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# The Intersection Local Time of Fractional Brownian Motion in the Plane

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We show how to renormalize the intersection local time of fractional Brownian motion of index $\beta$ in the plane, when $\frac{1}{2}<\beta<\frac{3}{4}$. When $\beta=\frac{1}{2}$, i.e., planar Brownian, such a renormalization is due to Varadhan. © 1987 Academic Press, Inc.

## I. INTRODUCTION

Fractional Brownian motion of index $\beta$, $0<\beta<1$, is the real valued Gaussian process $F_t$ indexed by $t \geqslant 0$, with mean zero and

$$\mathbb{E}(F_s F_t) = \tfrac{1}{2}(s^{2\beta} + t^{2\beta} - |s-t|^{2\beta}), \tag{1.1}$$

so that

$$\mathbb{E}(F_t - F_s)^2 = |s-t|^{2\beta}. \tag{1.2}$$

Fractional Brownian motion of index $\beta$ in the plane is simply

$$X_t = (F_t^{(1)}, F_t^{(2)}),$$

where $F_t^{(1)}$, $F_t^{(2)}$ are independent copies of the above real valued process. Note that $\beta=\frac{1}{2}$ corresponds to planar Brownian motion. Let

$$p(t,x) = \frac{e^{-|x|^2/2t}}{2\pi t} \tag{1.3}$$

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and set

$$\alpha(\varepsilon, B) = \int_B \int p(\varepsilon, X(s, t))\, ds\, dt, \tag{1.4}$$

where $X(s, t) = X_t - X_s$. Since $p(\varepsilon, x) \to \delta(x)$ as $\varepsilon \to 0$, formally, as $\varepsilon \to 0$, $\alpha(\varepsilon, B)$ should converge to

$$\int_B \int \delta(X_t - X_s)\, ds\, dt,$$

a measure of the amount of "time" the process spends intersecting itself.

In fact, if $\alpha(\varepsilon, \cdot)$ is restricted to Borel sets $B$ in $\{(s, t) \mid s < t\}$, then $\alpha(\varepsilon, \cdot)$ converges to a measure $\alpha(\cdot)$ supported on $L = \{(s, t) \mid X_t = X_s, s < t\}$. This has been used in Rosen [7] to compute the Hausdorff dimension of $L$, see also Cuzick [2] and Weber [10]. As $B$ approaches the diagonal $\{s = t\}$, $\alpha(\cdot)$ "explodes." We wish to explore this phenomenon here.

For the case of planar Brownian motion, $\beta = \frac{1}{2}$, Varadhan [12] has shown that, with $D_T = \{(s, t) \mid 0 \leqslant s \leqslant t \leqslant T\}$,

$$\alpha(\varepsilon, D_T) - \frac{T}{2\pi} \lg(1/\varepsilon)$$

converges in $L^2$, as $\varepsilon \to 0$. This result has recently received a great deal of attention, including new proofs and generalizations to other processes which share with Brownian motion such properties as independent increments or at least the Markov property, see Rosen [8–10], Le Gall [6], Dynkin [3–5], and Yor [13, 14].

In this paper we will show that for the planar fractional Brownian motion $X_t$, $\frac{1}{2} < \beta < \frac{3}{4}$,

$$\alpha(\varepsilon, D_T) - c(\beta)\, T \frac{1}{\varepsilon^{1 - 1/2\beta}} \tag{1.5}$$

converges in $L^2$ as $\varepsilon \to 0$, where $c(\beta) = (1/2\pi) \int_0^\infty (1/r^{2\beta} + 1)\, dr$.

As we will see, the fact that $X_t$ does not have independent increments is compensated to some extent by its property of local nondeterminism, a property discovered by Berman [1] and used by him in the study of (nonintersection) local times for the one-dimensional fractional Brownian motion $F_t$.

It is, above all, the fact that $X_t$ is a Gaussian process which allows us, by direct computations, to overcome the lack of a Markovian structure.

It is hoped that this will be but the beginning of the study of intersection local times for Gaussian processes.

## 2. Proof

We will prove

**Theorem 1.** *Let* $\frac{1}{2} < \beta < \frac{3}{4}$, *then*

$$\alpha(\varepsilon, D_T) - \mathbb{E}(\alpha(\varepsilon, D_T))$$

*converges in* $L^2$ *as* $\varepsilon \to 0$.

*Remark.* Notice that

$$\begin{aligned}
\mathbb{E}(\alpha(\varepsilon, D_T)) &= \int_0^T \int_0^t \frac{1}{(2\pi)^2} \int e^{-\varepsilon p^2}\, \mathbb{E}(e^{ip \cdot X(s,t)})\, d^2p\, ds\, dt \\
&= \int_0^T \int_0^t \frac{1}{(2\pi)^2} \int e^{-(\varepsilon + |t-s|^{2\beta}) p^2/2}\, d^2p\, ds\, dt \\
&= \frac{1}{2\pi} \int_0^T \int_0^t \frac{1}{|t-s|^{2\beta} + \varepsilon}\, ds\, dt \\
&= \frac{1}{2\pi} \int_0^T \int_s^T \frac{1}{|t-s|^{2\beta} + \varepsilon}\, dt\, ds = \int_0^T \int_s^\infty - \int_0^T \int_T^\infty \\
&= \frac{T}{2\pi} \int_0^\infty \frac{1}{t^{2\beta} + \varepsilon}\, dt - \int_0^T \int_{T-s}^\infty \frac{1}{t^{2\beta} + \varepsilon}\, dt\, ds \\
&= c(\beta)\, T \frac{1}{\varepsilon^{1 - 1/2\beta}} + O(1),
\end{aligned} \tag{2.1}$$

since

$$\int_0^T \int_{T-s}^\infty \frac{1}{t^{2\beta} + \varepsilon} \leqslant \frac{1}{2\beta - 1} \int_0^T \frac{1}{(T-s)^{2\beta - 1}}\, ds < \infty.$$

Thus, (1.5) follows from Theorem 1.

*Proof of Theorem* 1.

$$\begin{aligned}
&\mathbb{E}(\alpha(\varepsilon, D_T) - \mathbb{E}(\alpha(\varepsilon, D_T)))^2 \\
&\quad = \int_{D_T \times D_T} \int \frac{1}{(2\pi)^4} \iint e^{-\varepsilon(p^2 + q^2)/2} \{ \mathbb{E}(e^{ip\ X(s,t) + iq\ X(s',t')}) \\
&\qquad - \mathbb{E}(e^{ip\ X(s,t)})\, \mathbb{E}(e^{iq\ X(s',t')}) \} \\
&\quad = \int_{D_T \times D_T} \int \frac{1}{(2\pi)^4} \iint e^{-\varepsilon(p^2 + q^2)/2} \{ e^{-\mathrm{Var}(p \cdot X(s,t) + q \cdot X(s',t'))/2} \\
&\qquad - e^{-p^2|t-s|^{2\beta}/2 - q^2|t'-s'|^{2\beta}/2} \}.
\end{aligned} \tag{2.2}$$

We will first show how to bound (2.2) uniformly in $\varepsilon$. The variance in (2.2) depends on the relative positions of $s$, $t$, $s'$, $t'$. We can assume that $s < s'$ and distinguish three cases.

*Case* I. $s < s' < t < t'$.

We will use the property of local nondeterminism, which says that for

$$t_1 < t_2 < \cdots < t_n$$
$$\operatorname{Var}\left(\sum_{i=2}^{n} u_i \cdot X(t_i, t_{i-1})\right) \geqslant k \sum_{i=2}^{n} |u_i|^2 |t_i - t_{i-1}|^{2\beta} \tag{2.3}$$

for some constant $k > 0$. See Berman [1]. Thus

$$e^{-\operatorname{Var}(p \cdot X(s,t) + q \cdot X(s',t'))/2} \leqslant e^{-k(p^2 a^{2\beta} + (p+q)^2 b^{2\beta} + q^2 c^{2\beta})}, \tag{2.4}$$

where $a = s' - s$, $b = t - s'$, and $c = t' - t$. We integrate out $da$, $db$, and $dc$ using the simple bound

$$\int_0^T e^{-p^2 t^{2\beta}}\, dt \leqslant \frac{k}{1 + p^{1/\beta}} \tag{2.5}$$

to bound our integral by

$$\iint \frac{1}{1 + p^{1/\beta}} \frac{1}{1 + (p+q)^{1/\beta}} \frac{1}{1 + q^{1/\beta}}\, d^2q\, d^2p$$
$$\leqslant k \int \frac{1}{1 + p^{1/\beta}} \frac{1}{1 + p^{2/\beta - 2}}\, d^2p < \infty \tag{2.6}$$

if $3/\beta - 2 > 2$, i.e., $\beta < \frac{3}{4}$.

The other term in (2.2) is even easier:

$$\int_{D_T \times D_T} \iiint e^{-p^2|t-s|^{2\beta}/2}\, e^{-q^2|t'-s'|^{2\beta}/2}$$
$$\leqslant k \iiint \frac{1}{(a+b)^{2\beta}} \frac{1}{(c+b)^{2\beta}}\, da\, db\, dc$$
$$\leqslant k \int_0^T \frac{1}{(b^{2\beta-1})^2}\, db < \infty \tag{2.7}$$

since $4\beta - 2 < 4(\frac{3}{4}) - 2 = 1$.

*Case* II. $s < s' < t' < t$.

We try to use local nondeterminism as in Case I:

$$e^{-\operatorname{Var}(p \cdot X(s,t) + q \cdot X(s',t'))} \leqslant e^{-k(p^2(a^{2\beta} + c^{2\beta}) + (p+q)^2 \beta^{2\beta})}, \tag{2.8}$$

where $a=s'-s$, $b=t'-s'$, and $c=t-t'$. We integrate (2.8) first with respect to $q$, then with respect to $p$, and obtain

$$\frac{1}{a^{2\beta}+c^{2\beta}}\frac{1}{b^{2\beta}}. \tag{2.9}$$

Assume that $b \geqslant \delta a$ for some fixed, but arbitrary $\delta$. Since $2\beta > 1$,

$$\int_{\delta a}^{T}\frac{1}{b^{2\beta}}\,db \leqslant k\frac{1}{a^{2\beta-1}} \tag{2.10}$$

and we now can bound

$$\int_0^T \frac{da}{a^{2\beta-1}}\int_0^T \frac{dc}{a^{2\beta}+c^{2\beta}} \leqslant \int_0^T \frac{da}{a^{4\beta-2}}\int_0^\infty \frac{dc}{1+c^{2\beta}} < \infty,$$

since as before, $4\beta-2<1$.

Similarly the double expectation term in (2.2) gives

$$\frac{1}{(a+b+c)^{2\beta}}\frac{1}{b^{2\beta}} \leqslant \frac{1}{(a+c)^{2\beta}}\frac{1}{b^{2\beta}}.$$

Integrating out $db$ as in (2.10) we need only bound

$$\int_0^T \frac{da}{a^{2\beta-1}}\int_0^T \frac{dc}{(a+c)^{2\beta}} \leqslant k\int_0^T \frac{da}{a^{4\beta-2}} < \infty.$$

We can proceed similarly if $b \geqslant \delta c$. However, if both $b \leqslant \delta a$ and $b \leqslant \delta c$, local nondeterminism is insufficient and we must proceed more carefully, making use of the subtraction term in (2.2).

We first write

$$\begin{aligned}
&\operatorname{Var}(p\cdot X(s,t)+q\cdot X(s',t'))\\
&\quad = p^2|t-s|^{2\beta}+2p\cdot q\mathbb{E}(X(s,t)\,X(s',t'))+q^2|t'-s'|^{2\beta}\\
&\quad = p^2(a+b+c)^{2\beta}+2p\cdot qv+q^2b^{2\beta},
\end{aligned} \tag{2.11}$$

where, by (1.1),

$$\begin{aligned}
v &\doteq \mathbb{E}(X(s,t)\cdot X(s',t'))\\
&= (a+b)^{2\beta}-a^{2\beta}+(c+b)^{2\beta}-c^{2\beta}\\
&= a^{2\beta}\left[\left(1+\frac{b}{a}\right)^{2\beta}-1\right]+c^{2\beta}\left[\left(1+\frac{b}{c}\right)^{2\beta}-1\right]\\
&\leqslant k(a^{2\beta-1}b+c^{2\beta-1}b)
\end{aligned} \tag{2.12}$$

for $\delta$ sufficiently small.

We now write out the full integrand appearing in (2.2)

$$e^{-p^2((a+b+c)^{2\beta}+\varepsilon)/2}(e^{p\cdot qv}-1)\,e^{-q^2(b^{2\beta}+\varepsilon)/2} \tag{2.13}$$

and integrate with respect to $q$ to obtain

$$\int (e^{p\cdot qv}-1)\,e^{-q^2(b^{2\beta}+\varepsilon)/2}d^2q$$

$$=\frac{(e^{p^2v^2/2(b^{2\beta}+\varepsilon)}-1)}{b^{2\beta}+\varepsilon} \qquad \text{(which is positive)}$$

$$\leqslant \frac{e^{p^2v^2/2b^{2\beta}}-1}{b^{2\beta}}. \tag{2.14}$$

By (2.12),

$$v^2/b^{2\beta} \leqslant k(a^{4\beta-2}b^{2-2\beta}+c^{4\beta-2}b^{2-2\beta})$$

$$\leqslant k\left(a^{2\beta}\left(\frac{b}{a}\right)^{2-2\beta}+c^{2\beta}\left(\frac{b}{c}\right)^{2-2\beta}\right)\leqslant \frac{1}{4}(a+b+c)^{2\beta} \tag{2.15}$$

for $\delta$ small, which allows us to write

$$e^{-p^2(a+b+c)^{2\beta}/2}\,\frac{e^{p^2v^2/2b^{2\beta}}-1}{b^{2\beta}}$$

$$=e^{-p^2(a+b+c)^{2\beta}/4}\left[\frac{e^{-p^2((a+b+c)^{2\beta}/2-v^2/b^{2\beta})/2}-e^{-p^2(a+b+c)^{2\beta}/4}}{b^{2\beta}}\right]$$

$$\leqslant ke^{-p^2(a+b+c)^{2\beta}/4}p^2v^2/b^{4\beta} \tag{2.16}$$

using

$$|e^{-x}-e^{-y}|\leqslant 2\,|x-y|, \qquad x, y\geqslant 0. \tag{2.17}$$

By (2.12), (2.14) is bounded by

$$e^{-p^2(a+b+c)^2\beta/4}p^2\,\frac{(a^{4\beta-2}+c^{4\beta-2})}{b^{4\beta-2}}. \tag{2.18}$$

We integrate with respect to $p$ and need only bound

$$\iiint \frac{a^{4\beta-2}+c^{4\beta-2}}{(a+b+c)^{4\beta}}\cdot\frac{1}{b^{4\beta-2}}\,da\,db\,dc$$

$$\leqslant k\iiint \frac{1}{(a+b+c)^2}\cdot\frac{1}{b^{4\beta-2}}\,da\,db\,dc$$

$$\leqslant k\int (1+\lg(b))\frac{1}{b^{4\beta-2}}\,db<\infty, \tag{2.19}$$

Since as before, $4\beta-2<1$.

*Case* III. $s < t < s' < t'$.

Once again we first trye to use local nondeterminism which shows that our integrand is bounded by

$$e^{-k(p^2a^{2\beta}+q^2c^{2\beta})},$$

where $a = t - s$, $b = s' - t$, and $c = t' - s'$. After integrating with respect to $p$ and $q$ we need to bound

$$\iiint \frac{1}{a^{2\beta}} \frac{1}{c^{2\beta}}\, da\, db\, dc.$$

In general this will diverge. However, in the region $a \geqslant Mb$, $c \geqslant Nb$, and $N$, $M$ fixed but arbitrary, we have

$$\int_0^T db \left(\int_{Mb}^T \frac{da}{a^{2\beta}}\right)\left(\int_{Nb}^T \frac{dc}{c^{2\beta}}\right) \leqslant k \int_0^T \frac{db}{b^{4\beta-2}} < \infty$$

as before.

For relatively large $b$ we will have to proceed more carefully. Consider

$$\operatorname{Var}(p \cdot X(s, t) + q \cdot X(s', t')) = p^2a^{2\beta} + 2p \cdot qv + q^2c^{2\beta}, \tag{2.20}$$

where

$$v = \mathbb{E}(X(s, t)\, X(s', t')) = (a+b+c)^{2\beta} - (a+b)^{2\beta} - (c+b)^{2\beta} + b^{2\beta}. \tag{2.21}$$

Let us first suppose that both $a \leqslant \delta_1 b$, and $c \leqslant \delta_2 b$, where $\delta_1$ and $\delta_2$ small will be specified later. From our last equation we see that

$$\begin{aligned} v &= b^{2\beta}\left[\left(1+\frac{a}{b}+\frac{c}{b}\right)^{2\beta} - \left(1+\frac{a}{b}\right)^{2\beta} - \left(1+\frac{c}{b}\right)^{2\beta} + 1\right] \\ &\leqslant kb^{2\beta}\frac{a}{b}\frac{c}{b} \\ &= kb^{2\beta-2}ac. \end{aligned} \tag{2.22}$$

As in Case II we integrate with respect to $q$ first,

$$\int (e^{p \cdot qv} - 1)\, e^{-q^2(c^{2\beta}+\varepsilon)/2}\, d^2q \leqslant \frac{e^{p^2v^2/2c^{2\beta}} - 1}{c^{2\beta}}. \tag{2.23}$$

By (2.22),

$$\begin{aligned} v^2/c^{2\beta} &\leqslant kb^{4\beta-4}a^2c^2/c^{2\beta} \\ &= ka^{2\beta}\left(\frac{a}{b}\right)^{2-2\beta}\left(\frac{c}{b}\right)^{2-2\beta} \\ &\leqslant \frac{1}{4}a^{2\beta} \qquad \text{if} \quad \delta_1, \delta_2 \leqslant \delta_0 \text{ small.} \end{aligned} \tag{2.24}$$

As in Case II this allows us to bound

$$\begin{aligned} &\int e^{-p^2a^{2\beta}/2}\left(\frac{e^{p^2v^2/2c^{2\beta}}-1}{c^{2\beta}}\right) d^2p \\ &\leqslant k\int e^{-p^2a^{2\beta}/4}p^2\, d^2p \cdot v^2/c^{4\beta} \\ &\leqslant k\,\frac{v^2}{a^{4\beta}c^{4\beta}} \leqslant k\,\frac{b^{4\beta-4}}{a^{4\beta-2}c^{4\beta-2}} \qquad \text{by (2.22).} \end{aligned} \tag{2.25}$$

Now $4-4\beta > 4-3 = 1$ so that

$$\int_{\sqrt{ac}} \frac{db}{b^{4-4\beta}} \leqslant \frac{k}{(ac)^{3/2-2\beta}}.$$

Taking into account (2.25), we need only bound

$$\left(\int \frac{da}{a^{2\beta-1/2}}\right)\left(\int \frac{dc}{c^{2\beta-1/2}}\right) < \infty,$$

since $2\beta - \frac{1}{2} < 2(\frac{3}{4}) - \frac{1}{2} = 1$.

This handles the case $a \leqslant \delta_1 b$ and $c \leqslant \delta_2 b$. We finally consider $c \leqslant \delta_3 b$ and $a \geqslant \delta_4 b$. Return to (2.21) and note

$$\begin{aligned} v &= (a+b+c)^{2\beta} - (a+b)^{2\beta} - (c+b)^{2\beta} + b^{2\beta} \\ &\leqslant |(a+b+c)^{2\beta} - (a+b)^{2\beta}| + |(c+b)^{2\beta} - b^{2\beta}| \\ &\leqslant k(a+b)^{2\beta-1}c, \qquad (\text{recall } 2\beta - 1 > 0). \end{aligned} \tag{2.26}$$

Thus instead of (2.24) we have

$$\begin{aligned} v^2/c^{2\beta} &\leqslant k(a+b)^{4\beta-2}c^{2-2b} = k(a+b)^{2\beta}\left(\frac{c}{a+b}\right)^{2-2\beta} \\ &\leqslant k(a+b)^{2\beta}\,\delta_3^{2-2\beta} \leqslant ka^{2\beta}\left(1+\frac{1}{\delta_4^{2\beta}}\right)\delta_3^{2-2\beta} \\ &\leqslant \tfrac{1}{4}a^{2\beta} \qquad \text{if } \delta_3 \leqslant \delta_4^3 \text{ and } \delta_4 \leqslant \bar{\delta}_0 \text{ small.} \end{aligned} \tag{2.27}$$

This follows from the fact that $2\beta < \frac{3}{2}$, while $2 - 2\beta > \frac{1}{2}$.

As before we need only integrate

$$\iiint \frac{v^2}{a^{4\beta}c^{4\beta}} \leqslant \iiint \frac{(a+b)^{4\beta-2}}{a^{4\beta}c^{4\beta-2}}$$
$$\leqslant k\int_0^T \frac{dc}{c^{4\beta-2}}\int_{c/\delta_3}^T db \int_{\delta_4 b}^T \frac{da}{a^2}$$
$$\leqslant k\int_0^T \frac{\lg(c)}{c^{4\beta-2}}\,dc < \infty \tag{2.28}$$

as before.

All that remains in order to show uniform boundedness in Case III is to verify that by an appropriate choice of $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$ we can handle all possible cases. Take $\delta_4$ smaller than $\delta_0^3 \wedge \bar{\delta}_0^3 < \delta_0 \wedge \bar{\delta}_0$, where $\delta_0$ and $\bar{\delta}_0$ are described in (2.24) and (2.27). If $a \geqslant \delta_4 b$ then either

$$c \leqslant \delta_4^3 b$$

or

$$c \geqslant \delta_4^3 b,$$

both of which are covered by the above analysis.

On the other hand, if $a \leqslant \delta_4 b$ then either

$$c \leqslant \delta_4^{1/3} b$$

or

$$c \geqslant \delta_4^{1/3} b.$$

Since $\delta_4^{1/3} \leqslant \bar{\delta}_0$, these cases are also covered.

This shows that (2.2) is uniformly bounded in $\varepsilon$, and convergence will follow from a careful use of the dominated convergence theorem. Actually, we can show

$$\mathbb{E}([\alpha(\varepsilon, D_T) - \mathbb{E}(\alpha(\varepsilon, D_T)] - [\alpha(\varepsilon', D_T) - \mathbb{E}(\alpha(\varepsilon', D_T))])^2$$
$$\leqslant k|\varepsilon - \varepsilon'|^{\delta} \qquad \text{for some } \delta > 0$$

by using the bound

$$|e^{-p^2\varepsilon/2} - e^{-p^2\varepsilon'/2}| \leqslant kp^{2\delta}|\varepsilon - \varepsilon'|^{\delta}$$

and following the lines of our proof of boundedness.

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