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A RENORMALIZED LOCAL TIME FOR MULTIPLE INTERSECTIONS OF PLANAR BROWNIAN MOTION

by

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Abstract: We present a simple prescription for 'renormalizing' the local time for n-fold intersections of planar Brownian motion, generalizing Varadhan's formula for n = 2. In the latter case, we present a new proof that the renormalized local time is jointly continuous.

1. Introduction

If W_t is a planar Brownian motion with transition density function

$$(1.1) \qquad p_t(x) = \frac{e^{-|x|^2/2t}}{2\pi t} ,$$

then, with $W(s,t) = W_t - W_s$,

$$(1.2) \qquad \alpha(B) = \lim_{\varepsilon\to 0} \int_B p_\varepsilon(W(t_1,t_2))\dots p_\varepsilon(W(t_{n-1},t_n))dt_1\dots dt_n$$

defines a measure on

$$(1.3) \qquad R^n_\delta = \{(t_1,\dots,t_n) | \forall t_i \geq 0 \text{ and } \inf|t_j - t_k| \geq \delta\}$$

supported on

$$\{(t_1,\dots,t_n) | W_{t_1} = \dots = W_{t_n}\} ,$$

which has been applied in Rosen [1984a] to study the n-fold intersections of the path W. The measure $\alpha(\cdot)$ is called the n-fold intersection local time.

If we drop the condition $\inf|t_j - t_k| \geq \delta$ in (1.3), $\alpha(\cdot)$ 'blows up'. The main contribution of this paper is the following theorem which tells how to 'renormalize' (1.2). We use the notation $\{X\} = X - E(X)$.

THEOREM 1. Let

$$(1.4) \qquad I_\varepsilon(B) = \int_B \{p_\varepsilon(W(t_1,t_2))\}\dots\{p_\varepsilon(W(t_{n-1},t_n))\}dt_1\dots dt_n .$$

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Then $I_\varepsilon(B)$ converges in L^2 for all bounded Borel sets B in

$$R^n_{\leq} = \{(t_1,\ldots,t_n) | 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n\}\ .$$

REMARK 1. For $n = 2$, this theorem goes back to Varadhan [1969] and has recently seen several alternate proofs, see Rosen [1984b], Yor [1985a], [1985b], Le Gall [1985] and Dynkin [1985].

For $n = 3$ this theorem has recently been established independently by M. Yor and the present author using stochastic integrals.

A different type of renormalized local time has recently been obtained for general n by E. Dynkin.

REMARK 2. A full proof of Theorem 1 is given in Section 2. For general n the notation becomes fairly complicated, so that we felt it would be useful to illustrate our method of proof by looking carefully at the case $n = 2$.

We use the Fourier representation

$$(1.5) \qquad p_\varepsilon(x) = \frac{1}{(2\pi)^2} \int e^{ipx} e^{-\varepsilon|p|^2/2} dp$$

to write

$$(1.6) \qquad \mathbf{E}(I^2_\varepsilon(B)) = \frac{1}{(2\pi)^4} \int_{B\times B}\int e^{-\varepsilon(|p|^2+|q|^2)/2} \mathbf{E}\{e^{ipW(t_1,t_2)}\}\{e^{iqW(s_1,s_2)}\}\ .$$

Note that

$$(1.7) \qquad \mathbf{E}\{e^{ipW(t_1,t_2)}\}\{e^{iqW(s_1,s_2)}\} = \mathbf{E}(e^{ipW(t_1,t_2) + iqW(s_1,s_2)}) - \mathbf{E}(e^{ipW(t_1,t_2)})\mathbf{E}(e^{iqW(s_1,s_2)})$$

depends on the relative positions of s_1,s_2,t_1,t_2. We distinguish three possible cases.

CASE I: The intervals $[s_1,s_2]$, $[t_1,t_2]$ are disjoint. In this case, because W has independent increments, (1.7) vanishes.

CASE II: The intervals $[s_1,s_2]$, $[t_1,t_2]$ overlap, but neither one contains the other. For definiteness let us say

$$s_1 < t_1 < s_2 < t_2.$$

(1.7) becomes

$$(1.8) \qquad e^{-|q|^2\ell_1/2 - |p+q|^2\ell_2/2 - |p|^2\ell_3/2} - e^{-|q|^2\ell_1/2 - (|p|^2+|q|^2)\ell_2/2 - |p|^2\ell_3/2}$$
$$\leq 2e^{-|q|^2\ell_1/2 - |p+q|^2\ell_2/4 - |p|^2\ell_3/2}$$

where $\ell_1 = t_1 - s_1$, $\ell_2 = s_2 - t_1$, $\ell_3 = t_2 - s_2$.

We now integrate with respect to the variables s_1, s_2, t_1, t_2 using

$$\int_0^T e^{-v^2 s} ds \le \frac{c}{1+v^2} \tag{1.9}$$

to find that in Case II

$$\mathbb{E}(I_\varepsilon^2(B)) \le c \int (1+|q|^2)^{-1}(1+|p+q|^2)^{-1}(1+|p|^2)^{-1} dpdq. \tag{1.10}$$

This is easily seen to be finite and the dominated convergence theorem shows L^2 convergence.

<u>CASE III</u>: One of the intervals $[s_1,s_2]$, $[t_1,t_2]$ strictly contains the other. For definiteness, say

$$t_1 < s_1 < s_2 < t_2.$$

In such a case we refer to $[s_1,s_2]$ as an <u>isolated interval</u> and to q as an <u>isolated variable</u>.

If we attempt to use the method of Case II, we find instead of (1.10), the integral

$$\int (1+|p|^2)^{-1}(1+|p+q|^2)^{-1}(1+|p|^2)^{-1} dpdq$$

which diverges, since q appears only in one factor (hence the terminology isolated variable.)

We proceed more carefully. In Case III, (1.7) becomes

$$\begin{aligned} & e^{-|p|^2\ell_{1/2} - |p+q|^2\ell_{2/2} - |p|^2\ell_{3/2}} \\ & \quad - e^{-|p|^2\ell_{1/2} - (|p|^2+|q|^2)\ell_{2/2} - |p|^2\ell_{3/2}} \\ & \quad = e^{-|p|^2\ell_{1/2}}(e^{-|p+q|^2\ell_{2/2}} - e^{-(|p|^2+|q|^2)\ell_{2/2}})e^{-|p|^2\ell_{3/2}} \end{aligned} \tag{1.11}$$

where $\ell_1 = s_1 - t_1$, $\ell_2 = s_2 - s_1$, $\ell_3 = t_2 - s_2$.

The key step is now to integrate first with respect to the isolated variable q in (1.6).

We use

$$\begin{aligned} & \int e^{-\varepsilon|q|^2/2}(e^{-|p+q|^2\ell/2} - e^{-(|p|^2+|q|^2)\ell/2})dq \\ & \quad = e^{-|p^2|\ell/2}\int (e^{-p\cdot q\ell} - 1)e^{-|q|^2(\ell+\varepsilon)/2}dq \ = \end{aligned} \tag{1.12}$$

$$= e^{-|p|^2\ell/2}\,\frac{\left(e^{p^2\frac{\ell^2}{2(\ell+\varepsilon)}}-1\right)}{\ell+\varepsilon} \doteq F_\varepsilon(p,\ell) \geq 0 .$$

The remaining integrand in (1.6) is now positive, and monotone increasing as $\varepsilon \downarrow 0$. We can use the bound

$$F_\varepsilon(p,\ell) \leq F_0(p,\ell) = \frac{(1-e^{-|p|^2\ell/2})}{\ell} \leq c|p|^{2\delta}\ell^{-1+\delta} \tag{1.13}$$

for any $0 < \delta < 1$. We then integrate with respect to s_1,s_2,t_1,t_2 using (1.9) for $|p|^2$, to obtain, instead of (1.10) the bound

$$c\int(1+|p|^2)^{-1}|p|^{2\delta}(1+|p|^2)^{-1}dp < \infty. \tag{1.14}$$

As before, the dominated convergence theorem gives L^2 convergence.

REMARK 3. With a bit more work we can show

$$\mathbf{E}(I_\varepsilon(B) - I_{\varepsilon'}(B))^2 \leq c|\varepsilon-\varepsilon'|^\delta \tag{1.15}$$

for some $\delta > 0$. To do this we note that the expectation in (1.15) differs from (1.6) in that the factor

$$e^{-\varepsilon(|p|^2+|q|^2)/2} \tag{1.16}$$

is replaced by

$$(e^{-\varepsilon|p|^2/2} - e^{-\varepsilon'|p|^2/2})(e^{-\varepsilon|q|^2/2} - e^{-\varepsilon'|q|^2/2}) . \tag{1.17}$$

For any non-isolated variables we use the bound

$$|e^{-\varepsilon|p|^2/2} - e^{-\varepsilon'|p|^2/2}| \leq c|p|^{2\delta}(\varepsilon-\varepsilon') . \tag{1.18}$$

This suffices to show (1.15).

For use in discussing general n, we note that we can also obtain a useful bound from an isolated variable. Note:

$$\begin{aligned} F_\varepsilon(p,\ell) - F_{\varepsilon'}(p,\ell) &= \frac{(e^{-|p|^2\frac{\ell\varepsilon}{2(\ell+\varepsilon)}} - e^{-|p|^2\ell/2})}{\ell+\varepsilon} - \frac{(e^{-|p|^2\frac{\ell\varepsilon'}{2(\ell+\varepsilon')}} - e^{-|p|^2\ell/2})}{\ell+\varepsilon'} \\ &= (e^{-|p|^2\frac{\ell\varepsilon}{2(\ell+\varepsilon)}} - e^{-|p|^2\ell/2})\left(\frac{1}{\ell+\varepsilon} - \frac{1}{\ell+\varepsilon'}\right) \\ &\quad + (e^{-|p|^2\frac{\ell\varepsilon}{2(\ell+\varepsilon)}} - e^{-|p|^2\frac{\ell\varepsilon'}{2(\ell+\varepsilon')}})\cdot\frac{1}{\ell+\varepsilon'} . \end{aligned} \tag{1.19}$$

Therefore

$$(1.20)\qquad |F_\varepsilon(p,\ell) - F_{\varepsilon'}(p,\ell)| \le c|p|^{2\beta}\left(\left|\frac{\ell\varepsilon}{\ell+\varepsilon} - \ell\right|^\beta \frac{|\varepsilon' - \varepsilon|}{(\ell+\varepsilon)(\ell+\varepsilon')} + \left|\frac{\ell\varepsilon}{\ell+\varepsilon} - \frac{\ell\varepsilon'}{\ell+\varepsilon'}\right|^\beta \frac{1}{\ell+\varepsilon'}\right)$$

$$= c|p|^{2\beta}\left(\frac{\ell^{2\beta}}{(\ell+\varepsilon)^\beta} \frac{(\varepsilon' - \varepsilon)}{(\ell+\varepsilon)(\ell+\varepsilon')} + \left|\frac{\ell^2(\varepsilon' - \varepsilon)}{(\ell+\varepsilon)(\ell+\varepsilon')}\right|^\beta \frac{1}{\ell+\varepsilon'}\right).$$

Since $|\varepsilon - \varepsilon'| \le \max(\ell+\varepsilon, \ell+\varepsilon')$ we always have

$$(1.21)\qquad \frac{|\varepsilon' - \varepsilon|}{(\ell+\varepsilon)(\ell+\varepsilon')} \le \frac{|\varepsilon' - \varepsilon|^\delta}{\ell^{1+\delta}}$$

so that returning to (1.20), we have

$$(1.22)\qquad |F_\varepsilon(p,\ell) - F_{\varepsilon'}(p,\ell)| \le c|p|^{2\beta}(|\varepsilon' - \varepsilon|^\delta \ell^{-1+\beta-\delta} + |\varepsilon - \varepsilon|^{\delta\beta}\ell^{-(1+\delta\beta)+\beta}).$$

Taking $0 < \delta < \beta$ we find

$$(1.23)\qquad \int |F_\varepsilon(p,\ell) - F_{\varepsilon'}(p,\ell)|d\ell \le c|p|^{2\beta}|\varepsilon' - \varepsilon|^{\delta\beta}.$$

For $x = (x_1,\dots,x_{n-1})$, $x_i \in R^2$ we now define

$$(1.24)\qquad I_\varepsilon(x,T) = \int\limits_{0\le t_1\le\dots\le t_n\le T}\int \{p_\varepsilon(W(t_1,t_2)-x_1)\}\dots\{p_\varepsilon(W(t_{n-1},t_n)-x_{n-1})\}.$$

Without the brackets, and in the region (1.3), $\lim_{\varepsilon\to 0} I_\varepsilon(x,\cdot)$ is the occupation density of the random field

$$X(t) = (W(t_1,t_2),\dots,W(t_{n-1},t_n)),$$

and studied in Rosen [1984]. The limit of $I_\varepsilon(x,T)$ as $\varepsilon \to 0$ is a renormalized version of the occupation density, and we would like to know that with probability one it is continuous in x,T - as is known to be true when $n = 2$, see Rosen [1984b], Le Gall [1985].

Unfortunately, the bounds we find in the proof of Theorem 1 do not suffice to establish (pathwise) continuity.

The next theorem refers to the known case $n = 2$. The proof given here is new, and related to the proof of Theorem 1. It is offered in the hope that it will lead to a proof for general n - and be useful in studying other processes with an independence structure similar to Brownian motion, e.g. Lévy processes and Brownian sheets.

THEOREM 2. $I_\varepsilon(x,T) = \int_0^T\int_0^t \{p_\varepsilon(W(s,t) - x)\}dsdt$ converges as to a limit process $I(x,T)$ which is jointly continuous in x and T.

Let us now define

$$\alpha_\varepsilon^{(n)}(T) = \int\limits_{0\le t_1\le\ldots\le t_n\le T}\cdots\int p_\varepsilon(W(t_1,t_2))\ldots p_\varepsilon(W(t_{n-1},t_n))dt_1\ldots dt_n$$

without brackets, so that we know $\alpha_\varepsilon^{(n)}(T) \to \infty$ as $\varepsilon \to 0$.

In case $n = 2$ or 3 we can be more explicit.

$$\alpha_\varepsilon^{(2)}(T) \sim T\,\frac{\ell g(1/\varepsilon)}{2\pi}$$

$$\alpha_\varepsilon^{(3)}(T) \sim T\left(\frac{\ell g(1/\varepsilon)}{2\pi}\right)^2 + 2\left(\frac{T\ell gT-T}{2\pi} + \gamma(T)\right)\ell g(1/\varepsilon)$$

where

$$\gamma(T) = \lim_{\varepsilon\to 0} I_\varepsilon(0,T)$$
$$= \lim_{\varepsilon\to 0}\int_0^T\int_0^t \{p_\varepsilon(W(s,t))\}dsdt$$

of Theorem 2.

It would be nice to have a similar asymptotic expansion for $\alpha_\varepsilon^{(n)}(T)$ for general n. We have not yet succeeded in finding this, but mention that E. Dynkin has found such an expansion for his renormalized local time.

2. Proof of Theorem 1

Our proof for general n is similar to the proof for $n = 2$ given in the introduction. We first integrate over isolated variables where the 'bracket' is essential.

Here are the details. Let $W(s,t) = W_t - W_s$, $i^* = i+1$, and every $\prod$ or Σ is over all possible values of the indices, unless specified otherwise. We have

$$\text{(2.1)}\qquad \mathbb{E}(I_\varepsilon^2(B)) = \int\limits_{B\times B}\int dsdt\; dpdqG_\varepsilon(p,q)\mathbb{E}(\prod\{e^{ip_jW(t_j,t_{j^*})}\}\cdot\{e^{iq_jW(s_j,S_{j^*})}\})$$

where

$$\text{(2.2)}\qquad G_\varepsilon(p,q) = e^{-\varepsilon(\Sigma|p_j|^2 + |q_j|^2)/_2}\,.$$

By additivity, it will suffice to consider integrals of the above form where $B \times B$

is replaced by a Borel set

$$A \subseteq [0,T]^{2n}$$

in which the values of the $2n$ coordinates have a fixed relative ordering. Thus, e.g., if for some point in A the third component is larger than the second, then this will be true for all points in A. We rename the coordinates $r_1, r_2, \ldots, r_{2n}$ so that

$$0 < r_1 < r_2 < r_3 < \ldots < r_{2n} < T.$$

Throughout A, each s_i or t_j is uniquely identified with one of the r_k.

We say that an interval $[r_i, r_{i*}]$ is isolated if either

$$[r_i, r_{i*}[= [s_\ell, s_{\ell*}] \qquad \text{for some } \ell,$$

or

$$[r_i, r_{i*}] = [t_m, t_{m*}] \qquad \text{for some } m.$$

Let

$$I = \{i \mid [r_i, r_{i*}] \text{ is isolated}\}$$

$$I_S = \{\ell \mid [s_\ell, s_{\ell*}] \text{ is isolated}\}$$

$$I_T = \{k \mid [t_k, t_{k*}] \text{ is isolated}\} .$$

Note that the 'brackets' in (2.1) assure us that our integral will vanish unless 1, $2n-1$ are not in I.

In (2.1) we expand the bracket, $\{X\} = X - E(X)$ for all non-isolated intervals, obtaining many terms, each of which will be bounded separately.

We first consider the term

$$(2.3) \qquad \int_A\!\!\int \iint dpdq\; G_\varepsilon(p,q)\mathbf{E}\left(e^{i\sum_{I_T^c} p_j W(t_j, t_{j*}) + \sum_{I_S^c} q_j W(s_j, s_{j*})} \cdot \prod_{I_T} \left\{ e^{ip_j W(t_j, t_{j*})} \right\} \prod_{I_S} \left\{ e^{iq_j W(s_j, s_{j*})} \right\} \right) dsdt .$$

Write

$$(2.4) \qquad \sum_{I_T^c} p_j W(t_j, t_{j*}) + \sum_{I_S^c} q_j W(s_j, s_{j*}) = \sum_{i=1}^{2n-1} u_i W(r_i, r_{i*}) .$$

The u_i are linear combinations of the p's and q's. More precisely, if either $i = 1,\ 2n-1$ or $i \in I$, then u_i is equal to one of the p_j or q_j. Otherwise, u_i will be the sum of exactly one p_j and one q_k.

If $i \in I$ and $[r_i, r_{i*}] = [s_\ell, s_{\ell*}]$ set $v_i = q_\ell$, while if $[r_i, r_{i*}] = [t_m, t_{m*}]$ set $v_i = p_m$. v_i is called an isolated variable. Taking expectations, (2.3) becomes

$$(2.5) \qquad \int_A\!\int \iint dp\,dq\; G_\varepsilon(p,q) e^{-\sum_{I^c} |u_i|^2 \ell_i/2} \cdot \prod_I \left(e^{-|u_i+v_i|^2 \ell_i/2} - e^{-(|u_i|^2+|v_i|^2)\ell_i/2} \right) ds\,dt$$

where $\ell_i = r_{i+1} - r_i$ is the length of the i'th interval. We now integrate over the isolated variables v_i, and by (1.15) we find that (2.5) is equal to

$$(2.6) \qquad \int_A\!\int \iint d\hat{p}\,d\hat{q}\; G\,(\hat{p},\hat{q}) e^{-\sum_{I^c} |u_i|^2 \ell_i/2} \prod_I F_\varepsilon(u_i,\ell_i)\,ds\,dt$$

where $\hat{p},\hat{q}$ denote the remaining, i.e. non-isolated variables.

The integrand in (2.6) is now positive, and as in the introduction we use the bound (1.9), (1.10) to see that (2.6) is bounded by

$$(2.7) \qquad \int\!\!\int \prod_{I^c} (1 + |u_i|^2)^{-1} \prod_I |u_j|^{2\delta}\, d\hat{p}\,d\hat{q}\ .$$

From the discussion following (2.4) we see that the set $\{u_i\}_{i \in I^c}$ will span the set $\{u_j\}_{j \in I}$, and by choosing $\delta > 0$ small enough, it suffices to bound

$$(2.8) \qquad \int\!\!\int \prod_{I^c} (1 + |u_i|^2)^{-1+\beta}\, d\hat{p}\,d\hat{q}\ .$$

Each non-isolated variable will occur as a summand in precisely two (necessarily successive) factors in (2.8). For a variable occurring in one could not be non-isolated, while if it occurred in more than two - say u_i, u_j, u_k - the other component of u_j could not be non-isolated. The upshot of this is that if $|I^c| = k$, then any $k - 1$ vectors from the set $\{u_i\}_{i \in I^c}$ will span the set of non-isolated variables. (Remember, $i = 1, 2n-1$ are both in I^c, and both u_1, u_{2n-1} are exactly equal to a non-isolated variable.) We can now use Hölder's inequality to bound (2.8).

$$(2.9) \qquad \int\!\!\int \prod_{I^c} (1 + |u_i|^2)^{-1+\beta} d\hat{p}\,d\hat{q} = \int\!\!\int \prod_{i \in I^c} \left(\prod_{\substack{I^c \\ j \neq i}} (i + |u_j|^2)^{-1+\beta} \right)^{1/k-1} d\hat{p}\,d\hat{q} \ \leq$$

$$\leq \prod_{i \in I^c} \left\| \prod_{\substack{I^c \\ j \neq i}} (1 + |u_j|^2)^{-\frac{1+\beta}{k-1}} \right\|_k < \infty$$

as long as

$$\frac{2(1 - \beta)k}{k - 1} > 2$$

i.e.

$$\beta < \frac{1}{k} .$$

This shows that the term (2.3) is uniformly bounded. The other terms which come from our expanding the 'bracket' for non-isolated intervals, can be obtained from (2.3) by replacing some factors by their expectations. As in the introduction the resulting integrals can be bounded similarly to (2.3). Thus $\mathbf{E}(I_\varepsilon^2(B))$ is uniformly bounded, and L^2 convergence follows easily from the dominated convergence theorem.

If we wish we can even obtain

$$\mathbf{E}(I_\varepsilon(B) - I_{\varepsilon'}(B))^2 \leq C|\varepsilon - \varepsilon'|^\delta$$

for some $\delta > 0$, by following Remark 3 of the introduction.

3. Proof of Theorem 2

The reader is advised to go through the proof of Lemma 2 in Rosen [1983] in order to appreciate the constructions introduced here.

We will show that for some $\delta > 0$, and all m even

$$\text{(3.1)} \qquad \mathbf{E}(I_\varepsilon(x,T) - I_{\varepsilon'}(x',T'))^m \leq c_m|(\varepsilon,x,T) - (\varepsilon',x',T')|^{m\delta} ,$$

where the constant c_m can be chosen independent of $\varepsilon,\varepsilon' > 0$ and x,x',T,T' in any bounded set. Kolmogorov's theorem then assures us that, with probability one, for any $\beta < \delta$

$$\text{(3.2)} \qquad |I_\varepsilon(x,T) - I_{\varepsilon'}(x',T')| \leq c|(\varepsilon,x,T) - (\varepsilon',x',T')|^\beta,$$

first for all rational arguments in a bounded set as described - but then for all such parameters since $I_\varepsilon(x,T)$ is clearly continuous as long as $\varepsilon > 0$.

(3.2) shows that

$$\text{(3.3)} \qquad I(x,T) \doteq \lim_{\varepsilon \to 0} I_\varepsilon(x,T)$$

exists and is continuous in x,T.

It remains to prove (3.1). We concentrate first on bounding

$$\mathbf{E}(I_\varepsilon(x,T)^m) = \int_B \cdots \int dsdt \int dp G_\varepsilon(x,p) \, \mathbf{E}\left(\prod_{j=1}^m \left\{ e^{ip_j W(t_j,s_j)} \right\} \right) \tag{3.4}$$

where

$$G_\varepsilon(x,p) = \prod_{j=1}^m e^{-ip_j x - \varepsilon |p_j|^2/2} \tag{3.5}$$

$$B = \{(s,t) | 0 \le s \le t \le t\}^m . \tag{3.6}$$

It suffices, by additivity, to replace B by a region $A \subseteq [0,T]^{2m}$ in which the values of the coordinates have a fixed relative ordering. Let $r_1, r_2, \ldots, r_{2m}$ relabel the coordinates so that

$$0 < r_1 < r_2 < \ldots < r_{2m} < T \quad .$$

Thus, throughout A each r_j is uniquely identified with one of the s_ℓ or t_m.

In general, $\bigcup_i [r_i, r_{i*}]$ will have several components. Using independence, it is clear that in bounding (3.4) we can assume that there is only one component.

In analogy with our proof of Theorem 1, we will say that $[r_i, r_{i*}]$ is isolated if $[r_i, r_{i*}] = [s_j, t_j]$ for some j, in which case we set $v_i = p_j$ and refer to v_i as an isolated variable. Let

$$I = \{i | [r_i, r_{i*}] \quad \text{isolated}\}$$

$$J = \{j | [s_j, t_j] \quad \text{isolated}\}.$$

We note again that $1, 2m-1$ are not in I.

We now expand the 'brackets' in (3.4) for all non-isolated intervals $[s_j, t_j]$. We obtain many terms, of which we first consider

$$\int_A \cdots \int dsdt \int dp \; G_\varepsilon(x,p) \mathbf{E}\left(\prod_{J^c} e^{ip_j W(s_j,t_j)} \cdot \prod_J \left\{ e^{ip_j W(s_j,t_j)} \right\} \right) . \tag{3.7}$$

We now write

$$\sum_{J^c} p_j W(s_j,t_j) = \sum_{i=1}^{2m-1} u_i W(r_i, r_{i*}) \quad . \tag{3.8}$$

Taking expectations in (3.7) gives

$$\int_A \cdots \int dsdt \int dp \; G_\varepsilon(x,p) e^{-\sum_{I^c} |u_i|^2 \ell_i/2} \prod_I \left(e^{-|u_j+v_j|^2 \ell_j/2} - e^{-(|u_j|^2+|v_j|^2)\ell_j/2} \right) \tag{3.9}$$

where again $\ell_i = r_{i+1} - r_i$ is the length of the i'th interval.

We first integrate over isolated variables using

$$\text{(3.10)}\qquad \int G_\varepsilon(x,v)\left(e^{-|u+v|^2\ell/2} - e^{-(|u|^2+|v|^2)\ell/2}\right)dv$$

$$= e^{-|u|^2\ell/2}\int e^{ixv}(e^{-uv\ell}-1)e^{-|v|^2(\ell+\varepsilon)/2}dv$$

$$= e^{-|u|^2\ell/2}\, e^{-|x|^2/2(\ell+\varepsilon)}\,\frac{(e^{-ixu(\frac{\ell}{\ell+\varepsilon}) + |u|^2/2(\frac{\ell^2}{\ell+\varepsilon})} - 1)}{\ell+\varepsilon}$$

$$= \frac{e^{-x^2/2(\ell+\varepsilon)}}{\ell+\varepsilon}\left(e^{-ixu(\frac{\ell}{\ell+\varepsilon}) - |u|^2/2(\frac{\ell\varepsilon}{\ell+\varepsilon})} - e^{-|u|^2\ell/2}\right)$$

$$= \frac{e^{-x^2/2(\ell+\varepsilon)}}{\ell+\varepsilon}\left[e^{-ixu(\frac{\ell}{\ell+\varepsilon})}\left(e^{-|u|^2/2(\frac{\ell\varepsilon}{\ell+\varepsilon})} - e^{-|u|^2\ell/2}\right)\right.$$

$$\left. + \left(e^{-ixu(\frac{\ell}{\ell+\varepsilon})} - 1\right)e^{-|u|^2\ell/2}\right]$$

$$= \frac{A(x,\varepsilon)}{\ell+\varepsilon}[B(x,\varepsilon)(C(\varepsilon) - e^{-|u|^2\ell/2}) + (B(x,\varepsilon)-1)e^{-|u|^2\ell/2}]$$

where

$$A(x,\varepsilon) = e^{-|x|^2/2(\ell+\varepsilon)}$$

$$B(x,\varepsilon) = e^{-ixu(\frac{\ell}{\ell+\varepsilon})}$$

$$C(\varepsilon) = e^{-|u|^2/2(\frac{\ell\varepsilon}{\ell+\varepsilon})} \quad .$$

We use the following bounds

$$\text{(3.11)}\qquad \left|A(x,\varepsilon)B(x,\varepsilon)\,\frac{(C(\varepsilon) - e^{-|u|^2\ell/2})}{\ell+\varepsilon}\right| \le \frac{(C(\varepsilon) - e^{-|u|^2\ell/2})}{\ell+\varepsilon}$$

$$\le \frac{(1 - e^{-|u|^2\ell/2})}{\ell} \le c|u|^{2\delta}\ell^{-1+\delta}$$

and

$$\text{(3.12)}\qquad \left|A(x,\varepsilon)\,\frac{(B(x,\varepsilon)-1)}{\ell+\varepsilon}\, e^{-|u|^2\ell/2}\right| \le A(x,\varepsilon)\,\frac{|x|^{2\delta}|u|^{2\delta}}{\ell+\varepsilon}$$

$$\le A(x,\varepsilon)\,\frac{|x|^{2\delta}}{(\ell+\varepsilon)^\delta}\,|u|^{2\delta}\ell^{-1+\delta} \le c|u|^{2\delta}\ell^{-1+\delta}$$

since

$$(3.13) \qquad A(x,\varepsilon)\frac{|x|^{2\delta}}{(\ell+\varepsilon)^{\delta}} \le \sup_{a\ge 0}\,[e^{-a/2}a^{\delta}] < \infty.$$

To summarize, an integral (3.10) over an isolated variable v_j is bounded by

$$c|u_j|^{2\delta}\ell_j^{-1+\delta}.$$

We now integrate out $dsdt$ to find (3.9) bounded by

$$(3.14) \qquad \int \prod_{I^c} (1+|u_j|^2)^{-1} \prod_{I} |u_i|^{2\delta} d\hat{p}$$

where $\hat{p}$ denotes again the non-isolated variables.

We note that in our present set-up every isolated interval is immediately preceded by a non-isolated interval. Thus (3.14) is bounded by

$$(3.15) \qquad \int \prod_{I^c} (1+|u_j|^2)^{-1+\gamma} d\hat{p}$$

where $\gamma = 2\delta$.

Each u_j, $u \in I^c$, is a sum of certain non-isolated variables, see (3.8), called the <u>components</u> of u_j.

Let

$$F = \{i \mid i \in I^c \text{ and } r_i = s_j \text{ for some } j\}\ .$$

Thus, for $i \in F$, some non-isolated p_j appears as a component of u_i for the first time, i.e. p_j is not a component of u_ℓ for any $\ell < i$. Since every non-isolated variable must appear for a first time, it is clear that $\{u_i\}_{i\in F}$ spans the set of non-isolated variables.

Let

$$D = I^c - F = \{i \mid i \in I^c \text{ and } r_i = t_j,\ \text{some } j\}\ .$$

Lemma 4 of Rosen [1983] uses a simple induction argument to show that the set of vectors $\{u_i\}_{i\in D}$ spans the set of all its components. This does not necessarily mean that $\{u_i\}_{i\in D}$ spans the set of all non-isolated variables. The trouble comes from a non-isolated p_j such that $[s_j,t_j]$ contains only points of the form s_ℓ, i.e. no t_k's, so that p_j will not appear as a component in any u_i, $i \in D$.

Let R denote the set of such indices j. Since p_j is non-isolated, there will be at least one s_ℓ between s_j and t_j - so that, by (3.8) p_j will appear as a component in at least two u_k's, $k \in F$. Pick two such, and denote them by v_j and w_j. Note that all components of v_j and w_j other than p_j appear in u_i, where $r_i = t_j$, so $i \in D$ and therefore each of $\{u_i\}_{i\in D} \cup v_j$ and $\{u_i\}_{i\in D} \cup w_j$ contain p_j in their span. Also, as a consequence of the above, distinct indices j in R give rise to distinct $\{v_j, w_j\}$.

We therefore have

$$(3.16) \qquad \prod_{I^c} (1 + |u_j|^2)^{-1} = \prod_F (1 + |u_j|^2)^{-1} \prod_D (1 + |u_j|^2)^{-1}$$

$$\leq \prod_F (1 + |u_j|^2)^{-5/8} \prod_R (1 + |v_j|^2)^{-3/8} (1 + |w_j|^2)^{-3/8}$$

$$\cdot \prod_D (1 + |u_j|^2)^{-1} .$$

Using Hölder's inequality we see that (3.15) squared is less then

$$(3.17) \qquad \int \prod_F (1 + |u_j|^2)^{-\frac{10}{8}(1-\gamma)} d\hat{p}$$

$$\int \prod_R [(1 + |v_j|^2)(1 + |w_j|^2)]^{-\frac{6}{8}(1-\gamma)} \prod_D (1 + |u_j|^2)^{-2+2\delta} d\hat{p} \quad .$$

The first integral in (3.17) is clearly bounded for

$$\frac{20}{8}(1 - \gamma) > 2, \quad \text{i.e.} \quad \gamma < \frac{1}{5}$$

while, using Hölder once more we find the second integral squared is bounded by

$$(3.18) \qquad \int \prod_R (1 + |v_j|^2)^{-\frac{12}{8}(1-\gamma)} \prod_D (1 + |u_j|^2)^{-2+2\delta} d\hat{p}$$

times a similar integral with v_j replaced by w_j. Since our previous considerations show that each of

$$\{u_i\}_{i\in D} \cup \{v_j\}_{j\in R}$$

and

$$\{u_i\}_{i\in D} \cup \{w_j\}_{j\in R}$$

span the set of non-isolated variables, (3.18) is also finite if $\gamma < \frac{1}{5}$.

This completes our proof that the term (3.7) is uniformly bounded, and as in Theorem 1 all other terms can be handled similarly. Thus $\mathbf{E}(I_\varepsilon(x,T)^m)$ is uniformly bounded.

To establish (3.1) we use

$$\mathbf{E}(I_\varepsilon(x,T) - I_{\varepsilon'}(x',T'))^m \le c[\mathbf{E}(I_\varepsilon(x,T) - I_\varepsilon(x',T))^m + \mathbf{E}(I_\varepsilon(x',T) - I_{\varepsilon'}(x',T))^m + \mathbf{E}(I_{\varepsilon'}(x',T) - I_{\varepsilon'}(x',T'))^m] \tag{3.19}$$

and will bound each term separately.

Consider first the term

$$\mathbf{E}(I_\varepsilon(x,T) - I_\varepsilon(x',T))^m \tag{3.20}$$

which is similar to (3.4), except that in $G_\varepsilon(x,p)$, e^{ip_jx} is replaced by $e^{ip_jx} - e^{ip_jx'}$.

For each non-isolated variable we use the bound

$$|e^{ipx} - e^{ipx'}| \le c|p|^\delta|x - x'|^\delta \tag{3.21}$$

while for isolated variables v, using (3.10) we need to bound

$$(A(x,\varepsilon)B(x,\varepsilon) - A(x',\varepsilon)B(x',\varepsilon))\,\frac{c(\varepsilon) - e^{-|u|^2\ell/2}}{\ell + \varepsilon} + A(x,\varepsilon)\frac{(B(x,\varepsilon) - 1)}{\ell + \varepsilon} - A(x',\varepsilon)\frac{(B(x',\varepsilon) - 1))}{\ell + \varepsilon}\,e^{-|u|^2\ell/2} \quad . \tag{3.22}$$

By (3.11),

$$\left|\frac{c(\varepsilon) - e^{-|u|^2\ell/2}}{\ell + \varepsilon}\right| \le c|u|^{2\delta}\ell^{-1+\delta} \ . \tag{3.23}$$

So the first term in (3.22) is bounded by

$$\begin{aligned} &|A(x,\varepsilon) - A(x',\varepsilon)||B(x,\varepsilon)||u|^{2\delta}\ell^{-1+\delta} + A(x',\varepsilon)|B(x,\varepsilon) - B(x',\varepsilon)||u|^{2\delta}\ell^{-1+\delta} \\ &\le c(|x - x'|^\beta|u|^{2\delta}\ell^{-1+\delta-\beta} + |x - x'|^\beta|u|^{2\delta+\beta}\ell^{-1+\delta}) \end{aligned} \tag{3.24}$$

while the second term, if $|x'| \ge |x|$ we write as

$$\begin{aligned} &(A(x,\varepsilon) - A(x',\varepsilon))\,\frac{(B(x,\varepsilon) - 1)}{\ell + \varepsilon} - A(x',\varepsilon)\,\frac{(B(x',\varepsilon) - B(x,\varepsilon))}{\ell + \varepsilon} \\ &\le \left|1 - e^{-(|x'|^2-|x|^2)/2(\ell+\varepsilon)}\right| e^{-|x|^2/2(\ell+\varepsilon)}\,\frac{|u|^\delta|x|^\delta}{\ell + \varepsilon} \\ &\quad + e^{-|x'|^2/2(\ell+\varepsilon)}\,\frac{|u|^\delta|x-x'|^\delta}{\ell + \varepsilon} \\ &\le c|u|^\delta(|x - x'|^\beta\ell^{-1+\delta-\beta} + |x - x'|^{\delta-\beta}\ell^{-1+\beta}) \quad , \end{aligned} \tag{3.25}$$

using (3.13) and $|x - x'| \leq 2|x'|$.

If $|x| \leq |x'|$ we proceed similarly. These suffice to show that (3.20) is bounded $\leq c|x - x'|^{\alpha m}$ for some $\alpha > 0$.

We next turn to

$$(3.26) \qquad \mathbf{E}(I_\varepsilon(x,T) - I_{\varepsilon'}(x,T))^m$$

which is similar to (3.4) except that in $G_\varepsilon(x,p)$, $e^{-\varepsilon|p_j|^2/2}$ is replaced $e^{-\varepsilon|p_j|^2/2} - e^{-\varepsilon'|p_j|^2/2}$.

For non-isolated variables we use the bound (1.20) while for isolated variables we need to bound the difference of (3.10) and a similar expression with ε replaced by ε'.

Bound first

$$(3.27) \qquad \left|A(x,\varepsilon)B(x,\varepsilon)\frac{(C(\varepsilon) - e^{-|u|^2\ell/2})}{\ell + \varepsilon} - A(x,\varepsilon')B(x,\varepsilon')\frac{(C(\varepsilon') - e^{-|u|^2\ell/2})}{\ell + \varepsilon}\right|$$
$$\leq \left|\frac{(C(\varepsilon) - e^{-|u|^2\ell/2})}{\ell + \varepsilon} - \frac{(C(\varepsilon') - e^{-|u|^2\ell/2})}{\ell + \varepsilon'}\right|$$
$$+ |A(x,\varepsilon)B(x,\varepsilon) - A(x,\varepsilon')B(x,\varepsilon')|c|u|^{2\delta}\ell^{-1+\delta}$$

by (3.23). The first term in (3.27) is handled by (1.22) while the second is bounded by

$$(3.28) \qquad |B(x,\varepsilon) - B(x,\varepsilon')||u|^{2\delta}\ell^{-1+\delta} + |A(x,\varepsilon) - A(x,\varepsilon')||u|^{2\delta}\ell^{-1+\delta}$$
$$\leq |u|^{2\delta+\beta}|x|^\beta\left|\frac{\ell}{\ell+\varepsilon} - \frac{\ell}{\ell+\varepsilon'}\right|^\beta \ell^{-1+\delta} + |u|^{2\delta}|x|^{2\beta}\left|\frac{1}{\ell+\varepsilon} - \frac{1}{\ell+\varepsilon'}\right|^\beta \ell^{-1+\delta}$$
$$\leq |u|^{2\delta+\beta}|\varepsilon - \varepsilon'|^\beta \ell^{-1+\delta-\beta} + |u|^{2\delta}|\varepsilon - \varepsilon'|^\beta \ell^{-1+\delta-2\beta}.$$

We are left with bounding

$$(3.29) \qquad \left|A(x,\varepsilon)\frac{(B(x,\varepsilon) - 1)}{\ell + \varepsilon} - A(x,\varepsilon')\frac{(B(x,\varepsilon') - 1)}{\ell + \varepsilon'}\right|$$

if say, $\varepsilon < \varepsilon'$, we bound this by

$$(3.30) \qquad \left|(A(x,\varepsilon) - A(x,\varepsilon'))\frac{(B(x,\varepsilon') - 1)}{\ell + \varepsilon'} + \cdot A(x,\varepsilon)\right|\left[\frac{(B(x,\varepsilon) - 1)}{\ell + \varepsilon} - \frac{(B(x,\varepsilon') - 1)}{\ell + \varepsilon'}\right]\Bigg|.$$

The first term is bounded by

$$(3.31) \qquad \left(1 - e^{-\frac{|x|^2}{2}\left(\frac{1}{\ell+\varepsilon} - \frac{1}{\ell+\varepsilon'}\right)}\right)A(x,\varepsilon')\frac{(B(x,\varepsilon') - 1)}{\ell + \varepsilon'}$$
$$\leq |x|^{2\alpha}(\varepsilon - \varepsilon)^\alpha|u|^{2\delta}\ell^{-1+\delta-2\alpha},$$

by (3.12) while the second is bounded by

$$(3.32)\qquad \left|A(x,\varepsilon)(B(x,\varepsilon)-1)\left(\frac{1}{\ell+\varepsilon}-\frac{1}{\ell+\varepsilon'}\right)\right| + A(x,\varepsilon)\left|\frac{B(x,\varepsilon)-B(x,\varepsilon')}{\ell+\varepsilon'}\right|$$

$$\leq A(x,\varepsilon)\frac{(B(x,\varepsilon)-1)}{\ell+\varepsilon}\frac{(\varepsilon'-\varepsilon)}{\ell+\varepsilon'} + A(x,\varepsilon)\frac{|x|^{2\delta}}{(\ell+\varepsilon)^{\delta}}|u|^{2\delta}\ell^{-1+\delta}\left(\frac{\varepsilon'-\varepsilon}{\ell+\varepsilon'}\right)^{2\delta}$$

$$\leq c|u|^{2\delta}(\varepsilon'-\varepsilon)^{\alpha}\ell^{-1+\delta-\alpha}, \quad \text{since} \quad \varepsilon' > \varepsilon.$$

This completes the proof that (3.26) is less than $c|\varepsilon - \varepsilon'|^{\alpha m}$ for some $\alpha > 0$.

We turn to

$$(3.33)\qquad \mathbf{E}(I_{\varepsilon}(x,T) - I_{\varepsilon}(x,T'))^{m},$$

which, assuming $T' > T$ is of the same form as (3.4) except that B is replaced by

$$B_{T,T'} = \{(s,t)\,|\,0 \leq s \leq t, T \leq t \leq T'\}^{m}.$$

It clearly suffices to show that an integral of the form (3.7) is bounded by $c\,\mathrm{Vol}(A)^{\delta}$ for some $\delta > 0$.

To this end, we first integrate all isolated variables, as before, then use the bound

$$(3.34)\qquad \int\left(\int_A F(\hat{p},r)\,ds\,dt\right)d\hat{p} \leq \mathrm{Vol}(A)^{\delta}\int\left(\int_A F(\hat{p},r)^{\frac{1}{1-\delta}}ds\,dt\right)^{1-\delta}d\hat{p}\ .$$

It is clear from our considerations so far, that for $\delta > 0$ sufficiently small this integral converges.

This completes the proof of Theorem 2.

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