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TANAKA'S FORMULA FOR MULTIPLE INTERSECTIONS OF PLANAR BROWNIAN MOTION

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We establish a Tanaka-like formula relating the local times of r and $r+1$ fold self-intersections of a Brownian path in the plane.

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1. Introduction

It is well known that, for any r, planar Brownian motion W_t has r-multiple points, i.e. points $x\in\mathbb{R}^2$ with $x=W_{t_1}=W_{t_2}=\cdots=W_{t_r}$ for distinct $t_1,\ldots,t_r$. (Dvoretzky, Erdös and Kakutani, 1954). As a purely formal measure of such r-fold intersections we study

$$\int\cdots_B\int \delta(W_{t_2}-W_{t_1})\cdots\delta(W_{t_r}-W_{t_{r-1}})\,\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.1}$$

In a previous paper (Rosen, 1984) we showed how to interpret (1.1) as the local time of the random field

$$X_r(t_1,\ldots,t_r)=(W_{t_2}-W_{t_1},\ldots,W_{t_r}-W_{t_{r-1}}). \tag{1.2}$$

Recall that $X:R_+^r\to R^{2(r-1)}$ induces a measure $\mu_B(\cdot)$ on $\mathbb{R}^{2(r-1)}$, the occupation measure of X on $B\subseteq R^r$ defined by

$$\mu_B(A)=\lambda_r(X^{-1}(A)\cap B) \tag{1.3}$$

where λ_r denote Lesbesgue measure on $\mathbb{R}^r$. If $\mu_B\ll\lambda_{2(r-1)}$ we say that X has a local time α_r relative to B, defined by

$$\alpha_r(x,B)=\frac{\mathrm{d}\mu_B}{\mathrm{d}\lambda_{2(r-1)}}(x). \tag{1.4}$$

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By definition we have

$$\int_{\mathbb{R}^{2(r-1)}} f(x)\alpha_r(x, B)\, \mathrm{d}\lambda(x)_{2(4-1)}$$

$$= \int_B \cdots \int f(W_{t_2} - W_{t_1}, \cdots, W_{t_r} - W_{t_{r-1}})\, \mathrm{d}t_{t_1} \cdots \mathrm{d}_{t_r} \tag{1.5}$$

for all bounded Borel functions on $\mathbb{R}^{2(r-1)}$. If we formally take f to be the 'δ-function', we find that $\alpha_r(0, B)$ should give (1.1).

In (Rosen, 1984) we showed that if B is a product of disjoint intervals, $B = \times_{i=1}^{r} [a_i, b_i]$, X has a local time relative to B, and if we write $\alpha_r(x, a_1, b_1, \ldots, a_r, b_r) \doteq (x, B)$ then α_r can be taken to be a jointly continuous function of its arguments $(x, a_1, b_1, \ldots, a_r, b_r)$. We will sometimes write $I_j = [a_j, b_j]$ and will always assume that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{r+1} < b_{r+1}. \tag{1.6}$$

Properties of a measure analogous to (1.1) for r-fold intersections of independent planar Brownian motions have been established in (German, Horowitz and Rosen, 1984) and applied by Le Gall to study intersections of Wiener Sausages (1985) and derive important information on the Hausdorff measure of r-multiple points.

The goal of this paper is to present an explicit formula for α_{r+1} in terms of α_r analogous to Tanaka's formula for the local time of one dimensional Brownian motion, and to our own formula for double points (Rosen, 1985). Our formula for double points has been analyzed and extended in (Yor, 1985), and applied in (Rosen, 1986) and (Yor, 1985) to study the asymptotics of α_2.

To describe our formula, let

$$q_t(x) = \mathrm{e}^{-t}\, \mathrm{e}^{-|x|^2/2t}/2\pi t \tag{1.7}$$

be the transition density for killed planar Brownian motion, and set

$$K(x) = \int_0^\infty q_t(x)\, \mathrm{d}t. \tag{1.8}$$

It will be seen that $\alpha_r(x, a_1, b_1, \ldots, a_r, s)$ is a continuous increasing function of s, and since K is positive and measurable the integral

$$K_r(x) = \int_{I_r} K(x - W_s)\alpha(0, a_1, b_1, \ldots, a_r, \mathrm{d}s) \tag{1.9}$$

is well defined, although a priori it may be infinite. We will show below that K_r is, in fact, continuously differentiable, and we can state Tanaka's formula for α_{r+1}:

Theorem 1. *With probability one,*

$$-\alpha_{r+1}(0, a_1, \ldots, a_{r+1}, b_{r+1}) = K_r(W_{b_{r+1}}) - K_r(W_{a_{r+1}})$$

$$- \int_{I_{r+1}} \nabla K_r(W_t)\, \mathrm{d}W_t - \int_{I_{r+1}} K_r(W_t)\, \mathrm{d}t. \tag{1.10}$$

Remark 1. What makes this work is the identity

$$\left(-\frac{\Delta}{2}+1\right) * K = \delta.$$

2. It is easy to modify (1.10) to obtain a formula for $\alpha_{r+1}(x, \cdot)$.

3. $\alpha_r(0, a_1, \ldots, a_r, \mathrm{d}s)$ induces a measure ν, supported on the r-multiple points in $\mathbb{R}^2$, via the formula

$$\int_{R^2} g(x)\,\mathrm{d}\nu(x) = \int_{I_r} g(W_s)\alpha_r(0, a_1, \ldots, a_r, \mathrm{d}s).$$

K_r is just the 1-potential of ν.

If we abolish condition (1.6) and take all I_j equal, the resulting local time $\alpha_r(x)$ will be discontinuous at $x = 0$.

As alluded to above, Rosen (1986) and Yor (1985) apply Tanaka's formula for α_2 to determine the nature of the singularity of $\alpha_2(x)$ at $x = 0$, a problem also discussed in Varadhan (1969), Dynkin (1985) and Le Gall (1985). The formula of this paper has recently been applied by M. Yor (1985c) to analyze the singularity of $\alpha_3(x)$ as $x \to 0$.

An important pedagogical contribution of this paper is a new and greatly simplified proof of the joint continuity of α_r. There already exist a series of proofs for the joint continuity of loal times of random fields (Pitt, 1978; Geman and Horowitz, 1980; and Geman, Horwitz and Rosen, 1984). These proofs are long, making them difficult to follow and unfortunately allowing errors to creep in. Our proof, stated for α_r but easily extendable to other random fields, has the virtue of simplicity. It also yields stronger results. The weak convergence used in (3.8), a key step in our proof of Theorem 1, seems unobtainable with previous proofs. Our new proof owes much to a conversation with J. Cuzick.

I would also like to thank Joseph Horowitz for many helpful remarks.

2. Joint continuity of the local time α_r

For any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{r-1}) \in (0, 1]^{r-1}$, $x = (x_1, \ldots, x_{r-1}) \in \mathbb{R}^{2(r-1)}$,

$$e \in E_\gamma \doteq \{(a_1, b_1, \ldots, a_r, b_r) \,|\, a_{i+1} - b_i \geq \gamma, b_r \leq 1\},$$

let us define

$$\alpha_r^\varepsilon(x, e) = \int_{I_1} \cdots \int_{I_r} q_{\varepsilon_1}(x_1 - W_{t_2} + W_{t_1}) \cdots q_{\varepsilon_{r-1}}(x_{r-1} - W_{t_r} + W_{t_{r-1}})\,\mathrm{d}t_1 \cdots \mathrm{d}t_r. \tag{2.1}$$

Lemma 1. *For any m even, $\alpha < 1/2(r-1)$, we have*

$$\mathbb{E}((\alpha_r^\varepsilon(x, e) - \alpha_r^{\varepsilon'}(x', e'))^m) \leq c_{m,\alpha} |(\varepsilon, x, e) - (\varepsilon', x', e')|^{m\alpha} \tag{2.2}$$

for all ε, $\varepsilon' \in (0, 1]^{r-1}$, e, $e' \in E_\gamma$, x, $x' \in R^{2(r-1)}$.

Using (2.2), the multiparameter version of Kolmogorov's lemma (Meyer, 1979/80) shows that, for $\alpha < 1/2(r-1)$,

$$|\alpha_r^\varepsilon(x, e) - \alpha_r^{\varepsilon'}(x', e')| \leq c_\omega |(\varepsilon, x, e) - (\varepsilon', x', e')|^\alpha, \tag{2.3}$$

first for all *rational* $\varepsilon, \varepsilon' \in (0, 1]^{r-1}$, $e, e' \in E_\gamma$, x, x' in a bounded set (say $\{y \mid \|y\| \leq n\}$), but then for *all* $\varepsilon, \varepsilon', e, e', x, x'$ in the above sets, since, by (2.1), $\alpha_r^\varepsilon(x, e)$ is clearly a continuous function of its arguments.

Now take $x' = x$, $e' = e$ in (2.3) to conclude the existence of the limit

$$\alpha_r(x, e) \doteq \lim_{\varepsilon \to 0} \alpha_r^\varepsilon(x, e). \tag{2.4}$$

Again by (2.3) the convergence is locally uniform:

$$|\alpha_r^\varepsilon(x, e) - \alpha(x, e)| \leq c_\omega |\varepsilon|^\alpha \tag{2.5}$$

and the limit is continuous

$$|\alpha_r(x, e) - \alpha_r(x', e')| \leq c_\omega |(x, e) - x', e')|^\alpha. \tag{2.6}$$

We now identify this α_r with local time: Let $f(x)$ be a continuous function of compact support on $\mathbb{R}^{2(r-1)}$.

$$\int f(x)\alpha_r(x, e)\, \mathrm{d}\lambda_{2(r-1)}(x) = \lim_{\varepsilon \to 0} \int f(x)\alpha_r^\varepsilon(x, e)\, \mathrm{d}\lambda_{2(r-1)}(x) \tag{2.7}$$

by the local uniform convergence (2.5), while from (2.1)

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int f(x)\alpha^\varepsilon(x, e)\, \mathrm{d}\lambda_{2(r-1)}(x) \\ &= \lim_{\varepsilon \to 0} \int_{I_1} \cdots \int_{I_r} f * q_\varepsilon(W_{t_2} - W_{t_1}, \cdots, W_{t_r} - W_{t_{r-1}})\, \mathrm{d}r_1, \cdots \mathrm{d}t_r \\ &= \int_{I_1} \cdots \int_{I_r} f(W_{t_2} - W_{t_1}, \cdots, W_{t_r} - W_{t_{a1}})\, \mathrm{d}t_1 \ldots \mathrm{d}t_r \end{aligned} \tag{2.8}$$

since $f * q_\varepsilon \to f$ uniformly. Together with (2.7) this establishes the jointly continuous $\alpha_r(x, e)$ as the local time (1.4) for $B = \bigtimes_{i=1}^r [a_i, b_i]$.

Lemma 1 is proven in Section 4.

3. Tanaka's Formula for α_{r+1}

Set

$$K^\varepsilon(x) = \int_\varepsilon^\infty q_t(x)\, \mathrm{d}t. \tag{3.1}$$

K^ε is a positive C^∞ function, which decreases exponentially as $|x| \to \infty$. We have

$$K^\varepsilon(x) \uparrow K(x) \quad \text{for all } x, \text{ as } \varepsilon \downarrow 0. \tag{3.2}$$

In analogy with (1.9) let

$$K_r^\varepsilon(x)=\int_{I_r} K^\varepsilon(x-W_s)\alpha_r(0,a_1,\ldots,a_r\,\mathrm{d}s). \tag{3.3}$$

In Section 4 we will show

Lemma 2. *For all m even and* $\alpha<1$

$$\mathbb{E}((K_r^\varepsilon(x)-K_r^{\varepsilon'}(x'))^m)\leq c_m|(\varepsilon,x)-(\varepsilon',x')|^{\alpha m}. \tag{3.4}$$

As in Section 2 this implies the existence and continuity of the limit $\lim_{\varepsilon\to 0} K_r^\varepsilon(x)$ — but by (3.2) and the monotone convergence theorem this limit is $K_r(x)$ of (1.9).

We need yet another lemma.

Lemma 3. *For all m even,* $\alpha<\frac{1}{2}$,

$$\mathbb{E}((\nabla K_r^\varepsilon(x)-\nabla K_r^{\varepsilon'}(x'))^m)\leq c_m|(x,\varepsilon)-(x',\varepsilon')|^{\alpha m} \tag{3.5}$$

$$\mathbb{E}((K_r^\varepsilon(y)-K_r^\varepsilon(x)-(y-x)\cdot\nabla K_r^\varepsilon(x))^m)\leq c_m|x-y|^{3m/2}. \tag{3.6}$$

As before, (3.5) shows the existence and continuity of the limit $\lim_{\varepsilon\to 0}\nabla K_r^\varepsilon(x)$, while with (3.6) after first taking the $\varepsilon\to 0$ limit (uniform integrability) then a là Kolmogorov we can identity $\lim_{\varepsilon\to 0}\nabla K_r(x)$ with $\nabla(K_r(x))$. This establishes

Proposition 4. *$K_r(x)$ is a continuously differentiable function of x, with probability one.*

Now apply Itô's formula to the C^∞ function of x, $K_r^\varepsilon(x)$.

$$\begin{aligned}K_r^\varepsilon(W_{b_{r+1}})-K_r^\varepsilon(W_{a_{r+1}})-\int_{I_{r+1}}\nabla K_r^\varepsilon(W_t)\cdot\mathrm{d}W_t-\int_{I_{r+1}}K_r^\varepsilon(W_t)\,\mathrm{d}t\\ =-\int_{I_{r+1}}\left(-\frac{\Delta}{2}+1\right)K_r^\varepsilon(W_t)\,\mathrm{d}t\\ =-\int_{I_{r+1}}\int_{I_r}q_\varepsilon(W_t-W_s)\alpha_r(0,a_1,\ldots,a_r\,\mathrm{d}s)\end{aligned} \tag{3.7}$$

since $(-\Delta/2+1)K^\varepsilon(x)=q_\varepsilon(x)$.

Because of (2.5) and (2.6) we have weak convergence of the measures

$$\alpha_r^{(\varepsilon_1,\ldots,\varepsilon_{r-1})}(0,a_1,\ldots,a_r\,\mathrm{d}s)\to\alpha_r(0,a_1,\ldots,a_r\,\mathrm{d}s)$$

on I_r. Hence

$$\int_{I_{r+1}}\int_{I_r} q_\varepsilon(W_t-W_s)\alpha_r(0,a_1,\ldots,a_r,\mathrm{d}s)=$$
$$=\lim_{(\varepsilon_1,\ldots,\varepsilon_{r-1})\to 0}\int_{I_{r+1}}\int_{I_r} q_\varepsilon(W_t-W_s)\alpha_r^{(\varepsilon_1,\ldots,\varepsilon_{r-1})}(0,a_1,\ldots,a_r,\mathrm{d}s)$$
$$=\lim_{(\varepsilon_1,\ldots,\varepsilon_{r-1})\to 0}\int_{I_1}\cdots\int_{I_{r+1}} q_{\varepsilon_1}(W_{t_2}-W_{t_1})\cdots$$
$$\cdots q_{\varepsilon_{r-1}}(W_{t_r}-W_{t_{r-1}})q_\varepsilon(W_{t_{r+1}}-W_{t_r})\,\mathrm{d}t_1\cdots\mathrm{d}t_{r+1} \tag{3.8}$$

by (2.1).

(3.8) and (2.4) now show

$$\lim_{\varepsilon\to 0}\int_{I_{r+1}}\int_{I_r} q_\varepsilon(W_t-W_s)\alpha_r(0,a_1,\ldots,a_r,\mathrm{d}s)$$
$$=\alpha_{r+1}(0,a_1,\ldots,a_r,b_r,a_{r+1},b_{r+1}). \tag{3.9}$$

Since the earlier considerations of this section show that $K_r^\varepsilon(x)\to K_r(x)$ uniformly on compacts, taking the limit $\varepsilon\to 0$ in (3.7), together with the following lemma will complete the proof of Theorem 1.

Lemma 5. *For some subsequence $\varepsilon_n\to 0$,*

$$\int_{I_{r+1}}\nabla K_r^{\varepsilon_n}(W_t)\cdot\mathrm{d}W_t\to\int_{I_{r+1}}\nabla K_r(W_t)\cdot\mathrm{d}W_t\quad a.s. \tag{3.10}$$

4. Estimates

Proof of Lemma 1. We have

$$\mathbb{E}((\alpha_r^\varepsilon(x,e)-\alpha_r^{\varepsilon'}(x',e'))^m)$$
$$\leqslant c\mathbb{E}(\alpha_r^\varepsilon(x,e)-\alpha_r^\varepsilon(x',e))^m$$
$$+\mathbb{E}(\alpha_r^\varepsilon(x',e)-\alpha_r^{\varepsilon'}(x',e))^m+\mathbb{E}(\alpha_r^{\varepsilon'}(x',e)-\alpha_r^{\varepsilon'}(x',e'))^m. \tag{4.1}$$

We will bound each of these three terms separately. By (2.1),

$$\alpha_r^\varepsilon(x,e)=\int_{I_1\times\cdots\times I_r}\int_{R^{2(r-1)}}\prod_{j=1}^{r-1} e^{\mathrm{i}p_j\cdot(x_j-W_{t_{j+1}}+W_{t_j})}\,\mathrm{e}^{-\varepsilon_j(|p_j|^2+1)/2}\,\mathrm{d}p\,\mathrm{d}t \tag{4.2}$$

so that

$$\mathbb{E}((\alpha_r^\varepsilon(x,e)-\alpha_r^\varepsilon(x',e))^m$$
$$=\int_{R^{2(r-1)m}}\prod_{l=1}^m(\mathrm{e}^{\mathrm{i}p^l\cdot x}-\mathrm{e}^{\mathrm{i}p^l\cdot x'})\,\mathrm{e}^{-\sum_{j=1}^{r-1}(|p_j^l|^2+1)\varepsilon_j/2}$$
$$\times\int_{(I_1\times\cdots\times I_r)^m}\mathbb{E}(\mathrm{e}^{-\mathrm{i}\sum_{l=1}^m\sum_{j=1}^{r-1}p_j^l\cdot(W_{t_{j+1}}^l-W_{t_j}^l)})\,\mathrm{d}t\,\mathrm{d}p. \tag{4.3}$$

Of course

$$\mathbb{E}\left(e^{-i\sum_{l=1}^{m}\sum_{j=1}^{r-1}p_j^l\cdot(W^l_{t_{j+1}}-W^l_{t_j})}\right)=e^{-\mathbb{V}(\sum_{l=1}^{m}\sum_{j=1}^{r-1}p_j^l\cdot(W^l_{t_{j+1}}-W^l_{t_j}))/2} \tag{4.4}$$

Let $\pi', \ldots, \pi^r$ be r not necessarily distinct permutations of $\{1, \ldots, m\}$ and let

$$\Delta(\pi^1, \ldots, \pi^r) = [\{t_j^l\} \in (I_1 \times \cdots \times I_r)^m \mid t_j^{\pi^j(l)} \leqq t_j^{\pi^j(l+1)}] \tag{4.5}$$

where we define $t_j^{\pi^j(m+1)} = t_{j+1}^{\pi^{j+1}(1)}$. On $\Delta(\pi^1, \ldots, \pi^r)$, using the fact that Brownian motion has independent increments we have

$$\mathbb{V}\left(\sum_{l=1}^{m}\sum_{j=1}^{r-1} p_j^l\cdot(W_{t_{j+1}}^l - W_{t_j}^l)\right) = \sum_{j=1}^{r}\sum_{l=1}^{m} |u_j^l|^2 \bar{t}_j^l \tag{4.6}$$

where

$$u_j^l = \sum_{k\leq l} p_j^{\pi^j(k)} + \sum_{k>l} p_{j-1}^{\pi^j(k)} \quad (p_0' \doteq 0), \tag{4.7}$$

$$\bar{t}_j^l = t_j^{\pi^j(l+1)} - t_j^{\pi^j(l)}. \tag{4.8}$$

We now integrate out the $\bar{t}_j^l$ in (4.3), over $\Delta(\pi^1, \ldots, \pi^r)$, and use the bounds

$$\int_0^1 e^{-tv^2}\,dt \leq c(1+v^2)^{-1}, \qquad \int_\gamma^1 e^{-tv^2}\,dt \leq c(1+v^2)^{-2} \quad (\text{for } \bar{t}_j^m \geq \gamma)$$

to obtain

$$\int_{\Delta(\pi',\ldots,\pi^r)} e^{-\sum_{j=1}^{r}\sum_{l=1}^{m}|u_j^l|^2\bar{t}_j^l/2}\,dt \leq c\prod_{j=1}^{r} A_j(p) \tag{4.9}$$

where

$$A_j(p) \doteq \prod_{l=0}^{m}(1+|u_j^l|^2)^{-1} \quad (u_j^0 = u_{j-1}^m).$$

Using the generalized Hölder's inequality we have

$$\int_{\mathbb{R}^{2(-1)m}}\prod_{j=1}^{r} A_j(p)\,dp = \int_{R^{2(r-1)m}}\prod_{i=1}^{r}\left(\prod_{j\neq i} A_j^{1/r-1}(p)\right)dp$$

$$\leq \prod_{i=1}^{r}\left(\int_{R^{2(r-1)m}}\prod_{j\neq i} A_j^{r/r-1}(p)\right)^{1/r} < \infty \tag{4.10}$$

since, by (4.7), for each i,

$$\{u_j^l \mid 0 \leq l \leq m, j \neq i\}$$

form a linear set of coordinates for $R^{2(4-1)m}$.

Returning to (4.3), we use

$$|e^{ip\cdot x} - e^{ip\cdot x'}| \leq c(|p\|x-x'|)^\alpha \quad \text{for } \alpha < 1 \tag{4.11}$$

and find

$$\mathbb{E}(\alpha_r^\varepsilon(x, e) - \alpha_r^\varepsilon(x', e))^m$$

$$\leq c|x-x'|^{m\alpha}\sum_{\pi',\ldots,\pi^r}\int_{R^{2(r-1)m}}\left(\prod_{l=1}^{m}|p^l|^\alpha\right)\prod_{j=1}^{r} A_j(p)\,dp. \tag{4.12}$$

Since for any i

$$|p^l| \leq c \sum_{j\neq i} |p_j^l - p_{j-1}^l| \leq c \prod_{j\neq i} (1+|p_j^l - p_{j-1}^l|) \leq c \prod_{j\neq i} (1+|u_j^{\bar\pi^j(l)} - u_j^{\bar\pi^j(l)-1}|), \tag{4.13}$$

the latter by (4.7) where $\bar\pi^j = (\pi^j)^{-1}$, we have

$$\prod_{l=1}^m |p^l| \leq c \prod_{i=1}^r \left(\prod_{l=1}^m \prod_{j\neq i} (1+|u_j^l|^2) \right)^{1/r}. \tag{4.14}$$

Now (4.12) is bounded as was (4.10), if $2r/(r-1)-2\alpha>2$, i.e. if $\alpha<1/(r-1)$. The second expectation in (4.1) is bounded similarly using

$$|e^{-|p^l|^2\cdot\varepsilon} - e^{-|p^l|^2\cdot\varepsilon'}| \leq c|p^l|^{2\alpha}|\varepsilon-\varepsilon'|^\alpha \tag{4.15}$$

instead of (4.11), if $\alpha<1/(2(r-1))$.

Finally, to control the third term in (4.1), let $I_1\times\cdots\times I_r$ correspond to e, $I_1'\times\cdots\times I_r'$ to e', and let S denote the symmetric difference of these two sets. In the analogue of (4.3) we must integrate over S^m. Recall (4.9):

$$\begin{aligned}\int_{\Delta(\pi_1',\ldots,\pi^r)\cap S^m} & e^{-\sum_{j=1}^r \sum_{l=1}^m |u_j^l|^2 \bar t_j^l/2}\, dt \\ &\leq |S|^{m/q}\left(\int_{\Delta(\pi',\ldots,\pi^r)} e^{-q'\sum_{j=1}^r \sum_{l=1}^m |u_j^l|^2 \bar t_j^l/2}\, dt\right)^{1/q'} \\ &\leq c|S|^{m/q} \prod_{j=1}^r A_j^{1/q'}(p)\end{aligned} \tag{4.16}$$

for any q, q' with $1/q+1/q'=1$. As before this is integrable in p if

$$\frac{2r}{r-1}\cdot 1/q'>2, \quad \text{i.e. } \frac{1}{q'}>\frac{r-1}{r}, \quad \text{i.e. } \frac{1}{q}=1-\frac{1}{q'}<1-\frac{r-1}{r}=\frac{1}{r}.$$

Thus the third term in (4.1) is bounded by $|e-e'|^{m\alpha}$ if $\alpha<1/r$. This completes the proof of Lemma 1.

Proof of Lemma 2. In light of (2.5) and (2.6), $\alpha_r^\delta(0, a_1, \ldots, a_r, ds)$ converges to $\alpha_r(0, a_1, \ldots, a_r, ds)$ as measures on I_r, hence

$$K_r^{\varepsilon,\delta}(x) = \int_{I_r} K^\varepsilon(x-W_s)\alpha_r^\delta(0, a_1, \ldots, a_r, ds) \tag{4.17}$$

converges to $K_r^\varepsilon(x)$ as $\delta\to 0$ for almost all paths. In fact this convergence takes place in all L^m by uniform integrability: K^ε is bounded, and by (2.2) the $\alpha_r^\delta(0, a_1, \ldots, a_r, b_r)$ are uniformly bounded in any L^p norm. Hence it suffices to show

$$\mathbb{E}(K_r^{\varepsilon,\delta}(x) - K_r^{\varepsilon',\delta}(x'))^m \leq c|(\varepsilon, x)-(\varepsilon', x')|^{\alpha m} \tag{4.18}$$

with c independent of δ.

By definition

$$K_r^{\varepsilon,\delta}(x)=\int_{I_r} K^\varepsilon(x-W_s)\int_{I_1}\cdots\int_{I_{r-1}} q_\delta(W_{t2}-W_{t1})\cdots q_\delta(W_{t_{r-1}}-W_s)\,\mathrm{d}t_1\ldots\mathrm{d}t_r\,\mathrm{d}s. \tag{4.19}$$

Since

$$K^\varepsilon(x)=\int \mathrm{e}^{-\mathrm{i}p\cdot x}\left(\frac{\mathrm{e}^{-\varepsilon(|p|^2+1)}}{|p|^2+1}\right)\mathrm{d}\lambda_2(p),$$

we can write this as

$$K_r^{\varepsilon,\delta}(x)=\int_{I_1\times\cdots\times I_r}\int_{\mathbb{R}^{2r}} H(p,x)(|p_r|^2+1)^{-1} \times \mathrm{e}^{\mathrm{i}\sum_{j=1}^{r-1}p_j(W_{t_{j+1}}-W_{t_j})+\mathrm{i}p_r\cdot W_{t_r}}\,\mathrm{d}p\,\mathrm{d}t \tag{4.20}$$

where

$$H(p,x)=\mathrm{e}^{-\mathrm{i}p_r\cdot x}\,\mathrm{e}^{-\varepsilon(|p_r|^2+1)}\,\mathrm{e}^{-\delta\sum_{j=1}^{r-1}(|p_j^2|+1)}.$$

Using the change of variables $q_0=p_r$, $q_j=p_j+p_r$, we have

$$\mathbb{E}(K_r^{\varepsilon,\delta}(x))^m=\int_{(I_1\times\cdots\times I_r)^m}\int_{R^{2mr}}\left(\prod_{l=1}^m H(p^l,x)(|q_0^l|^2+1)^{-1}\right) \times\mathbb{E}(\mathrm{e}^{\mathrm{i}\sum_{j=0}^{r-1}\sum_{l=1}^m q_j^l(W_{t_{j+1}}-W_{t_j})})\,\mathrm{d}t\,\mathrm{d}p$$

where $t_0\doteq 0$. As before, on $\Delta(\pi^1,\ldots,\pi^r)$

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\sum_{l=1}^m\sum_{j=0}^{r-1}q_j(W_{t_{j+1}}-W_{t_j})})=\mathrm{e}^{-\sum_{j=1}^r\sum_{l=0}^{m-1}|v_j^l|^2\bar{t}_j^l/2} \tag{4.22}$$

where now

$$v_j^l=\sum_{k\le l}q_j^{\pi^j(k)}+\sum_{k>l}q_{j-1}^{\pi^j(k)}.$$

Integrating over $\mathrm{d}\bar{t}_j$ on $\Delta(\pi^1,\ldots,\pi^r)$ and using $|H(p,x)|\le 1$, we find (4.21) bounded by

$$\sum_{\pi^1,\ldots,\pi^r}\int_{R^{2rm}} B_0^2(q)\ldots B_r^2(q)\,\mathrm{d}q \tag{4.23}$$

where

$$B_0^2(q)=\prod_{l=0}^m(|q_0^l|^2+1)^{-1},\qquad B_j^2(q)=\prod_{l=0}^m(|v_j^l|^2+1)^{-1}\quad(v_j^m=v_{j+1}^0)$$

as before.

$$\int_{R^{2rm}}\prod_{j=0}^r B_j^2(q)\,\mathrm{d}q\le\int\prod_{i=0}^r\left(\prod_{j\ne i}B_j^{2/r}\right)\mathrm{d}q \le\prod_{i=0}^r\left(\int_{R^{2rm}}\prod_{j\ne i}B_j^{2(r+1)/r}\,\mathrm{d}q\right)^{1/r+1}<\infty. \tag{4.24}$$

As in the proof of Lemma 1 these ideas suffice to prove Lemma 2.

Proof of Lemma 3. This is similar to the proof of Lemma 2, but here we use

$$|e^{ip_r\cdot y}-e^{ip_r\cdot x}-ip_r\cdot(x-y)\,e^{ip_r\cdot x}|\leq c(|p_r\|x-y|)^{3/2}$$

and instead of (4.24) we need to bound

$$\begin{aligned}\int_{R^{2rm}} & B_0^{1/2}(q)\prod_{j=1}^{r}B_j^2(q)\,dq\\ &=\int_{R^{2rm}}\left[\prod_{i=1}^{r}\left(\prod_{\substack{j=0\\ j\neq i}}^{r}B_j(q)\right)^{1/2r}\right](B_1\cdots B_r)^{2-(r-1)/2r}\,dq\\ &\leq\left(\prod_{i=1}^{r}\left\|\prod_{\substack{j=0\\ j\neq i}}^{r}B_j^{1/2r}\right\|_{4r+1}\right)\|(B_1\cdots B_r)^{2-(r-1)/2r}\|_{(4r+1)/(3r+1)}\end{aligned} \tag{4.25}$$

since

$$\frac{r}{4r+1}+\frac{3r+1}{4r+1}=1,$$

while

$$\left\|\prod_{\substack{j=0\\ j\neq i}}^{r}B_j^{1/2r}\right\|_{4r+1}^{4r+1}=\int\prod_{\substack{j=0\\ j\neq i}}^{r}B_j^{2+1/2r}\,dq<\infty$$

and

$$\begin{aligned}\|(B_1\cdots B_r)^{2-(r-1)/2r}\|_{(4r+1)/(3r+1)}^{(4r+1)/(3r+1)}&=\int(B_1\cdots B_r)^{((3r+1)/2r)\cdot(4r+1)/(3r+1))}\,dq\\ &=\int(B_1\cdots B_r)^{2+1/2r}\,dq<\infty.\end{aligned}$$

This completes the proof of (3.6), and (3.5) is similar.

Proof of Lemma 5. It suffices to show L^2 convergence in (3.10), and for this it suffices to show

$$\int_{I_{r+1}}\mathbb{E}(|\nabla K_r^{\varepsilon}(W_t)-\nabla K_r(W_t)|^2)\,dt\to 0. \tag{4.26}$$

Since $\nabla K_r^{\varepsilon}(W_t)\to\nabla K_r(W_t)$ for a.e. path and fixed t, it suffices to show $\nabla K_r^{\varepsilon}(W)$ Cauchy in $L^2(d\mathbb{P}\times dt)$.

As in the proof of Lemma 3, we find

$$\begin{aligned}\int_{I_{r+1}} & \mathbb{E}(|\nabla K_r^{\varepsilon}(W_t)-\nabla K_r^{\varepsilon'}(W_t)|^2)\,dt\\ &\leq c|\varepsilon-\varepsilon'|^{1/8}\int_{R^{4r}}\frac{|p_r^1\cdot p_r^2||p_r^1|^{1/4}|p_r^2|^{1/4}}{(1+|p_r^1|^2)(1+|p_r^2|^2)}\cdot\sum_{\pi}\int_{\Delta(\pi',\dots,\pi_r)}e^{-\sum_{j=1}^{r}\sum_{l=1}^{2}|Z_j^l|^2\bar{t}_j^l}\\ &\leq|\varepsilon-\varepsilon'|^{1/8}\int_{R^{4r}}D_1^2(p)\cdots D_r^2(p)Q^{3/4}(p)\,dp\end{aligned} \tag{4.27}$$

where

$$Q(p)=(1+|p_r^1|)^{-1}(1+|p_r^2|)^{-1}, \qquad D_j(p)=(1+|Z_{j-1}^2|)^{-1}(1+|Z_j^1|)^{-1}(1+|Z_j^2|)^{-1}$$

and

$$Z_j^l=\sum_{k\leq l} p_j^{\pi^j(k)}+\sum_{k>l} p_{j-1}^{\pi^j(k)}.$$

Write

$$\int \prod_{j=1}^r D_j^2 Q^{3/4}\,\mathrm{d}p=\int\left[\prod_{i=1}^r\left(\prod_{j\neq i} D_jQ\right)^{3/4r}\right](D_1\cdots D_r)^{(5r+3)/4r}\,\mathrm{d}p$$

$$\leq \prod_{i=1}^r\left\|\left(\prod_{j\neq i} D_jQ\right)^{3/4r}\right\|_{(8r+3)/3}\|(D_1\cdots D_r)^{(5r+3)/4r}\|_{(8r+3)/5r+3)}<\infty,$$

as before since

$$\frac{3r}{8r+3}+\frac{5r+3}{8r+3}=1.$$

References

A. Dvoretzky, P. Erdös and S. Kakutani, Multiple points of paths of Brownian motion in the plane, Bull. Res. Council Israel F3 (1954) 364–371.

E. Dynkin, Random fields associated with multiple points of the Brownian motion, J. Funct. Anal. 62 (1985) 3.

D. Geman and J. Horowitz, Occupation densities, Ann. Probab. 8 (1980) 1–67.

D. Geman, J. Horowitz and J. Rosen, A local time analysis of intersections of Brownian paths in the plane, Ann. Probab. 12 (1984) 86–107.

J.F. Le Gall, Sur la mesure de Hausdorff des points multiples du mouvement Brownien plan, C.R. Acad. Sc. Paris t. 299, Serie I, No. 13 (1984a) 627–630.

J.F. Le Gall, Sur la saucisse de Wiener et les points multiples du mouvement Brownien, Preprint, Laboratoire de Calcul des Probabilities, 4 Place Jussieu, 75252 Paris Cedex 05, France (1984b).

J.F. Le Gall, Sur la temps local d'intersection du mouvement Brownien plan et la methode de renormalization de Varadhan, Seminaire de Probab. XIX, Lecture Notes in Mathematics, 1123 (Springer, New York, 1985).

P.A. Meyer, Appendix p. 116, Seminaire de Prob. XV. (Springer Lecture Notes in Math, 850, Springer, New York, 1979/1980).

L. Pitt, Local times for Gaussian vector fields, Indiana Univ. Math. J. 27 (1978) 309–330.

J. Rosen, A representation for the intersection local time of Browniam motion in space, Ann. Probab. 13 (1) (1958) 145–153.

J. Rosen, Self intersections of random fields, Ann. Probab. 12 (1984) 108–119.

J. Rosen, Tanaka's formula and renormalization for intersections of planar Brownian motion, Ann. Probab. (1986) to appear.

S. Varadhan, Appendix to Euclidean Quantum Field Theory by K. Symanzk, in: R. Jost, ed., Local Quantum Theory (Academic Press, New York, 1969).

M. Yor, Complements aux formules de Tanaka-Rosen, Seminaire de Prob. XIX, Lecture Notes in Mathematics, 1123 (Springer, New York, 1985a).

M. Yor, Renormalization et convergence en loi pour les temps locaux d'intersection du mouvement Brownian dans R^3, Seminaire de Prob. XIX, Lecture Notes in Mathematics, 1123 (Springer, New York, 1985b).

M. Yor, Renormalization for some triple integrals of two dimensional Brownian motion, Preliminary version, Laboratoire de Calcul des Probabilities, 4 Place Jussieu, 75252 Paris Cedex 05, France (1985c).