Stochastics, Vol. 23, pp. 219–240
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Printed in Great Britain

Limit Laws for the Intersection Local Time of Stable Processes in $\mathbb{R}^2$

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(*Received 21 August 1986; in final form 3 August 1987*)

We prove renormalization results for self-intersections of a stable process of index β in the plane. If $1<\beta\leqq 4/3$ we show that a suitably renormalized version of the intersection local time converges in law to a Brownian motion.

KEY WORDS: Stable processes, intersection local time, renormalization.

AMS 1980 subject classification: 60F05, 60J30, 60J55.

1. INTRODUCTION

X_t will denote a symmetric stable process of index $\beta>1$ in $\mathbb{R}^2$, with transition density function

$$f_\varepsilon(x)=\frac{1}{(2\pi)^2}\int e^{ipx-\varepsilon p^\beta}\,d^2p. \tag{1.1}$$

Here and throughout the paper, we use the abbreviation p^β for $|p|^\beta$. If

$$\alpha_\varepsilon(B)=\iint\limits_B f_\varepsilon(X_t-X_s)\,ds\,dt \tag{1.2}$$

*This work partially supported by NSF MCS-8302081.

is restricted to sets $B \subseteq \mathbb{R}_+^2$ away from the diagonal, then we know from Rosen [1985B], that as $\varepsilon \to 0$, $\alpha_\varepsilon(\cdot)$ converges weakly to a measure, the intersection local time, supported on $\{(s,t) | X_s = X_t\}$.

If we let

$$\alpha_\varepsilon(T) = \int_0^T \int_0^t f_\varepsilon(X_t - X_s)\, ds\, dt \tag{1.3}$$

then $\alpha_\varepsilon(T)$ will blow up as $\varepsilon \to 0$. In particular we note

$$\mathbb{E}(\alpha_\varepsilon(T)) = \frac{\Gamma(2/\beta)}{\beta} \frac{T}{2\pi} \cdot \frac{1}{\varepsilon^{2/\beta - 1}} + 0(1). \tag{1.4}$$

The following theorems describe the asymptotic behavior of $\alpha_\varepsilon(T)$ more precisely. B_t will denote a generic Brownian motion in $\mathbb{R}$, and convergence in law in the following theorems refers to convergence as processes in $C(R_+, R^2)$ equipped with the topology of compact convergence.

THEOREM 1 *If* $1 < \beta < 4/3$, *then*

$$\varepsilon^{2/\beta - 3/2}[\alpha_\varepsilon(T) - \mathbb{E}(\alpha_\varepsilon(T))] \tag{1.5}$$

converges in law to $c(\beta) B_T$ *where the constant* $0 < c(\beta) < \infty$ *is given by*

$$c^2(\beta) = \frac{1}{(2\pi)^4} \iint \left(\frac{1}{p^\beta} + \frac{1}{q^\beta}\right)^2 \left(\frac{1}{|p+q|^\beta} - \frac{1}{p^\beta + q^\beta}\right) e^{-(p^\beta + q^\beta)}\, dp\, dq. \tag{1.6}$$

THEOREM 2 *If* $\beta = 4/3$, *then*

$$\frac{1}{\sqrt{\lg(1/\varepsilon)}} [\alpha_\varepsilon(T) - \mathbb{E}(\alpha_\varepsilon(T))] \tag{1.7}$$

converges in law to $c(4/3)\, B_T$, where

$$c^2(4/3) = \frac{3}{16\pi} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3. \tag{1.8}$$

THEOREM 3 *If* $\beta > 4/3$, *then*

$$\alpha_\varepsilon(T) - \mathbb{E}(\alpha_\varepsilon(T))$$

converges pathwise to a finite random variable.

Theorem 3 comes from Rosen [1985B], where the limit, the renormalized intersection local time, is studied in detail.

The main purposes of this paper is to prove Theorems 1 and 2. These theorems were inspired by the following theorem of M. Yor.

THEOREM 4 (Yor, [1985]) *For Brownian motion in* $\mathbb{R}^3$,

$$\frac{1}{\sqrt{\lg(1/\varepsilon)}}\left[\alpha_\varepsilon(T) - \mathbb{E}(\alpha_\varepsilon(T))\right] \tag{1.10}$$

converges in law to $1/\sqrt{2}\pi\, B_T$. □

Yor's elegant proof uses stochastic integrals. In order to prove Theorems 1 and 2, we are forced to develop a completely different approach. We will see that our method also yields Theorem 4.

2. SKETCH OF THE PROOF

We will use the method of moments. With the notation

$$\{Y\} = Y - \mathbb{E}(Y) \tag{2.1}$$

and $X(s,t) = X_t - X_s$, we have

$$\mathbb{E}(\{\alpha_\varepsilon(T)\}^{2n}) = \frac{1}{(2\pi)^{4n}} \iint_{D_T^{2n}} e^{-\varepsilon \Sigma p_j^\beta}\, \mathbb{E}\left(\prod_{j=1}^{2} \{e^{ip_j X(s_j,t_j)}\}\right) ds_j\, dt_j\, d^2 p_j \tag{2.2}$$

where $D_T = \{(s,t) | 0 \leqq s \leqq t \leqq T\}$, and an analogous expression for (2.2) in the case of Brownian motion in $\mathbb{R}^3$.

The precise form of the expectation on the right hand side of (2.2) will depend on the relative positions of the s and t's.

In general, the set $\bigcup_j [s_j, t_j]$ will have several components, and the expectation in (2.2) will factor.

A component which contains m intervals of the form $[s_j, t_j]$ will be said to have order m. The main point of our proof is that the dominant contribution to (2.2) comes from regions with n components of order 2.

More precisely, for each ordering of the s_j's, if we hold fixed the initial points

$$u_1 < u_2 < \cdots < u_n$$

of the n components, each component will contribute a factor

$$\frac{h(\varepsilon)}{2} + o(h(\varepsilon)) \tag{2.3}$$

where

$$h(\varepsilon) = \begin{cases} c^2(\beta)\lg(1/\varepsilon) & \text{if } \beta = 4/3 \\ c^2(\beta)\dfrac{1}{\varepsilon^{4/\beta - 3}} & \text{if } \beta < 4/3 \\ \dfrac{1}{2\pi^2}\lg(1/\varepsilon) & \text{if } d = 3 \end{cases} \tag{2.4}$$

where $d=3$ refers to Brownian motion in $\mathbb{R}^3$. (The dependence on the initial points u_i is essentially contained in the $o(h(\varepsilon))$ term. For a precise statement, see the remark following Lemma 1.)

Thus for each ordering of the s_j's, the n component term in (2.2) will contribute

$$\left(\frac{h(\varepsilon)}{2} + o(h(\varepsilon))\right)^n \int_{0 \le u_1 < \cdots < u_n \le T} \cdots \int du_1 \ldots du_n = h^n(\varepsilon)\frac{T^n}{2^n n!} + o(h^n(\varepsilon)). \tag{2.5}$$

Since there are $(2n)!$ ways to order the s_j's, the total contribution to (2.2) from n component terms will be

$$\frac{(2n)!}{2^n n!}(h(\varepsilon)T)^n + o(h^n(\varepsilon)). \tag{2.6}$$

We will show that terms with less than n components are $0(h^n(\varepsilon))$, thus

$$\mathbb{E}(\{\alpha_\varepsilon(T)\}^{2n})=\frac{(2n)!}{2^n n!}(h(\varepsilon)T)^n+o(h^n(\varepsilon)), \tag{2.7}$$

so that

$$\mathbb{E}\left(\left\{\frac{\alpha_\varepsilon(T)}{\sqrt{h(\varepsilon)}}\right\}^{2n}\right)\to\frac{(2n)!}{2^n n!}T^n \qquad \text{as } \varepsilon\to 0 \tag{2.8}$$

which is precisely the $2n$th moment of a Brownian motion B_T.

Furthermore, it follows from Rosen [1985B] that if $b\leqq c$

$$\alpha_\varepsilon([a,b]\times[c,d])$$

converges as $\varepsilon\to 0$ to a random variable (having bounded moments), so that $\{\alpha_\varepsilon(T)\}/\sqrt{h(\varepsilon)}$ has asymptotically independent increments. (2.8) now shows that all its finite dimensional distributions converge to those of B_T, and our theorem will follow by standard methods. (See the remarks at the end of the proof.)

3. ASYMPTOTICS FOR COMPONENTS OF ORDER TWO

We have

$$\mathbb{E}(\{e^{ipX(s,t)}\}\{e^{iqX(s',t')}\})=\mathbb{E}(e^{ipX(s,t)+iqX(s',t')})-\mathbb{E}(e^{ipX(s,t)})\mathbb{E}(e^{iqX(s',t')}). \tag{3.1}$$

We assume $s<s'$ and distinguish two cases

I: $s<s'<t'<t$, so that (3.1) becomes

$$e^{-p^\beta(a+c)}(e^{-|p+q|^\beta b}-e^{-(p^\beta+q^\beta)b}) \tag{3.2}$$

with $a=s'-s$, $b=t'-s'$, $c=t-t'$.

II: $s<s'<t<t'$, and (3.1) becomes

$$e^{-p^\beta a}(e^{-|p+q|^\beta b}-e^{-(p^\beta+q^\beta)b})e^{-q^\beta c} \tag{3.3}$$

with $a=s'-s$, $b=t-s'$, $c=t'-t$.

The following three lemmas will yield the asymptotics for components of order two, (2.3), (2.4). Here $\bar{s}$ denotes the initial point of the next component or T if we are studying the last.

LEMMA 1

$$\frac{1}{(2\pi)^4}\iiint_{a+b+c\leqq \bar{s}-s} da\,db\,dc \iint e^{-p^\beta(a+\varepsilon)}e^{-|p+q|^\beta b}e^{-q^\beta(c+\varepsilon)}dp\,dq \tag{3.4}$$

$$= \begin{cases} \left[\dfrac{3}{32\pi^2}\left(\dfrac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3 + o(1)\right]\lg(1/\varepsilon), & \beta = 4/3 \\[2ex] \left[\dfrac{1}{(2\pi)^4}\displaystyle\iint \frac{1}{p^\beta}\frac{1}{|p+q|^\beta}\frac{1}{q^\beta}e^{-(p^\beta+q^\beta)}dp\,dq + o(1)\right]\dfrac{1}{\varepsilon^{4/\beta-3}}, & \beta < 4/3 \end{cases}$$

and the analogous expression for Brownian motion in $\mathbb{R}^3$ *gives*

$$\left[\frac{1}{(2\pi)^2} + o(1)\right]\lg(1/\varepsilon). \qquad \Box \tag{3.5}$$

Remark In this and the following lemmas convergence will be uniform for $\bar{s}-s\geqq\delta>0$, for each fixed $\delta>0$, while a bound of the order of the right hand side will be obtained independent of δ. This is sufficient for our purposes.

Proof We shall often make use of the simple bound

$$\int_0^T e^{-tv^\beta}dt \leqq \frac{c}{1+v^\beta}. \tag{3.6}$$

Step 1 We can always assume that $|p|\geqq 1$ in (3.4), for the contribution from $|p|\leqq 1$ is $O(1)$ as follows from (3.6) and the Cauchy–Schwartz inequality:

$$\int \frac{1}{1+|p+q|^\beta}\frac{1}{1+q^\beta}d^2q \leqq \int \frac{1}{(1+q^\beta)^2}d^2q < \infty.$$

Similarly, we can insert or remove at will a condition $|q|\geqq 1$.

Step 2 We can remove the condition $a+b+c \leqq \bar{s}-s \doteq \gamma$, since the contribution from $a+b+c \geqq \gamma$ is $0(1)$. For the latter condition implies that one of a, b, or c is $\geqq \gamma/3$, which allows us to bound one of the dp or dq integrals independently of ε, while the other integral can be controlled in Step 1.

As a result (3.4) is

$$\frac{1}{(2\pi)^4} \iint_{|p| \geqq 1} \frac{1}{p^\beta} \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} e^{-\varepsilon(p^\beta + q^\beta)} dp\, dq + 0(1). \tag{3.7}$$

Step 3 If $\beta < 4/3$, we scale (3.7) to obtain

$$\frac{1}{\varepsilon^{4/\beta - 3}} \frac{1}{(2\pi)^4} \iint_{|p| \geqq \varepsilon^{1/\beta}} \frac{1}{p^\beta} \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} e^{-(p^\beta + q^\beta)} dp\, dq. \tag{3.8}$$

We now show that the integrand in (3.8) is integrable—i.e. without the condition $|p| \geqq \varepsilon^{1/\beta}$, which establishes Lemma 1 for $\beta < 4/3$.

$$\int \frac{1}{(p+q)^\beta} \frac{1}{q^\alpha} dq = \pi^{d/2} \frac{\Gamma\left(\frac{\alpha+\beta-d}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(d - \frac{\alpha+\beta}{2}\right)\Gamma(\alpha/2)\Gamma(\beta/2)} \frac{1}{p^{\alpha+\beta-d}} \tag{3.9}$$

for $\alpha + \beta \geqq d$, Donoghue [1966, p. 158], so that

$$\begin{aligned} \iint \frac{1}{p^\beta} \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} e^{-(p^\beta + q^\beta)} dp\, dq &< \int \frac{e^{-p^\beta}}{p^\beta} \left(\int \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} dq \right) dp \\ &\leqq c \int \frac{e^{-p^\beta}}{p^{3\beta-2}} d^2 p < \infty \end{aligned}$$

since $3\beta - 2 < 3(4/3) - 2 = 2$.

If $\beta = 4/3$, we first show that we can eliminate the factor $e^{-\varepsilon q^\beta}$ in (3.7). We use

$$|1 - e^{-\varepsilon q^\beta}| \leqq \varepsilon^{\delta/\beta} q^\delta \tag{3.10}$$

and (3.9) to bound

$$\int_{|p|\geq 1} \frac{e^{-\varepsilon p^\beta}}{p^\beta} \int \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} |1-e^{-\varepsilon q}|^\beta \, dq\, dp$$

$$\leq \varepsilon^{\delta/\beta} \int_{|p|\geq 1} \frac{e^{-\varepsilon p^\beta}}{p^\beta} \int \frac{1}{|p+q|^\beta} \frac{1}{q^{\beta-\delta}} dq\, dp$$

$$\leq c\varepsilon^{\delta/\beta} \int_{|p|\geq 1} \frac{e^{-\varepsilon p^\beta}}{p^{2-\delta}} d^2 p$$

$$= c \int \frac{e^{-p^\beta}}{p^{2-\delta}} d^2 p < \infty.$$

A similar analysis applies to Brownian motion in $\mathbb{R}^3$.

Returning to $\beta = 4/3$, we use (3.9) to evaluate

$$\frac{1}{(2\pi)^4} \int_{|p|\geq 1} \frac{e^{-\varepsilon p^\beta}}{p^\beta} \left(\int \frac{1}{|p+q|^\beta} \frac{1}{q^\beta} dq \right) dp = \frac{\pi}{(2\pi)^4} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 \int_{|p|\geq 1} \frac{e^{-\varepsilon p^\beta}}{p^2} d^2 p$$

$$= \frac{\pi}{(2\pi)^4} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 2\pi \int_{\varepsilon^{1/\beta}}^{\infty} e^{-r^\beta} \frac{dr}{r}$$

$$= \frac{\pi}{(2\pi)^4} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 \frac{2\pi}{\beta} \lg\left(\frac{1}{\varepsilon}\right) + o(1) \tag{3.11}$$

which establishes Lemma 1 for $\beta = 4/3$.

Finally, we evaluate the analogous expression for Brownian motion in $\mathbb{R}^3$. We have a factor $1/(2\pi)^6$, since $d = 3$, and the convention of having $p^2/2$ in the exponents leads us to

$$\frac{8}{(2\pi)^6} \int_{|p|\geq 1} \frac{e^{\varepsilon - p^2/2}}{p^2} \left(\int \frac{1}{|p+q|^2} \frac{1}{q^2} dq \right) dp$$

$$= \frac{8\pi^3}{(2\pi)^6} \int_{|p|\geq 1} \frac{e^{-\varepsilon p^2/2}}{p^3} d^3 p, \qquad \text{(by (3.9))},$$

$$= \frac{8\pi^3 \cdot 4\pi}{(2\pi)^6} \int_{\sqrt{\varepsilon}}^{\infty} e^{-r^2/2} \frac{dr}{r} = \frac{1}{(2\pi)^2} \lg\left(\frac{1}{\varepsilon}\right) + o(1) \tag{3.12}$$

completing the proof of Lemma 1.

LEMMA 2

$$\frac{1}{(2\pi)^4}\iiint_{a+b+c\leqq \bar{s}-s} da\,db\,dc \iint -e^{p^\beta(a+\varepsilon)}e^{-(p^\beta+q^\beta)b}e^{-q^\beta(c+\varepsilon)}dp\,dq$$

$$=\begin{cases}\left[\dfrac{9}{64\pi}+o(1)\right]\lg(1/\varepsilon), & \beta=4/3\\[2ex] \left[\dfrac{1}{(2\pi)^4}\displaystyle\iint\frac{1}{p^\beta}\frac{1}{p^\beta+q^\beta}\frac{1}{q^\beta}e^{-(p^\beta+q^\beta)}dp\,dq+o(1)\right]\dfrac{1}{\varepsilon^{4/\beta-3}}, & \beta<4/3\end{cases}$$

and the analogous expression for Brownian motion in $\mathbb{R}^3$ *gives*

$$\left[\frac{4}{(2\pi)^3}+o(1)\right]\lg(1/\varepsilon). \tag{3.14}$$

Proof Arguing as in Lemma 1 we are reduced to evaluating

$$\frac{1}{(2\pi)^4}\iint_{p^\beta+q^\beta\geqq 1}\frac{1}{p^\beta}\frac{1}{p^\beta+q^\beta}\frac{1}{q^\beta}e^{-\varepsilon(p^\beta+q^\beta)}d^2p\,d^2q. \tag{3.15}$$

$\beta<4/3$ is handled precisely as before.

If $\beta=4/3$, change variables

$$\begin{cases}x=|p|^{2/3}\\ y=|q|^{2/3}\end{cases}$$

so that (3.15) becomes

$$\frac{1}{(2\pi)^4}\left(\frac{3}{2}\right)^2(2\pi)^2\iint_{\substack{x,y\geqq 0\\ x^2+y^2\geqq 1}}\frac{e^{-\varepsilon(x^2+y^2)}}{x^2+y^2}dx\,dy$$

$$=\frac{1}{(2\pi)^2}\left(\frac{3}{2}\right)^2\frac{\pi}{2}\int_1^\infty e^{-\varepsilon r^2}\frac{dr}{r}=\frac{9}{32\pi}\int_{\sqrt{\varepsilon}}^\infty e^{-r^2}\frac{dr}{r}$$

$$=\frac{9}{64\pi}\lg(1/\varepsilon)+o(1).$$

For Brownian motion in $\mathbb{R}^3$ we need to evaluate

$$\frac{8}{(2\pi)^6}\iint_{p^2+q^2\geq 1}\frac{1}{p^2}\frac{1}{p^2+q^2}\frac{1}{q^2}e^{-\varepsilon(p^2+q^2)/2}d^3p\,d^3q$$

$$=\frac{8}{(2\pi)^6}(4\pi)^2\iint_{\substack{x,y\geq 0\\ x^2+y^2\geq 1}}\frac{e^{-\varepsilon(x^2+y^2)/2}}{x^2+y^2}dx\,dy$$

$$=\frac{8}{(2\pi)^6}(4\pi)^2\frac{\pi}{4}\lg(1/\varepsilon)+0(1)$$

$$=\frac{4}{(2\pi)^3}\lg(1/\varepsilon)+0(1)$$

completing the proof of Lemma 2. $\square$

LEMMA 3:

$$\frac{1}{(2\pi)^4}\iiint_{a+b+c\leq\bar{s}-s}da\,db\,dc\iint(e^{-p^\beta(a+c+\varepsilon)}(e^{-|p+q|^\beta b}-e^{-(p^\beta+q^\beta)b})\,e^{-\varepsilon q^\beta}dq\,dq$$

$$=\begin{cases}\left[\dfrac{9}{64\pi}+o(1)\right]\lg(1/\varepsilon). & \beta=4/3\\[2ex] \left[\dfrac{1}{(2\pi)^4}\displaystyle\iint\frac{1}{p^{2\beta}}\left(\frac{1}{|p+q|^\beta}-\frac{1}{p^\beta+q^\beta}\right)e^{-(p^\beta+q^\beta)}dp\,dq+o(1)\right] & \\ \times\dfrac{1}{\varepsilon^{4/\beta-3}}, & \beta<4/3\end{cases}\tag{3.16}$$

and the analogous expression for Brownian motion in $\mathbb{R}^3$ gives

$$\left[\frac{4}{(2\pi)^3}+o(1)\right]\lg(1/\varepsilon),\tag{3.17}$$

Proof We first observe that for $x, y > 0$, arguing separately for $x \geqq y$ and $y \geqq x$, we have

$$\left|\int e^{-xt} - e^{-yt}\,dt\right| \leqq \int_0^\infty \left|e^{-xt} - e^{-yt}\right| dt \leqq \left|\frac{1}{x} - \frac{1}{y}\right| \tag{3.18}$$

while

$$\left|\frac{1}{|p+q|^\beta} - \frac{1}{p^\beta + q^\beta}\right| \leqq \frac{c}{q^\beta}\frac{|p|}{|q|}, \qquad \text{if } |p| \leqq |q|/4. \tag{3.19}$$

We first consider processes in $\mathbb{R}^2$. Brownian motion in $\mathbb{R}^3$ will be handled later by different means.

Step 1 We can assume $|p| \geqq 1$, for the contribution from $|p| \leqq 1$ is $0(1)$. To see this we may certainly take $|q| \geqq 4$. Then using (3.6), (3.18), (3.19) we bound our contribution by

$$\int_{|p| \leqq 1} \frac{|p|}{(1+p^\beta)^2}\left(\int_{|q| \geqq 4} \frac{c}{q^{\beta+1}}\,dq\right) dp < \infty.$$

Furthermore, since

$$\int_{|p| \geqq 1} \frac{1}{(1+p^\beta)^2} \int_{|q| \leqq M} \frac{1}{(p+q)^\beta}\,dq\,dp < 1$$

we can, if we wish, also assume $|q| \geqq M$.

Step 2 We can remove the condition $a+b+c \leqq \bar{s} - s \doteq \gamma$, since the complement contributes $0(1)$. To see this, note that it suffices to check $|p| < |q|/4$, since if $|q| < 4|p|$, we use

$$\frac{1}{p^{2\beta}} \leqq \frac{4}{p^\beta q^\beta} \tag{3.20}$$

and can then use the proof of Lemma 1. If a or $c \geqq \gamma/3$ we can use (3.18), (3.19) to obtain the bound

$$\int_{|p| \geqq 1} \frac{e^{-\gamma/3p^\beta}}{p^{2\beta}} |p| \int_{|q| \geqq M} \frac{1}{q^{\beta+1}}\,dq < \infty.$$

If $b \geqq \gamma/3$, we first integrate out dq:

$$\int e^{-b|p+q|^\beta} dq = \frac{c}{b^{2/\beta}}.$$

Step 3 If $\beta < 4/3$, we scale to obtain

$$\frac{1}{\varepsilon^{4/\beta-3}} \frac{1}{(2\pi)^4} \iint_{|p| \geqq \varepsilon^{1/\beta}} \frac{1}{p^{2\beta}} \left(\frac{1}{|p+q|^\beta} - \frac{1}{p^\beta + q^\beta} \right) e^{-(p^\beta+q^\beta)} dp\, dq. \quad (3.21)$$

We show that the integrand in (3.20) is integrable. If $|q| < 4|p|$ we can use (3.20) and the proof of Lemma 1, while if $|p| \leqq |q|/4$ we use (3.19) to bound by

$$\iint \frac{1}{p^{2\beta}} \frac{1}{q^\beta} \left(\frac{|p|}{|q|} \right)^{2/3} e^{-(p^\beta+q^\beta)} dp\, dq = \iint \frac{1}{p^{2\beta-2/3}} \frac{1}{q^{\beta+2/3}} e^{-(p^\beta+q^\beta)} dp\, dq < \infty$$

since

$$2\beta - 2/3 < 2(4/3) - 2/3 = 2$$

and

$$\beta + 2/3 < 4/3 + 2/3 = 2.$$

This completes the proof of Lemma 3 for $\beta < 4/3$.

If $\beta = 4/3$, we first show that the factor $e^{-\varepsilon q^\beta}$ can be dropped. As before, via Lemma 1, we can assume $|p| \leqq |q|/4$, so that using (3.19), (3.10)

$$\int_{|p| \geqq 1} \frac{e^{-\varepsilon p^\beta}}{p^{2\beta}} \left| \frac{1}{|p+q|^\beta} - \frac{1}{p^\beta + q^\beta} \right| |1 - e^{-\varepsilon q^\beta}| dp\, dq$$

$$\leqq c\varepsilon^{\delta/\beta} \int_{|p| \geqq 1} \frac{e^{-\varepsilon p^\beta}}{p^{2\beta}} |p| \left(\int_{|q| \geqq 4|p|} \frac{1}{q^{\beta+1-\delta}} dq \right) dp$$

$$\leqq c\varepsilon^{\delta/\beta} \int_{|p| \geqq 1} \frac{e^{-\varepsilon p^\beta}}{p^{2-\delta}} dp = 0(1).$$

We now calculate for $\beta=4/3$

$$\frac{1}{(2\pi)^4}\iint_{|p|\geqq 1}\frac{e^{-\varepsilon p^\beta}}{p^{2\beta}}\left(\int_0^\infty e^{-|p+q|^\beta b}-e^{-(p^\beta+q^\beta)b}\,db\right)dq\,dp. \tag{3.22}$$

We interchange the $dbdq$ integration which is justified by Fubini, using (3.18), (3.19) as above—as long as $\varepsilon>0$.

With

$$f\doteq\int e^{-q^\beta}d^2q=2\pi\int_0^\infty e^{-r^{4/3}}r\,dr=\frac{2\pi}{\beta}\int_0^\infty e^{-s}s^{2/\beta-1}\,ds=\frac{2\pi}{\beta}\Gamma(3/2)$$

$$=\frac{\pi\sqrt{\pi}}{\beta},\qquad (2/\beta=3/2),$$

we see that (3.22) equals

$$f\cdot\frac{1}{(2\pi)^4}\int_{|p|\geqq 1}\frac{e^{-\varepsilon p^\beta}}{p^{2\beta}}\left(\int_0^\infty\frac{1-e^{-p^\beta b}}{b^{2/\beta}}\,db\right)d^2p$$

$$=f\frac{1}{(2\pi)^4}\left(\int_0^\infty\frac{1-e^{-b}}{b^{2/\beta}}\,db\right)\int_{|p|\geqq 1}\frac{e^{-\varepsilon p^\beta}}{p^2}\,dp$$

$$=f\frac{1}{(2\pi)^4}\left(\int_0^\infty\frac{1-e^{-b}}{b^{2/\beta}}\,db\right)\frac{2\pi}{\beta}\lg(1/\varepsilon)+0(1).$$

We next recall the standard calculation

$$\int_0^\infty\frac{1-e^{-b}}{b^{3/2}}\,db=\int_0^1\int_0^\infty\frac{e^{-sb}}{\sqrt{b}}\,db\,ds=\int_0^1\frac{ds}{\sqrt{s}}\Gamma(1/2)=\int_0^1\frac{ds}{\sqrt{s}}\Gamma(1/2)=2\sqrt{\pi}.$$

Putting all this together, (3.22) gives

$$\frac{\pi\sqrt{\pi}}{\beta}\,\frac{1}{(2\pi)^4}\,\frac{2\pi}{\beta}2\sqrt{\pi}\lg(1/\varepsilon)+0(1)=\frac{1}{4\pi\beta^2}\lg(1/\varepsilon)+0(1)$$

$$=\frac{9}{64\pi}\lg(1/\varepsilon)+0(1)$$

which completes the proof of Lemma 3 for $\beta=4/3$.

For Brownian motion in $\mathbb{R}^3$, we proceed differently. We first integrate $dq\,dp$ to obtain

$$\frac{1}{(2\pi)^6}\iint e^{-p^2(a+c+\varepsilon)/2}(e^{-|p+q|^2b/2}-e^{-(p^2+q^2)b/2})e^{-\varepsilon q^2/2}\,ds\,dp$$

$$=\frac{1}{(2\pi)^6}\int e^{-p^2(a+b+c+\varepsilon)/2}\left(\int(e^{pqb}-1)e^{-q^2(b+\varepsilon)/2}\,dq\right)dp$$

$$=\frac{1}{(2\pi)^3}\frac{1}{(b+\varepsilon)^{3/2}}\left[\frac{1}{\left(a+b+c+\varepsilon-\dfrac{b^2}{b+\varepsilon}\right)^{3/2}}-\frac{1}{(a+b+c+\varepsilon)^{3/2}}\right]. \tag{3.24}$$

We next integrate out dc, then da to obtain first

$$\frac{2}{(2\pi)^3}\frac{1}{(b+\varepsilon)^{3/2}}\quad\frac{1}{\left(a+b+\varepsilon-\dfrac{b^2}{b+\varepsilon}\right)^{1/2}}-\frac{1}{(a+b+\varepsilon)^{1/2}}\quad+0(1)$$

and then

$$\frac{4}{(2\pi)^3}\frac{1}{(b+\varepsilon)^{3/2}}\left[\sqrt{b+\varepsilon}-\sqrt{b+\varepsilon-\frac{b^2}{b+\varepsilon}}\right]+0(1)$$

$$=\frac{4}{(2\pi)^3}\left[\frac{1}{b+\varepsilon}-\frac{\sqrt{2\varepsilon b+\varepsilon^2}}{(b+\varepsilon)^2}\right]+0(1).$$

Integrating the first term gives $(4/(2\pi)^3)\lg(1/\varepsilon)+0(1)$ while the second term is bounded by

$$\int_0^T\frac{\sqrt{\varepsilon}}{(b+\varepsilon)^{3/2}}+\frac{\varepsilon}{(b+\varepsilon)^2}\,db=0(1).$$

This completes the proof of Lemma 3.

Putting Lemmas 1, 2, 3 together we get (2.4) with

$$\frac{c^2(\beta)}{2} \doteq \frac{1}{(2\pi)^4} \iint \frac{1}{p^\beta}\left(\frac{1}{p^\beta}+\frac{1}{q^\beta}\right)\left(\frac{1}{(p+q)^\beta}-\frac{1}{p^\beta+q^\beta}\right) e^{-(p^\beta+q^\beta)}\,dp\,dq$$

$$=\frac{1}{2}\frac{1}{(2\pi)^4} \iint \left(\frac{1}{p^\beta}+\frac{1}{q^\beta}\right)^2\left(\frac{1}{(p+q)^\beta}-\frac{1}{p^\beta+q^\beta}\right) e^{-(p^\beta+q^\beta)}\,dp\,dq.$$

In the proof of our lemmas we have verified that (3.25) is integrable. We now show that $c^2(\beta)>0$. This follows from the following bound, where $r=(|p| \wedge |q|)/(|p| \vee |q|)<1$:

$$\frac{1}{1+r^\beta}<1<\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{1-r^2\sin^2\phi}}\right)^\beta$$

$$=\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{1-2r\cos\phi+r^2}}\right)^\beta$$

$$<\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{(\sqrt{1-2r\cos\phi+r^2})^\beta}$$

where the equality comes from Gradshetyn and Ryzhik [1980], p. 387.

4. BOUNDS ON COMPONENTS OF ORDER $\geqq 3$

In this section we will show that a component of order m contributes

$$o(h^{m/2}(\varepsilon)). \tag{4.1}$$

We can assume that our component contains the intervals $[s_j, t_j]$, $1 \leqq j \leqq m$ and we relabel these $2m$ points by

$$r_1 \leqq r_2 \leqq \cdots \leqq r_{2m}.$$

We write

$$\Sigma p_j X(s_j, t_j) = \Sigma u_i X(r_i, r_{i+1}) \tag{4.2}$$

so that each u_i is a linear combination of the p_j's. Using independence, we have

$$\mathbb{E}(e^{i\Sigma p_j X(s_j,t_j)}) = e^{-\Sigma u_i^\beta (r_{i+1}-r_i)}. \tag{4.3}$$

Using (3.6), it would suffice to bound

$$\int E(\varepsilon, p) \Pi \frac{1}{1+u_i^\beta} dp \tag{4.4}$$

by (4.1), where

$$E(\varepsilon, p) = e^{-\varepsilon \Sigma p_j^\beta}. \tag{4.5}$$

We begin by deriving such bounds whenever possible, but we will see that for certain orderings of the points s_i, t_j we will need a more delicate analysis (see (4.18)).

Let us define

$F = \{i \mid r_i = s_j \text{ for some } j\}$

$D = F^c = \{i \mid r_i = t_l \text{ for some } l\}$

$R = \{i \mid r_i = s_j, \text{ and } [s_j, t_j] \text{ contains only points of the form } r_l, l \in F\}$.

In the latter case, we say $j \in \hat{R}$, and shall refer to $[s_j, t_j]$ as an interval in $\hat{R}$, whose height is the number of points in $[s_j, t_j)$, so that e.g. if $r_1 = s_1$, $r_2 = s_2$, $r_3 = t_1$, then $[s_1, t_1]$ is an interval in $\hat{R}$ of height 2, with $1 \varepsilon \hat{R}$.

We will use notation such as u_F to denote $\{u_i\}_{i \in F}$. It is easy to see that u_F is a nonsingular linear transformation of the p_j's. It is shown in Rosen [1983] that

$$\operatorname{span} u_D = \operatorname{span} \{p_j\}_{j \notin \hat{R}}. \tag{4.6}$$

Consequently, if $R = \phi$, u_D is also a nonsingular linear transformation of the p_j's, and (4.4) is actually $0(1)$:

$$\int \prod \frac{1}{1+u_i^\beta} dp = \int \prod_{i \in F} \frac{1}{1+u_i^\beta} \prod_{j \in D} \frac{1}{1+u_j^\beta} dp$$

$$\leqq \left\| \prod_F \frac{1}{1+u_i^\beta} \right\|_2 \left\| \prod_D \frac{1}{1+u_j^\beta} \right\|_2 < \infty$$

since

$$\int \frac{1}{(1+u^{\beta})^2}\,du < \infty.$$

As we shall see when $R \neq \phi$, we shall have to shift some of the factors between the two products in (4.7) (we refer to this as "borrowing"), before performing the Cauchy–Schwartz bound of (4.7), which we will refer to as the initial Cauchy–Schwartz inequality (since in general it will be followed by other bounds). The details will become clearer as we proceed.

We will deal first with $\beta = 4/3$, in some sense the hardest case, and later explain what to do in the other cases.

Assume that $R = \{i\}$ and that the single interval $[s_j, t_j]$ in $\hat{R}$ is of height $k \geqq 4$. Let $L = \{i, i+1, \dots, i+k-1\}$ and write $(L^c = F - L)$,

$$\prod_F \frac{1}{1+u_i^{\beta}} \prod_D \frac{1}{1+u_j^{\beta}} \leq c\left(\prod_{L^c} \frac{1}{1+u_i^{\beta}} \prod_L \frac{1}{1+u_l^{1.03}}\right)\left(\prod_L \frac{1}{1+u_l^{0.3}} \prod_D \frac{1}{1+u_j^{\beta}}\right). \tag{4.8}$$

We apply the initial Cauchy–Schwartz inequality, and the first factor is bounded as before. To bound the second factor we use Hölder's inequality

$$\left(\int \prod_L \frac{1}{(1+u_l^{0.3})^2} \prod_D \frac{1}{(1+u_j^{\beta})^2}\,dp\right)^k \leq \prod_{l\in L} \int \frac{1}{(1+u_l^{0.3})^{2k}} \prod_D \frac{1}{(1+u_j^{\beta})^2}\,dp < \infty. \tag{4.9}$$

since, if $k \geqq 4$, $(0.3)2k > 2$, and each u_l, $l \in L$ contains p_j as a summand, so that $u_{D\cup l}$ is a non-singular linear transformation by (4.6).
If $|R| > 1$, with all intervals in $\hat{R}$ of height $\geqq 4$, we can handle each interval in $\hat{R}$ successively.

Assume now that $R = \{i\}$, and the single interval $[s_j, t_j]$ in $\hat{R}$ is of height 3. We first do the dp_j integral: for definiteness assume $u_{i+1} - u_i = p_k$, $u_{i+2} - u_{i+1} = p_l$

$$\int \frac{1}{1+u_i^{\beta}} \frac{1}{1+u_{i+1}^{\beta}} \frac{1}{1+u_{i+2}^{\beta}}\,dp_j$$

$$=\int \frac{1}{1+p_j^\beta} \frac{1}{1+(p_j+p_k)^\beta} \frac{1}{1+(p_j+p_k+p_l)^\beta} dp_j$$

$$\leqq c\int\left(\frac{1}{1+p_j^{\beta/2}} \frac{1}{1+(p_j+p_k)^{\beta/2}}\right)\left(\frac{1}{1+p_j^{\beta/2}} \frac{1}{1+(p_j+p_k+p_l)^{\beta/2}}\right)$$

$$\times\left(\frac{1}{1+(p_j+p_k)^{\beta/2}} \frac{1}{1+(p_j+p_k+p_l)^{\beta/2}}\right) dp_j$$

$$\leqq c\|\cdot\|_3\|\cdot\|_3\|\cdot\|_3$$

$$\leqq c\frac{1}{1+p_k^{2/3-\delta}} \frac{1}{1+p_l^{2/3-\delta}} \frac{1}{1+(p_k+p_l)^{2/3-\delta}} \doteq H(p_k, p_l) \tag{4.10}$$

since $3\beta/2=2$, and e.g.

$$\int \frac{1}{1+p_j^2} \frac{1}{1+(p_j+p_k)^2} dp_j \leqq \frac{1}{1+p_k^{2-\delta}}. \tag{4.11}$$

If we were now to apply the initial Cauchy–Schwartz inequality, we would be (barely) divergent—so we must borrow from u_{i+3} which contains both p_k, p_l as summands. Write

$$\left(\prod_{s\in L^c} \frac{1}{1+u_s^\beta} \cdot H(p_k, p_l) \cdot \frac{1}{1+u_{i+3}^{0.3}}\right)\left(\frac{1}{1+u_{i+3}^{1.03}} \prod_{\substack{t\in D\\ t\neq i+3}} \frac{1}{i+u_t^\beta}\right)$$

and now apply Cauchy–Schwartz to easily bound the resulting integral. (Note: we have previously integrated out dp_j).

As before $|R|>1$, with all intervals in $\hat{R}$ of height $\geqq 3$ presents no new problems.

Consider now $R=\{i\}$, with $[s_j, t_j]$, the single interval in $\hat{R}$, having height 2. We shall find that our integral (4.4) now diverges—but only as $0(\lg(1/\varepsilon))$.

Let $u_{i+1}-u_i=p_k$. We first integrate out dp_j for the bound

$$\int \frac{1}{1+u_i^\beta} \frac{1}{1+u_{i+1}^\beta} dp_j = \int \frac{1}{1+p_j^{4/3}} \frac{1}{1+(p_j+p_k)^{4/3}} dp_j \leqq c\frac{1}{1+p_k^{2/3}}. \tag{4.12}$$

We now borrow from u_{i+2}, which contains p_k as a summand:

$$\left(\prod_{L^c} \frac{1}{1+u_s^\beta} \frac{1}{1+p_k^{2/3}} \frac{1}{1+u_{i+2}^{1/3}}\right)\left(\prod_{\substack{D\\ t\neq i+2}} \frac{1}{1+u_t^\beta} \frac{1}{1+|u_{i+2}|}\right). \quad (4.13)$$

Before applying the Cauchy–Schwartz inequality we make a general comment: If $\{v_i\}$ is a non-singular linear transformation of the $\{p_j\}$, we have

$$\Sigma v_i^\beta \leqq c \Sigma p_j^\beta \quad (4.14)$$

since both sides are homogeneous of order β, continuous, and non-zero on the unit sphere. Hence we may always bound

$$F(\varepsilon, p) \leqq F(c\varepsilon, v). \quad (4.15)$$

With this tool, we apply the initial Cauchy–Schwartz inequality, then apply Hölder's inequality to the first factor to separate out p_k and u_{i+2} and finally use

$$\int \frac{e^{-\varepsilon p^\beta}}{1+p^2}\, dp = 0(\lg(1/\varepsilon)). \quad (4.16)$$

All this shows (4.4) is $0(\lg(1/\varepsilon))$.

As before we see that if $|R|>1$, if all intervals in $\hat{R}$ are of height $\geqq 2$, and if there are k intervals of height 2, (4.4) is bounded by

$$0(\lg^k(1/\varepsilon)). \quad (4.17)$$

This clearly satisfies (4.1)—unless m is even and $k=m/2$.

Notice, however, that for each interval $[s_j, t_j]$ of height 2, if $s_j < s_i < t_j$, then t_i cannot be involved in any interval of type 1 or 2, so that except when t_i is the last point in our component, we obtain extra convergence producing factors.

Finally, we turn to consider intervals in $\hat{R}$ of height 1 (the isolated intervals of Rosen [1985a]). It is here that the reduction to (4.4) is insufficient. We must recall that our original integral, (2.2), involves the subtractions $\{\ \}$. We first integrate out dp_j, if $[s_j, t_j]$ is an interval of $\hat{R}$ of height 1. After applying (3.18) we bound

$$\int \left| \frac{1}{|p+q|^{\beta}} - \frac{1}{p^{\beta}+q^{\beta}} \right| e^{-\varepsilon p^{\beta}} dp = 0(|q|^{2-\beta}) \tag{4.18}$$

as follows:

If $|p| \leqq 4|q|$ we use the bound

$$\int_{|p| \leqq 4|q|} \frac{1}{|p+q|^{\beta}} dp \leqq \int_{|p| \leqq 5|q|} \frac{d^2 p}{|p|^{\beta}} = 0(|q|^{2-\beta}). \tag{4.19}$$

While if $|q|/|p| < 1/4$ we use (3.19), (with p, q reversed!) to bound (4.14) by

$$c|q| \int_{|p| \geqq 4|q|} \frac{1}{p^{\beta+1}} d^2 p = 0(|q|^{2-\beta}). \tag{4.20}$$

Thus if $r_i = s_j$, the dp_j integral is $0(u_{i-1}^{2-\beta})$, but $u_{i+1} = u_{i-1}$, and these contribute a factor of $1/(1+u_{i-1}^{\beta})^2$—altogether $1/1+u_{i-1}^{3\beta-2}$. Since $3\beta-2>\beta$ (as long as $\beta>1$), the isolated interval actually increases convergence in comparison to a component from which it is excised. This shows that all our previous bounds hold (in fact are improved!), in case removal of all isolated intervals brings us to the cases considered above.

All that remains is to consider the case where removal of an isolated interval creates a new isolated interval (nested intervals), e.g. $s_1 < s_2 < s_3 < t_3 < t_2 < t_1 < \ldots$. If we have such a "tower" of nested intervals, we know from Lemma 3 (or (4.14)) that integrating the inner two will give a factor.

$$0(\lg(1/\varepsilon)),$$

while outer intervals will be convergent. It is clear that in a component of length $\geqq 3$ not all intervals can be in the form of doubly nested intervals—and this completes the proof of (4.1) in case $\beta = 4/3$, and therefore of Theorem 2.

When $\beta < 4/3$, the basic approach is similar, but now intervals $[s_j, t_j]$ in $\hat{R}$ of height $\geqq 3$ may be divergent. To control them write, instead of (4.8)

$$\left(\prod_{L^c}\frac{1}{1+u_i^\beta}\prod_L\frac{1}{1+|u_l|}\right)\left(\prod_L\frac{1}{1+u_l^{\beta-1}}\prod_D\frac{1}{1+u_i^\beta}\right).$$

Applying the initial Cauchy–Schwartz inequality, the first factor gives rise to an innocuous power of $\lg(1/\varepsilon)$, while the second factor as in (4.9) contributes

$$\left(\int\frac{e^{-\varepsilon u^\beta}}{(1+u^{\beta-1})^{2l}}du\right)^{l/2l}. \tag{4.21}$$

If $2l(\beta-1)\geqq 2$, (4.17) is at most $0\sqrt{\lg(1/\varepsilon)}$, while if $2l(\beta-1)<2$, we bound by

$$\left(\int\frac{e^{-\varepsilon u^\beta}}{u^{2l(\beta-1)}}du\right)^{1/2}=c\frac{1}{\varepsilon^{1/\beta-l(1-1/\beta)}}.$$

To show this is

$$o(h^{l/2}(\varepsilon))=o\left(\frac{1}{\varepsilon^{(2/\beta-3/2)l}}\right)$$

we need only show

$$(2/\beta-3/2)l>1/\beta-l(1-1/\beta)$$

i.e. $(l-1)/\beta>l/2$, i.e. $\beta<2(l-1)/l$ which is true if $l\geqq 3$. These ideas suffice to prove Theorem 1.

The case of Brownian motion in $\mathbb{R}^3$ can be treated almost identically with $\beta=4/3$—except that the analogue of (4.18)—

$$\int\left|\int(e^{-b(p+q)^2/2}-e^{-b(p^2+q^2)/2})e^{-p^2/2}d^3p\right|db=0(|q|) \tag{4.22}$$

must be proved differently.

The inner integral can be done explicitly:

$$e^{-bq^2/2}\int(e^{-bp\cdot q}-1)e^{-(b+\varepsilon)p^2/2}d^3p$$

$$=e^{-bq^2/2}(e^{q^2b^2/2(b+\varepsilon)}-1)/(b+\varepsilon)^{3/2}.$$

This is positive, and only increases as ε decreases to 0, hence the inner integral in (4.22) is bounded in absolute value by

$$\frac{1-e^{-bq^2/2}}{b^{3/2}},$$

so that the integral in (4.18) is bounded by

$$\int_0^\infty \frac{1-e^{-bq^2/2}}{b^{3/2}}\,db = |q| \int_0^\infty \frac{1-e^{-b/2}}{b^{3/2}}\,db$$

proving (4.18), since $\int_0^\infty ((1-e^{-b/2})/b^{3/2})\,db$ is finite, (in fact $=\sqrt{2\pi}$ as in the computation following (3.23)). This completes the proof of Theorem 4.

Remark By following the above ideas, and keeping track of S, T it is easy to see that for some $\gamma > 0$, and all n,

$$\mathbb{E}\left(\frac{\alpha_\varepsilon(T)-\alpha_\varepsilon(S)}{\sqrt{h(\varepsilon)}}\right)^{2n} \leqq C(T-S)^{\gamma n}.$$

This establishes tightness, Billingsley [1968, Theorem 12.3], and completes the proof of our theorems.

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