^{\ell(W)(2-\beta)}\right]$$

The dU integral has $\ell(U) = \ell(w)$ isolated integrals, and $\leq 2k-2\ell(W) \leq 2k-2$ others - hence the total dU, dW integral is $o(\epsilon^{-\alpha})$. This completes the proof of Theorem 3.

## 7. Proof of Lemma 1

**Proof of lemma 1:** We have

$$G(x) = \int_0^\infty e^{-\lambda t} p_t(x)\,dt \le \int_0^\infty p_t(x) = G_0(x) \tag{7.1}$$

which gives half of (a). We note that

$$p_t(x) = \frac{1}{(2\pi)^2}\int e^{ip\cdot x} e^{-tp^\beta} d^2p, \quad t > o \tag{7.2}$$

is a positive, $C^\infty$ function of x, and

$$p_t(x) \le ct^{-2/\beta} \tag{7.3}$$

If $|x| \neq 0$, say $x_1 \neq 0$, then integrating by parts in (7.2) in the $dp_1$ direction gives (say $x_1 = \max(x_i)$)

$$p_t(x) = \frac{-1}{(2\pi)^2}\,\frac{i}{x_1}\int e^{ip\cdot x} t\beta p_1 p^{\beta-2} e^{-tp^\beta} d^2p \tag{7.4}$$

Substituting this into (7.1) we have

$$G(x) = \frac{-1}{(2\pi)^2}\,\frac{i}{x_1}\int_0^\infty e^{-\lambda t}dt\left[\int e^{ip\cdot x} t\beta p_1 p^{\beta-2} e^{-tp^\beta} d^2p)\right] \tag{7.5}$$

$$= \frac{c}{x_1}\int e^{ip\cdot x} p_1 p^{\beta-2} dp\left[\int_0^\infty e^{-\lambda t} t e^{-tp^\beta} dt\right]$$

$$= \frac{c}{x_1}\int e^{ip\cdot x}\,\frac{p_1 p^{\beta-2}}{(\lambda+p^\beta)^2} d^2p$$

where interchanging the order of integration is easily justified by Fubini's theorem since $\beta>1$.

We write (7.5) as

$$G(x) = \frac{c}{x_1}\int e^{ip\cdot x}\, r_{\beta-1,\beta+1}(p)\,dp \tag{7.6}$$

where the notation $r_{a,b}(p)$ will remind us that

$$r_{a,b}(p) \le \begin{cases} cp^a & , |p|\le 1 \\ c\,\dfrac{1}{p^b} & , |p|\ge 1 \end{cases}$$

We integrate by parts twice more to find

(7.7) $$G(x) = \frac{c}{x_1^3}\int e^{ip\cdot x}\, r_{\beta-3,\beta+3}(p)\,d^2p$$

which completes the proof of (a), since $r_{\beta-3,\beta+3}(p)$ is integrable.

Furthermore, by (7.7)

(7.8) $$\nabla G(x) = \frac{c}{x_1^3}\int e^{ip\cdot x}\,\vec{p}\; r_{\beta-3,\beta+3}(p)\,d^2p + \frac{c}{x_1^4}\int e^{ip\cdot x}\, r_{\beta-3,\beta+3}\,dp$$

$$= \frac{c}{x_1^3}\int e^{ip\cdot x}\, r_{\beta-2,\beta+2}(p)\,d^2p \quad +$$

and we can integrate by parts once more to find

(7.9) $$\nabla G(x) = \frac{c}{x_1^4}\int e^{ip\cdot x} r_{\beta-3,\beta+3}(p)\,d^2p.$$ Note: $2|x_1| \ge |x|$

This procedure can be iterated, and shows that

(7.10) $$|\nabla^\ell G(x)| \le \frac{c}{x^{\ell+3}}$$

This will provide a good bound for large x. For small x, we recall (3.2):

(7.11) $$G(x) = G_0(x) - H(x).$$

Of course, we have

(7.12) $$|\nabla^\ell G_0(x)| \le \frac{c}{x^{2-\beta+\ell}}$$

and we intend to show that

(7.13) $$|\Delta^\ell_{a_1,\dots,a_\ell} H(x)| \le |a_1||a_2|\cdots|a_\ell|\,\frac{c}{x^\ell}$$

$$\text{for } |a_i| \le \epsilon,\quad |X| \ge 4\ell\epsilon$$

Altogether, this will give, for $|X| \ge 4\ell\epsilon$

(7.14) $$|\Delta^\ell_{a_1,\dots,a_\ell} G(x)| \le |a_1|\cdots|a_\ell|\,\frac{c}{X^{2-\beta+\ell}}$$ holds for $|a_i| \le \frac{|x|}{4k} \le 1$

Combined with (7.10) we have

$$(7.15) \qquad |\Delta^{\ell}_{a_1,\ldots,a_\ell} G(x)| \le \frac{|a_1|\cdots|a_\ell|}{X^{\ell}}\, r_{\beta-2,3}(x)$$

which is (2.11).

We note that $r_{\beta-2,3}(x)$ is integrable.

From (7.15) we have, for $|x| \ge 4\ell\epsilon$

$$(7.16) \qquad \sup_{|a_i| \le \epsilon} \prod_{i=1}^{\ell} G(a_i)\,|\Delta^{\ell}_{a_1,\ldots,a_\ell} G(x)|$$

$$\le c \sup_{|a_i|\le\epsilon} \prod_{i=1}^{\ell} \left[\frac{1}{a_i^{(2-\beta)}}\, \frac{|a_i|}{|x|}\right] r_{\beta-2,3}(x)$$

$$\le c \sup_{|a_i|\le\epsilon} \prod_{i=1}^{\ell} \left[\frac{|a_i|}{|x|}\right]^{\beta-1} G_0^{\ell}(x)\, r_{\beta-2,3}(x)$$

$$\le c\, G_0^{\ell}(x) \left[\frac{\epsilon}{|x|}\right]^{(\beta-1)\ell} r_{\beta-2,3}(x)$$

which is (2.12).

We now prove (7.13), (but we first remark that if $\beta > 3/2$, then $H(x)$ is $C^1$ and the following analysis can be simplified considerably).

$$H(x) = \frac{1}{(2\pi)^2} \int e^{ip\cdot x} \frac{\lambda}{p^{\beta}(\lambda+p^{\beta})}\, d^2p.$$

so that

$$(7.17) \qquad \Delta_a H(x) = c\int e^{ip\cdot x} \frac{(e^{ip\cdot a}-1)}{p^{\beta}(\lambda+p^{\beta})}\, d^2p$$

We integrate by parts in the $dp_1$ direction to find

$$(7.18) \qquad \Delta_a H(x) = \frac{c}{x_1}\int e^{ip\cdot x} \frac{d}{dp_1}\left[\frac{e^{ip\cdot a}-1}{p^{\beta}(\lambda+p^{\beta})}\right] d^2p$$

$$= c\,\frac{a_1}{x_1}\, H(x+a)$$

$$+ \frac{c}{x_1} \int e^{ip\cdot x}(e^{ip\cdot a}-1)\, r_{-\beta-1,2\beta+1}(p)\, d^2p$$

Since $|e^{ip\cdot a}-1| \leq 2\,|p|\,|a|$ we obtain (7.13) for $\ell = 1$. Write $F(x;a)$ for the integral in (7.18) so that

$$\Delta_a H(x) = c\,\frac{a_1}{x_1}\,H(x+a) + \frac{c}{x_1}\,F(x;a) \tag{7.19}$$

Then,

$$\begin{aligned}\Delta_b\Delta_a H(x) &= c\,a_1\left[\Delta_b\left[\frac{1}{x_1}\right]\right]H(x+a)\\ &+ c\,\frac{a_1}{x_1+b}\,\Delta_b\,H(x+a)\\ &+ c\left[\Delta_b\left[\frac{1}{x_1}\right]\right]F(x;a)\\ &+ \frac{c}{x_1+b}\,\Delta_b\,F(x;a)\end{aligned} \tag{7.20}$$

We study the last term

$$\Delta_b F(x;a) = \int e^{ip\cdot x}(e^{ip\cdot b}-1)(e^{ip\cdot a}-1)r_{-\beta-1,2\beta+1}(p)d^2p \tag{7.21}$$

Integrating by parts gives us

$$\begin{aligned}\Delta_b F(x;a) &= c\,\frac{b_1}{x_1}\,F(x+b;a)\\ &+ c\,\frac{a_1}{x_1}\,F(x+a;b)\\ &+ \frac{c}{x_1}\int e^{ip\cdot x}(e^{ip\cdot a}-1)(e^{ip\cdot b}-1)r_{-\beta-2,2\beta+2}(p)d^2p\end{aligned} \tag{7.22}$$

and as before this establishes (7.13) for $\ell=2$. Iterating this procedure proves (7.13) for all $\ell$, completing the proof of lemma 2.

## REFERENCES

[1] Dynkin, E. [1988A] Self-Intersection Gauge for Random Walks and For Brownian Motion. Annals of Probab., Vol. 16, No. 1, 1988.

[2] Dynkin, E.B. [1988B] Regularized Self-intersection Local Times of the Planar Browninan Motion, Ann. Probab., Vol. 16, No. 1, 1988.

[3] LeGall, J.-F. [1988] Wiener Sausage and Self-Intersection Local Times. Preprint.

[4] Rosen, J. [1986] A Renormalized Local Time for the Multiple Intersections of Planar Brownian Motion. Seminaire de Probabilities XX, Spring Lecture Notes in Mathematics, 1204.

[5] Rosen [1988] - Limit Laws for the Intersection Local Time of Stable Processes in $\mathbb{R}^2$. Stochastics, Vol. 23, 219-240.

[6] Yor, M. [1985] Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement Brownien dans $\mathbb{R}^3$, Seminaire de Probabilites XIX, 1983/4, J. Azema, M. Yor, Eds., Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 350-356 (1985).

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