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ASYMPTOTICS OF CERTAIN RANDOM FIELDS ON A CIRCLE

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ABSTRACT

We study the asymptotic behavior of sums S_n of random variables defined for certain mean-field type ferromagnetic systems on a circle. The probability distribution of S_n may be expressed in terms of a certain measure on a space $\mathscr{Y}$ of continuous functions. Suitably scaled and centered versions of this measure have limits, in terms of which the asymptotic behavior of S_n is determined. These function space limits depend crucially upon the minimum points of a nonlinear functional on $\mathscr{Y}$ related to the specific free energy of the ferromagnetic system. The study of these limits is related to work by M. Donsker, S. Varadhan, and other authors. The proofs of the function space limit results are lengthy and will appear elsewhere.

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1. INTRODUCTION

For each $n \in \{1, 2, \ldots\}$ we define a mean-field-type ferromagnetic system on the sites $\left\{\frac{j}{n}; j = 1, \ldots, n\right\}$ of a circle of circumference one. Let $\{X_j^{(n)}; j = 1, \ldots, n\}$ denote the random variables which measure the strengths of the magnetic moments at the sites $\left\{\frac{j}{n}\right\}$. The joint distribution of the $\{X_j^{(n)}\}$ – i.e., the Gibbs measure of the system – is defined to be

$$d\Gamma_n(X_1, \ldots, X_n) :=$$

$$:= \frac{1}{Z_n} \exp\left[\frac{1}{2n} \sum_{1 \leq j, l \leq n} J\left(\frac{j}{n} - \frac{l}{n}\right) X_j X_l + \right. \left. + \sum_{j=1}^{n} H\left(\frac{j}{n}\right) X_j\right] \prod_{j=1}^{n} d\rho(X_j), \tag{1.1}$$

where

$$Z_n := \int_{R^n} \exp\left[\frac{1}{2n} \sum_{1 \leq j \leq l \leq n} J\left(\frac{j}{n} - \frac{l}{n}\right) X_j X_l + \right. \left. + \sum_{j=1}^{n} H\left(\frac{j}{n}\right) X_j\right] \prod_{j=1}^{n} d\rho(X_j). \tag{1.2}$$

We have set $\beta := \frac{1}{kT}$ equal to one (k is Boltzmann's constant, T the absolute temperature). Among all Gibbs measures on the sites $\left\{\frac{j}{n}\right\}$ with isotropic pair interaction potentials J satisfying out hypotheses, $d\Gamma_n$ is essentially the only one which exhibits interesting limiting behavior. This is explained in Remark 4.2.

Let $\mathscr{Y}$ denote the space of all real-valued continuous functions on R which are periodic of period one. The function $J(t)$ in (1.1)-(1.2) is an even, positive, suitably smooth element of $\mathscr{Y}$ which also satisfies the technical hypotheses listed at the start of Section 3. The most important of these is that J be positive definite (i.e., each Fourier coefficient positive). We normalize J so that

$$\int_0^1 J(t)\, dt = 1. \tag{1.3}$$

The function H in (1.1) is a suitably smooth element of $\mathscr{Y}$ (see Section 3) while ρ is a Borel probability measure on R which is not a point mass and which satisfies

$$(1.4) \qquad \int_R \exp(cx^2)\, d\rho(x) < \infty \quad \text{for all } c > 0.$$

Thus, the integral defining the partition function Z_n in (1.2) converges. Because J is positive, (1.1)-(1.2) define a ferromagnetic system on the sites $\{\frac{j}{n};\ j = 1, \ldots, n\}$: the interaction strength between sites $\frac{j}{n}$ and $\frac{l}{n}$ is $n^{-1} J\left(\frac{j}{n} - \frac{l}{n}\right)$; an external magnetic field of strength $H\left(\frac{j}{n}\right)$ is applied at each site $\frac{l}{n}$; ρ is the distribution of a single spin in the absence of interactions with the other spins. Because of the factor $\frac{1}{n}$ multiplying $J\left(\frac{j}{n} - \frac{l}{n}\right)$, (1.1)-(1.2) define a mean-field-type model. The case $J = 1$, $H = h$, a constant, is the Curie – Weiss model (Brout [3], Kac [13] §3). The function J in (1.1) makes the present model highly nontrivial in comparison with the Curie – Weiss case, where each spin interacts equally with all the other spins. From a probabilistic viewpoint, the spin random variables in the Curie – Weiss model are, for each n, exchangeable (viz., for $J = 1$, $H = h$, the measure $d\Gamma_n$ in (1.1) is invariant under permutations of the X_j's) and thus identically distributed. This is not true for the present model.

The purpose of this paper is to discuss the asymptotics, as $n \to \infty$, of suitably normalized sums of the random variables $\{X_i^{(n)}\}$. To simplify the statements and proofs of certain results, we shall assume in some cases that H is a constant function (see discussion after the statement of Theorem 2.2). See Section 2 for the statement of the results. Similar asymptotics have already been treated in great detail in the Curie – Weiss case (Ellis – Newman [6], [8], Ellis – Newman – Rosen [9]). The probabilistic limit theorems of Section 2 are proved as a consequence of new limit theorems for certain measures on the function space $\mathscr{Y}$. In Section 3, the necessary functional analysis is developed. Section 4 contains statements of the function space limit theorems. In Section 5 the theorems of Section 2 are proved. Proofs of the theorems of Section 4 are lengthy and will appear elsewhere (Ellis – Rosen [10]).

We point out the relationship of our results to other results in the literature. The proof of Theorem 2.1 is an instance of Laplace's method in function space. Analogous problems have been studied by a number of authors (Donsker – Varadhan [5], Pincus [14], Schilder [15], Simon [16], Varadhan [18]). Our theorem is not covered by any of these papers and must be proved from scratch. Theorems 2.3-2.5 are proved by relating the distribution of S_n to a probability measure on $\mathscr{Y}$ and deriving the limits of suitably scaled and centered versions of this measure (see Theorems 4.5-4.7). The limit result which yields the case $\alpha = 1$ of Theorem 2.3 is contained in Varadhan [18] but the limit results which yield Theorems 2.4-2.5 as well as the cases $\alpha \geqslant 2$ of Theorem 2.3 are new.

The limit theorems in function space depend upon the minimum points of a nonlinear functional on $\mathscr{Y}$. In Schilder [15] there are related results. In a slightly different context, he proves, in the case of a unique minimum point which is non-degenerate, an asymptotic expansion for an integral analogous to (4.10). In Theorem 4.8, we give the first term of the asymptotic expansion of the integral (4.10); this is derived from a result which arises in the proof of Theorems 4.5-4.7. Theorem 4.8 is stated for a unique minimum point which is either non-degenerate or degenerate and constant (see Definition 3.1 for these terms). The entire asymptotic expansion of the integral (4.10) should be obtainable by our methods with extra work.

In this paper, we treat the case of degenerate minimum points only under the restrictive hypothesis that H is a constant function. This guarantees that any degenerate minimum points are constant functions and simplifies the statements and proofs of certain theorems. Degenerate minimum points in the case of general H will be treated in Ellis – Rosen [10].

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2. STATEMENT OF RESULTS

For this section, the functions J and H are assumed to satisfy the hypotheses listed at the start of Section 3. (H is not assumed to be a constant function unless otherwise specified.) Define $S_n := \sum_{j=1}^{n} X_j^{(n)}$. The limiting behavior as $n \to \infty$ of $\frac{S_n}{n}$ and of $\frac{S_n - nc}{n^{1-\gamma}}$ for suitable constants c and $\gamma > 0$ is precisely determined. Although the Curie – Weiss model is not covered by our methods ($J = 1$ is not positive definite), our results are analogous to those obtained in the Curie – Weiss case and also include new phenomena. For $W \in \mathscr{Y}$, define $S_{n,W} := \sum_{j=1}^{n} W\left(\frac{j}{n}\right) X_j^{(n)}$. In Theorem 4.9, we discuss the asymptotics of the random field $\{S_{n,W};\ W \in \mathscr{Y}\}$.

We define the various notions of convergence which arise (Chung [4] § 4.4). Given probability measures $\{\xi_n;\ n = 1, 2, \dots\}$ and a measure ξ on R, we say that the ξ_n tend weakly to ξ, and write $\xi_n \Rightarrow \xi$, if $\int \psi d\xi_n \to \int \psi d\xi$ for all $\psi \in C(R)$; we say that the ξ_n tend vaguely to ξ, and write $\xi_n \xrightarrow{v} \xi$, if $\int \psi d\xi \to \int \psi d\xi$ for all $\psi \in C(R)$ which vanish at $\pm\infty$. Given random variables $\{T_n;\ n = 1, 2, \dots\}$ with probability distributions $\{\xi_n;\ n = 1, 2, \dots\}$, we say that the T_n tend in distribution (respectively, vaguely) to ξ, and write $T_n \xrightarrow{\mathscr{D}} \xi$ (respectively, $T_n \xrightarrow{v} \xi$), if the ξ_n tend weakly (respectively, vaguely) to ξ. If $T_n \xrightarrow{\mathscr{D}} \xi$, then ξ must be a probability measure; if $T_n \xrightarrow{v} \xi$, then ξ may be defective: $\xi = b\zeta$, some $b \in [0, 1]$, ζ a probability measure on R. We denote a point mass at c by δ_c. If $\xi = \delta_c$, we write $T_n \xrightarrow{\mathscr{D}} c$.

Theorem 2.3 states that under suitable hypotheses there exist an integer $\theta \geqslant 1$, positive numbers $\{b_j;\ j = 1, \dots, \theta\}$ with $\sum b_j = 1$, and distinct real numbers $\{\bar{y}_j;\ j = 1, \dots, \theta\}$ so that

$$(2.1) \qquad \frac{S_n}{n} \xrightarrow{\mathscr{D}} \sum_{j=1}^{\theta} b_j \delta_{\bar{y}_j}.$$

If $\theta = 1$, then this is the law of large numbers for $\{X_j^{(n)}\}$: $\frac{S_n}{n} \xrightarrow{\mathscr{D}} \bar{y}_1$. Physically, $\theta = 1$ means that in the thermodynamic limit $n \to \infty$, the ferromagnet has a unique phase with magnetization per site $\bar{y}_1$ (see (Ellis – Newman [6], [8], Ellis – Newman – Rosen [9] for more physics). If $\theta \geqslant 2$, then (2.1) corresponds to the existence of multiple phases, where b_j is the probability of the system being in the j-th phase and $\bar{y}_j$ is the magnetization per site in the j-th phase.

Theorems 2.4-2.5 analyze the fluctuations of $\frac{S_n}{n}$ about each $\bar{y}_j$. Under suitable hypotheses, for each $j \in \{1, \ldots, \theta\}$ there exist an integer $k_j \geqslant 1$, a positive real number λ_j, and a probability measure ξ_{k_j, λ_j} so that

$$(2.2) \qquad \frac{S_n - n\bar{y}_j}{n^{1 - \frac{1}{2k_j}}} \xrightarrow{v} b_j \xi_{k_j, \lambda_j}.$$

The measure $\xi_{k, \lambda}$ is Gaussian if and only if $k = 1$, so that $\gamma = k = 1$ is the central limit theorem for $\{X_j^{(n)}\}$. The case $\theta = 1$, $k \geqslant 2$ corresponds to a phase transition. If $\theta = 1$, then $\xrightarrow{v}$ in (2.2) is replaced by $\xrightarrow{\mathscr{D}}$. The Gaussian case of (2.2) is treated in Theorem 2.4. The non-Gaussian case is treated in Theorem 2.5 under the assumption that H in (1.1)-(1.2) is a constant function. To avoid notational and other complications which arise in the general case, Theorems 2.4-2.5 state (2.2) under a special assumption (which implies $\theta = 1$). More general statements appear in Section 5.

Our first result concerns the specific free energy f for the system (1.1)-(1.2), defined by

$$(2.3) \qquad f := - \lim_{n \to \infty} \frac{1}{n} \ln Z_n .$$

Fact (2.7) will be proved in Section 5 in the special case that H is a constant function

Theorem 2.1. *Define the operator* $\mathscr{J}$ *on* $L^2[0, 1]$ *by*

$$(2.4) \qquad (\mathscr{J}Y)(t) := \int_0^1 J(t-s)\, Y(s)\, ds, \qquad Y \in L^2[0,1].$$

Then $\mathscr{J}$ *has a square root* $\sqrt{\mathscr{J}}$ *which is invertible and whose range is a subset of* $\mathscr{Y}$. *Write* $\mathscr{J}^{-\frac{1}{2}}$ *for* $(\sqrt{\mathscr{J}})^{-1}$. *Define the function* φ *on* $\boldsymbol{R}$ *by*

$$(2.5) \qquad \varphi(y) := \ln \int_{\boldsymbol{R}} \exp(yx)\, d\rho(x), \qquad y \in \boldsymbol{R},$$

and the functional G *on* $\mathscr{Y}$ *by*

$$(2.6) \qquad G(Y) := \begin{cases} \frac{1}{2} \| \mathscr{J}^{-\frac{1}{2}} \|_2^2 - \int_0^1 \varphi(Y+H)\, dt & \text{if} \quad Y \in \mathscr{D}(\mathscr{J}^{-\frac{1}{2}}), \\ + \infty & \text{if} \quad Y \in \mathscr{Y} \setminus \mathscr{D}(\mathscr{J}^{-\frac{1}{2}}), \end{cases}$$

where $\| - \|_2$ *denotes the* $L^2[0,1]$ *norm. Then* $\inf G$ *over* $\mathscr{Y}$ *is finite, it is attained at some point* Y^* *of* $\mathscr{D}(\mathscr{J}^{-\frac{1}{2}})$, *and*

$$(2.7) \qquad f := -\lim_{n \to \infty} \frac{1}{n} \ln Z_n = \min_{\mathscr{Y}} G = G(Y^*).$$

The asymptotics of S_n depend crucially upon the minimum points of G; these are points $Y^* \in \mathscr{Y}$ for which $G(Y^*) = \min G$ over $\mathscr{Y}$. (The physical meaning of other critical points of G is explored and corresponding limit theorems are proved in the Curie – Weiss case in Ellis – Newman – Rosen [9]). In Theorems 2.3-2.5, a key hypothesis is that these minimum points are finite in number. The next theorem locates all the minimum points and states that they are finite in number in the important special case that H is a constant function. Fixing h real, we define

$$(2.8) \qquad G_{\text{c. w.}}(y) = \frac{y^2}{2} - \varphi(y+h), \qquad y \in \boldsymbol{R}.$$

This function plays the same role for the Curie – Weiss model as G in (2.6) plays for the present model. Because of (1.3), if H in (2.6) equals h, a constant, and $Y = y$, a constant, then $G(Y) = G_{\text{c. w.}}(y)$.

Theorem 2.2. $G_{c.\,w.}(y)$ *has finitely many minimum points, say* $y_1^*, \ldots, y_\alpha^*$, *some* $\alpha \geq 1$. *If* H *in* (2.5) *equals the constant* h *in* (2.8), *then the minimum points of* G *are precisely the constant functions* $Y_1^* := y_1^*, \ldots, Y_\alpha^* := y_\alpha^*$ *and*

$$\min_{\mathscr{Y}} G = \min_R G_{\text{c.w.}}. \tag{2.9}$$

Conversely, if G *has a minimum point which is a constant function, then* H *is constant. Given* $\alpha \in \{1, 2, \ldots\}$ *and any points* $\tilde{y}_1, \ldots, \tilde{y}_\alpha$ *in* R, *the measure* ρ *in* (2.5) *may be chosen so that* (1.4) *holds and* $G_{\text{c.w.}}$ *has precisely* $\tilde{y}_1, \ldots, \tilde{y}_\alpha$ *as minimum points. Furthermore, the types of these minimum points may be arbitrarily prescribed (see Definition* 3.2 *for the term type).*

We now turn to the statements of the limit results (2.1)-(2.2). The forms of the limits depend upon the non-degeneracy or degeneracy of the minimum points and upon the numbers $\{k_j\}$ and $\{\lambda_j\}$, called the types and strengths, respectively, of the minimum points. These terms are defined in Definitions 3.1-3.2. Let Y^* be a minimum point. For now, it suffices to say that if Y^* is non-degenerate, then the type k of Y^* equals one; if Y^* is degenerate and isolated, then the type k of Y^*, which measures the extent of the degeneracy, is an integer exceeding one.

Limit theorems for degenerate minimum points are stated only in the case that H in (1.1), (1.2), (2.6) is a constant function or, equivalently, that the minimum points are all constant functions. As a result of this restriction, the statements and proofs of the corresponding limit theorems are greatly simplified. Only in the case of constant H do we have examples of degenerate minimum points. Furthermore, if H is non-constant, then the minimum points are non-constant, and we cannot guarantee that they are finite in number except under the condition of non-degeneracy. Limit theorems for non-degenerate minimum points will be stated for general H.

The next theorem discusses the limiting behavior $\frac{S_n}{n}$. It is obvious generalization of the Curie – Weiss result (Ellis – Newman [6], Thm. 3.8).

Theorem 2.3. *Assume that* G *has finitely many minimum points, say* $Y_1^*, \ldots, Y_\alpha^*$, *some* $\alpha \geqslant 1$. *If* $\alpha = 1$, *then*

$$(2.10) \qquad \frac{S_n}{n} \xrightarrow{\mathscr{D}} \int_0^1 Y_1^* \, dt.$$

If $\alpha \geqslant 1$, *assume either that each* Y_j^* *is non-degenerate or that* H *is a constant function and at least one* Y_j^* *is degenerate. For* $j = 1, \ldots, \alpha$, *let* k_j *denote the type of* Y_j^* *and define* $\bar{k} := \max\{k_j \mid j = 1, \ldots, \alpha\}$. *Let* $\bar{y}_1, \ldots, \bar{y}_\theta$ *be the distinct numbers in the set* $\left\{\int_0^1 Y_j^* \, dt \colon k_j = \bar{k},\ j = 1, \ldots, \alpha\right\}$. *Then there exist positive numbers* $\{b_j;\ j = 1, \ldots, \theta\}$ *so that*

$$(2.11) \qquad \frac{S_n}{n} \xrightarrow{\mathscr{D}} \sum_{j=1}^{\theta} b_j \delta_{\bar{y}_j}.$$

The numbers $\{b_j\}$ *are defined in* (5.6).

We note that if $\alpha = 1$, then (2.10) holds even if Y_1^* is a non-constant, degenerate minimum point.

For the other limit results, assume that G has a unique minimum point Y^*. Then (2.10) holds with $Y_1^* = Y^*$. The statements for arbitrary $\alpha \geqslant 2$ appear in Section 5. The following are obvious generalizations of the Curie – Weiss result (Ellis – Newman [6], Cor. 3.10). Given $\sigma^2 > 0$, we denote by dN_{σ^2} a Gaussian measure with mean 0 and variance σ^2; $N(\sigma^2)$ denotes the corresponding random variable.

Theorem 2.4. *Assume that* Y^* *is the unique minimum point and that* Y^* *is non-degenerate. Then*

$$(2.12) \qquad \frac{S_n - n \int_0^1 Y^* \, dt}{\sqrt{n}} \xrightarrow{\mathscr{D}} dN_{\sigma^2_{Y^*}}$$

where $\sigma^2_{Y^*}$ *is defined in* (5.8).

Theorem 2.5. *Assume that* H *is a constant function. Assume that* $Y^* = y^*$, *a constant, is the unique minimum point and that* Y^* *is degenerate. Then*

$$(2.13)\qquad \frac{S_n - ny^*}{n^{1-\frac{1}{2k}}} \xrightarrow{\mathscr{D}} \frac{\exp\left(-\frac{\lambda u^{2k}}{(2k)!}\right) du}{\int_R \exp\left(-\frac{\lambda u^{2k}}{(2k)!}\right) du}$$

where k *and* λ *denote the type and strength, respectively, of* Y^*.

3. FUNCTIONAL ANALYSIS

We first state our hypotheses on the functions J and H in (1.1). We then define the two kinds of minimum points (non-degenerate, degenerate) and the terms type and strength. Examples of the various kinds of minimum points will be given.

The function J is to be an even element of $\mathscr{Y}$, so that

$$(3.1)\qquad J(t) = \hat{J}(0) + 2 \sum_{p=1}^{\infty} \hat{J}(p) \cos 2\pi pt,$$

where

$$\hat{J}(p) := \int_0^1 J(t)(\cos 2\pi pt)\, dt.$$

The probabilistic analysis in Section 4 involves an approximation $\hat{J}_n(p)$ to $\hat{J}(p)$, defined by

$$(3.2)\qquad \hat{J}_n(p) := \sum_{j \in Z} \hat{J}(p + jn), \qquad p \in Z,\ n \in \{1, 2, \dots\},$$

where we define $\hat{J}(p) = \hat{J}(|p|)$ for $p \leqslant -1$. As a function on Z, $\hat{J}_n(\cdot)$ is even and periodic of period n.

Our hypotheses on J are that $J \in \mathscr{Y}$ is even; $J > 0$ on $[0, 1]$; $\hat{J}(0) = 1$, $\hat{J}(p) > 0$ for all $p \neq 0$ (i.e., J positive-definite); $J \in C^\infty$; and

$$(3.3)\qquad \begin{aligned} &\hat{J}(p) < \hat{J}_n(p) < L\hat{J}(p), \\ &|p| \leqslant \left[\tfrac{n}{2}\right], \qquad n = 1, 2, \dots, \quad \text{for some } L \in (1, \infty). \end{aligned}$$

Of the function H, we require that $H \in \mathscr{Y} \cap C^\infty$. H is not assumed to be a constant function unless otherwise specified.

We comment first on the hypothesis that $J \in C^\infty$; $H \in C^\infty$. Actually, to prove Theorem 2.1 and a few of the simpler probabilistic limit theorem, $\sum_{p \in Z} p\hat{J}(p) < \infty$ and $H \in \mathscr{Y}$ are enough. For the other probabilistic limit theorems, more smoothness is required: $J(p) \leqslant \dfrac{\text{const}}{|p|^{6+\epsilon}}$, some $\epsilon > 0$, all $p \in Z \setminus \{0\}$; in those cases where we do not require H constant, we need $|\hat{H}(p)| \leqslant \dfrac{\text{const}}{|p|^{10}}$, all $p \in Z \setminus \{0\}$. These bounds on $\hat{J}$ and $\hat{H}$ hold if $\mathscr{J}$ and H are C^∞. We assume $\hat{J}(0) = 1$, which is the same as (1.3). The lower bound in (3.3) is automatic because in (3.2) $\hat{J}(p + jn) > 0$, all $j \neq 0$.

We note that because of our assumptions on J,

(3.4) $\qquad 1 = \hat{J}(0) > \hat{J}(p) > 0 \quad \text{for all} \quad p \neq 0.$

Just check

(3.5) $\qquad \hat{J}(0) - \hat{J}(p) = \int_0^1 J(t)(1 - \cos 2\pi pt)\, dt > 0 \quad \text{for} \quad p \neq 0.$

19 This fact is an important element in the proof of Theorem 2.2.

We now turn to the functional analysis. We regard $\mathscr{Y} \subset L^2[0, 1]$ as spaces of real-valued functions: $\|-\|$ denotes the supremum norm on $\mathscr{Y}$; $(-, -)$ and $\|-\|_2$ denote the $L^2[0, 1]$ inner product and norm, respectively. For $Y \in L^2[0; 1]$ consider the Fourier expansion

(3.6) $\qquad Y = \sum_{p \in Z} \hat{Y}(p) \exp(2\pi i p t),$

where

$$\hat{Y}(p) := \int_0^1 Y(t) \exp(2\pi i p t)\, dt.$$

By taking real parts of (3.6), we express Y as an expansion in terms of the functions

$$(3.7) \qquad B_p(t) := \begin{cases} \sqrt{2}\cos 2\pi pt, & p \in \{1, 2, \ldots\}, \\ 1, & p = 0, \\ \sqrt{2}\sin 2\pi pt, & p \in \{-1, -2, \ldots\}. \end{cases}$$

We have for $Y \in L^2[0,1]$

$$(3.8) \qquad Y = \sum_{p \in Z} y(p) B_p(t),$$

where

$$y(p) := (Y, B_p),$$

$$(3.9) \qquad \| Y \|_2^2 = \sum_{p \in Z} (y(p))^2 .$$

In terms of the numbers $\{\hat{J}(p)\}$, the integral operator $\mathscr{J}$ in (2.4) assumes a simple form:

$$(3.10) \qquad \text{for } Y \text{ as in (3.8)}, \quad \mathscr{J} Y = \sum_{p \in Z} \hat{J}(p) y(p) B_p .$$

$\mathscr{J}$ is symmetric, positive-definite, and trace class. The square root $\sqrt{\mathscr{J}}$ of $\mathscr{J}$ is a symmetric, positive-definite, Hilbert – Schmidt integral operator with kernel $J_{\frac{1}{2}}(t-s)$, where

$$(3.11) \qquad J_{\frac{1}{2}}(t) := 1 + 2 \sum_{p=1}^{\infty} \sqrt{\hat{J}(p)} \cos 2\pi pt.$$

Alternatively, $\sqrt{\mathscr{J}}$ is defined as in (3.10) with $\{\sqrt{\hat{J}(p)}\}$ replacing $\{\hat{J}(p)\}$. Write $\mathscr{J}^{-\frac{1}{2}}$ for $(\sqrt{\mathscr{J}})^{-1}$. The domain $(\mathscr{J}^{-\frac{1}{2}})$ of $\mathscr{J}^{-\frac{1}{2}}$ is the set

$$(3.12) \qquad \mathscr{H}_{\mathscr{J}} := \Big\{ Y \mid Y \text{ as in (3.8) with } \sum_{p \in Z} (\hat{J}(p))^{-1} (y(p))^2 < \infty \Big\}.$$

$\mathscr{H}_{\mathscr{J}}$ is a Hilbert space with norm

$$(3.13) \qquad \| Y \|_{\mathscr{J}} := \Big\{ \sum_{p \in Z} \hat{J}(p))^{-1} (y(p))^2 \Big\}^{\frac{1}{2}} .$$

We claim $\mathscr{H}_{\mathscr{J}} \subset \mathscr{Y}$. This follows from the inequality

$$(3.14) \qquad \sum_{p \in Z} |y(p)| \leqslant \|Y\|_{\mathscr{J}} \sum_{p \in Z} \hat{J}(p) < \infty, \qquad Y \in \mathscr{H}_{\mathscr{J}}.$$

If $(\hat{J}(p))^{-1}$ is a polynomial in p^2, then $\mathscr{H}_{\mathscr{J}}$ is a periodic Sobolev space.

We rewrite the functional G defined in (2.6). Define the functional F on $\mathscr{Y}$ by

$$(3.15) \qquad F(Y) := -\int_0^1 \varphi(Y + H)\, dt, \qquad Y \in \mathscr{Y}.$$

Then

$$(3.16) \qquad G(Y) = \begin{cases} \frac{1}{2}\|Y\|^2_{\mathscr{J}} + F(Y), & Y \in \mathscr{H}_{\mathscr{J}} \\ +\infty, & Y \in \mathscr{Y} \setminus \mathscr{H}_{\mathscr{J}} \end{cases}$$

By (5.17) and $F \leqslant 0$ (since $\varphi > 0$ on $\boldsymbol{R}$), it follows that G is bounded on $\mathscr{H}_{\mathscr{J}}$.

We next define the various types of minimum points. Given $j \in \{1, 2, \ldots\}$, $\bar{Y}, Y_1, \ldots, Y_j \in \mathscr{H}_{\mathscr{J}}$, we define the Gateaux derivatives

$$(3.17) \qquad \begin{aligned} &D^j G(\bar{Y}, Y_1, \ldots, Y_j) := \\ &:= \frac{d^j}{dw_1 \ldots dw_j} G\big(Y + \sum_{i=1}^{j} w_i Y_i\big)\Big|_{w_1 = \ldots = w_j = 0}. \end{aligned}$$

If $Y_1 = \ldots = Y_j$, we write $D^j G(\bar{Y}, Y_1^j)$ instead. For all $\bar{Y}, Y \in \mathscr{H}_{\mathscr{J}}$, one finds

$$\text{(a)} \quad DG(\bar{Y}, Y) = (\mathscr{J}^{-1}\bar{Y} - \varphi'(\bar{Y} + H), Y),$$

$$(3.18) \ \text{(b)} \quad D^2 G(\bar{Y}, Y^2) = (\mathscr{J}^{-1} Y - \varphi''(\bar{Y} + H) Y, Y),$$

$$\text{(b)} \quad D^j G(\bar{Y}, Y^j) = (-\varphi^{(j)}(\bar{Y} + H), Y^j), \qquad j \geqslant 3.$$

Let Y^* be a minimum point of G. Then $Y^* \in \mathscr{H}_{\mathscr{J}}$ and $DG(Y^*, Y) = 0$ for all $Y \in \mathscr{H}_{\mathscr{J}}$ (B e r g e r [1], p. 301). Thus,

$$(3.19) \qquad \mathscr{J}^{-1} Y^* = \varphi'(Y^* + H).$$

This is the Euler – Lagrange equation for the minimization problem of Theorem 2.1. By (2.4), it may be expressed in the form of a Hammerstein integral equation (T r i c o m i [17]).

$$Y^*(t) = \int_0^1 J(t-s)\varphi'(Y^*(s) + H(s))\, ds. \tag{3.20}$$

We define $\mathscr{M}_{Y^*}$ as the operator

$$\mathscr{M}_{Y^*} Y := \varphi''(Y^* + H)Y, \qquad Y \in L^2[0,1]; \tag{3.21}$$

since $Y^*, H \in \mathscr{Y}$ and φ'' is smooth, $\mathscr{M}_{Y^*}$ is a bounded operator. We also note that $\varphi'' > 0$ on R ($\varphi''(y)$ is the variance of the measure $\dfrac{\exp(yx)\, d\rho(x)}{\int_R \exp(yx)\, d\rho(x)}$, which is not a point mass). Thus, $\mathscr{M}_{Y^*}$ is a positive operator. Because Y^* is a minimum point, $D^2G(Y^*, Y^2) \geqslant 0$ for all $Y \in \mathscr{H}_{\mathscr{J}}$. Thus

$$\mathscr{K}_{Y^*} := \mathscr{J}^{-1} - \mathscr{M}_{Y^*} \text{ is non-negative semi-definite on } \mathscr{D}(\mathscr{J}^{-1}). \tag{3.22}$$

For the rest of this section, we write $\mathscr{M}$ instead of $\mathscr{M}_{Y^*}$.

Definition 3.1. The minimum point Y^* is *non-degenerate* if $\mathscr{K}_{Y^*}$ is positive definite on $\mathscr{D}(\mathscr{J}^{-1})$. Y^* is *degenerate* if $\mathscr{K}_{Y^*}$ has a non-empty nullspace $\mathscr{N}(\mathscr{K}_{Y^*})$.

Since $\mathscr{N}(\mathscr{K}_{Y^*}) = \{Y\colon \mathscr{J}MY = Y\}$, the dimension of $\mathscr{N}(\mathscr{K}_{Y^*})$ is finite by the compactness of $\mathscr{J}M$. We have agreed to consider the degenerate case only for H a constant function. We state a useful criterion. If $H = h$, a constant, and $Y^* = y^*$, a constant, is a minimum point, then Y^* is either non-degenerate or degenerate according to whether

$$\varphi''(y^* + h) < (\hat{J}(0))^{-1} = 1 \text{ or } \varphi''(y^* + h) = (\hat{J}(0))^{-1} = 1; \tag{3.23}$$

$\varphi''(y^* + h)$ cannot exceed one because of (3.22). Also

$$\text{if } \varphi''(y^* + h) = 1, \text{ then } \mathscr{N}(\mathscr{K}_{Y^*}) \text{ is spanned by the function } 1. \tag{3.24}$$

The classification according to (3.23) and the fact (3.24) follow directly from (3.4), which is a consequence of the assumed positivity of J. One may use Theorem 2.2 to find measures ρ for which $\varphi''(y^* + h) = 1$, where $Y^* = y^*$ is a minimum point.

In the non-constant case, Y^* will be non-degenerate if

$$(3.25) \qquad \max_{t \in [0,1]} \varphi''(Y^*(t) + H(t)) < (\hat{J}(0))^{-1} = 1.$$

For (3.25) to hold, it suffices if the measure ρ is even, $\varphi''(0) = \int x^2 \, d\rho(x) < 1$, and ρ has the so-called GHS property: $\varphi'''(y) \leqslant 0$ for all $y \geqslant 0$. Then φ'' is decreasing so that $\varphi''(y) \leqslant \varphi''(0) < 1$ for all y real; (3.25) follows. A large class of measures satisfying the GHS property is known (Ellis – Newman [7]). In the non-constant case, we have no examples of degenerate minimum points.

We comment on the hypotheses of Theorems 2.3-2.5 that either the minimum points be finite in number or that there be a unique minimum point. The case of constant H is covered by Theorem 2.2. If H is non-constant, then there will be a unique minimum point Y^* if $(\hat{J}(0))^{-1} = 1 > \max \varphi''$ on R (Tricomi [17], p. 212). One can prove by the method of steepest descent (Berger [1], p. 128) that the minimum points of G are finite in number if they are all non-degenerate.

We next define the terms type and strength. Let Y^* be an isolated minimum point. If Y^* is non-degenerate, we define the type $k = k(Y^*)$ to be one. Now say Y^* is degenerate (we specialize to the constant case in a moment). Then for fixed $Y \in \mathscr{H}_{\mathscr{I}}$ and all real $\delta \neq 0$ sufficiently small, $G(Y^* + \delta Y) > G(Y^*)$. Since G is an analytic functional (Berger [1], § 2.3), there exists an integer $k \geqslant 2$ such that

$$(3.26) \qquad \begin{aligned} &D^j G(Y^*, Y^j) = 0 \quad \text{for } j = 2, 3, \ldots, 2k-1, \\ &D^{2k} G(Y^*, Y^{2k}) > 0. \end{aligned}$$

Definition 3.2. Let Y^* be an isolated minimum point of G. We define the *type* $k(Y^*)$ of Y^* by

$$(3.27)\qquad k(Y^*) := \begin{cases} 1 & \text{if } Y^* \text{ is non-degenerate,} \\ k^* & \text{if } Y^* \text{ is degenerate,} \end{cases}$$

where k^* is the smallest integer satisfying (3.26) as Y runs through $\mathcal{N}(\mathcal{K}_{Y^*})$. Let $H = h$, a constant, and assume that $Y^* = y^*$, a constant, is degenerate. Then $k(Y^*)$ is the integer $k \geqslant 2$ with the property that

$$(3.28)\qquad \begin{aligned} &\text{(a)}\quad \varphi''(y^* + h) = 1, \quad \varphi^{(j)}(y^* + h) = 0 \quad \text{for } j = 3, \dots, 2k - 1. \\ &\text{(b)}\quad \varphi^{(2k)}(y^* + h) < 0. \end{aligned}$$

The definition (3.28) of $k(Y^*)$ in the constant case follows from the fact that if Y^* is a degenerate, constant minimum point, then Y^* is isolated and $\mathcal{N}(\mathcal{K}_{Y^*})$ is spanned by the function 1. Again, in the case of constant H, examples of minimum points with different types are provided by Theorem 2.2.

We now define the term strength. If Y^* is a non-degenerate minimum point of G, then one may prove that the operator $\mathcal{K}_{Y^*}$ defined in (3.22) is invertible. We define

$$(3.29)\qquad \mathcal{J}_{Y^*} := (\mathcal{K}_{Y^*})^{-1}.$$

Definition 3.3. Let Y^* be a minimum point of G. We define the *strength* $\lambda(Y^*)$ of Y^* by

$$(3.30)\qquad \lambda(Y^*) := \begin{cases} [(\mathcal{J}_{Y^*}1, 1)]^{-1}, & \text{is } Y^* \text{ is non-degenerate,} \\ -\varphi^{(2k)}(y^* + h), & \text{if } H = h, \text{ a constant, and} \\ & Y^* = y^*, \text{ a constant, is degenerate with} \\ & k = k(Y^*) \geqslant 2. \end{cases}$$

For Y^* non-degenerate, the limiting variance $\sigma^2_{Y^*}$ in (2.12) is defined by $\sigma^2_{Y^*} = (\lambda(Y^*))^{-1} - 1$. Therefore, we need the following fact.

Lemma 3.4. *If Y^* is non-degenerate, then $(\lambda(Y^*))^{-1} > 1$ and $\sigma^2_{Y^*} > 0$.*

Proof. One may prove that if Y^* is non-degenerate, then

$$(3.31)\qquad \mathscr{J}_{Y^*} := (\mathscr{J}^{-1} - \mathscr{M})^{-1} = \sqrt{\mathscr{J}} \sum_{i=0}^{\infty} (\sqrt{\mathscr{J}}\, \mathscr{M} \sqrt{\mathscr{J}})^i \sqrt{\mathscr{J}}\,.$$

Writing $1 = \mathscr{J}^{-1} 1$, we have

$$\lambda_{Y^*}^{1} := (\mathscr{J}_{Y^*} \mathscr{J}^{-1} 1,\ \mathscr{J}^{-1} 1) =$$

$$(3.32)\qquad = ([\mathscr{J}^{-1} + \mathscr{M} + \mathscr{M}\mathscr{J}\mathscr{M} + \mathscr{M}\mathscr{J}\mathscr{M}\mathscr{J}\mathscr{M} + \ldots] 1, 1) >$$

$$> (\mathscr{J}^{-1} 1, 1) = 1;$$

the last step follows from the positivity of φ'' and of the kernel J of $\mathscr{J}$. ∎

Remark 3.5. The -1 appears in the definition of strength in the non-degenerate case to stay consistent with the definition in Ellis – Newman [6], [8], Ellis – Newman – Rosen [9].

4. PROBABILISTIC ANALYSIS IN FUNCTION SPACE

For this section, the function J and H are assumed to satisfy the hypotheses of Section 3. In Lemma 4.1, we represent the partition function Z_n in (1.2) as an expectation with respect to a Gaussian probability measure P_n on $\mathscr{Y}$. This representation is the key to proving (2.7) in Theorem 2.1. Next, in Lemma 4.2, we relate the distribution of $\frac{S_n - nc}{n^{1-\gamma}}$, c, γ constants, to a probability measure Q_n on $\mathscr{Y}$ which is absolutely continuous with respect to P_n. Theorems 4.5-4.7 give the limits of suitably scaled and centered versions of Q_n. In Section 5, we prove Theorems 2.3-2.5 from Lemma 4.2 and Theorems 4.5-4.7. Proofs of the latter theorems will appear in Ellis – Rosen [10]. In Theorem 4.8, we give the first term in the asymptotic expansion of the integral (4.10) which represents Z_n. The connection between this and Schilder [15] was pointed out in the Introduction. Finally, in Theorem 4.9 we state limit results for the random field $\{S_{n,W};\ W \in \mathscr{Y}\}$ defined at the beginning of Section 2. These limit theorems follow from an analogue of Lemma 4.2 and from Theorems 4.5-4.7.

The probability measures P_n and Q_n alluded to above are defined in terms of the Fourier analysis of Section 3. Matters are simplified if n is restricted to odd integers; even integers n can be accommodated by changes in notation.

Convention. Throughout Sections 4 and 5, the positive integer n representing the number of sites in the ferromagnetic system is always odd.

Let $n \geqslant 1$ be given. We define an n-dimensional subspace $\mathscr{T}_n$ of $L^2[0, 1]$ by

$$(4.1) \qquad \mathscr{T}_n := \Big\{ Y \mid Y = \sum_{|p| \leqslant \frac{n-1}{2}} y(p) B_p, \; y(p) \text{ real} \Big\}.$$

Denote by τ_n the orthogonal projection onto $\mathscr{T}_n$:

$$(4.2) \qquad \text{for } Y := \sum_{p \in Z} y(p) B_p \in L^2[0, 1], \qquad \tau_n Y := \sum_{|p| \leqslant \frac{n-1}{2}} y(p) B_p .$$

We define the finite-rank operator $\mathscr{J}_n$ by

$$(4.3) \qquad \begin{aligned} &\mathscr{J}_n Y = \sum_{|p| \leqslant \frac{n-1}{2}} \hat{J}_n(p) y(p) B_p \\ &\text{for } Y := \sum_{p \in Z} y(p) B_p \in L^2[0, 1], \end{aligned}$$

where the numbers $\{\hat{J}_n(p)\}$ are defined in (3.2). Clearly, $\mathscr{J}_n = \mathscr{J}_n \tau_n$, $\mathscr{J}_n : \mathscr{T}_n \to \mathscr{T}_n$, and $\mathscr{J}_n$ is positive-definite on $\mathscr{T}_n$. We define

$$(4.4) \qquad \overline{\mathscr{J}}_n := \mathscr{J}_n \upharpoonright \mathscr{T}_n .$$

The operators $\{\mathscr{J}_n\}$ approximate the operator $\mathscr{J}$ in (2.4): $\mathscr{J}_n \to \mathscr{J}$ in trace class norm.

We define P_n to be the Gaussian probability measure concentrated on $\mathscr{T}_n$ with mean 0 and covariance operator $\overline{\mathscr{J}}_n$. Equivalently, P_n is defined by

$$(4.5) \qquad \int_{L^2[0,1]} \psi(Y)\, dP_n(Y) :=$$

$$:= \prod_{|p| \leq \frac{n-1}{2}} (2\pi J_n(p))^{-\frac{1}{2}} \int_{R^n} \psi\Big(\sum_{|p| \leq \frac{n-1}{2}} y(p) B_p \Big) \times$$

$$\times \exp \Big[-\frac{1}{2} \sum_{|p| \leq \frac{n-1}{2}} \frac{(y(p))^2}{J_n(p)} \Big] \prod_{|p| \leq \frac{n-1}{2}} dy(p),$$

for all $\psi \in C(L^2[0,1])$.

We define P to be the Gaussian probability measure on $L^2[0,1]$ with mean 0 and covariance operator $\mathscr{J}$. P is concentrated on $\mathscr{Y}$. From now on we restrict consideration to $\mathscr{Y}$ since all measures will be concentrated on that space. One may prove

(4.6) $\quad P_n \Longrightarrow P,$

where $\Longrightarrow$ denotes weak convergence of measures $\big(\int \psi dP_n \to \int \psi dP$ for all $\psi \in C(\mathscr{Y})\big)$. We define the nonlinear functional $F_n : \mathscr{Y} \to \boldsymbol{R}$ by

(4.7) $\quad F_n(Y) := -\frac{1}{n} \sum_{j=1}^{n} \varphi\big((\tau_n Y)\big(\frac{j}{n}\big) + H\big(\frac{j}{n}\big)\big), \qquad Y \in \mathscr{Y}.$

If $\tau_n Y$ were replaced by Y, the resulting functionals would be a Riemann approximation to the functional F defined in (3.15). One can prove that

(4.8) $\quad$ if $\{Y_n\} \subset \mathscr{Y}, \quad Y \in \mathscr{H}_{\mathscr{J}}, \quad$ and $\quad \|Y_n - Y\| \to 0,$
then $F_n(Y_n) \to F(Y).$

Finally, we define the probability measure Q_n by

(4.9) $\quad dQ_n(Y) := \dfrac{\exp(-nF_n(Y))\, dP_n(\sqrt{n}\, Y)}{\int \exp(-nF_n(Y))\, dP_n(\sqrt{n}\, Y)}.$

Here and below, all integrals involving P_n, P, Q_n are integrals over $\mathscr{Y}$ unless otherwise noted. Using property (1.4) of the measure ρ, one may show that the integral in the denominator of Q_n converges.

The connection between all these definitions and the model in Section 1 is revealed by the following lemma.

Lemma 4.1. *The partition function* Z_n *defined in* (1.2) *has the representation*

(4.10) $$Z_n = \int \exp(-nF_n(Y))\, dP_n(\sqrt{n}\,Y).$$

Remark 4.2. We discuss what would happen if the factor $\frac{1}{n}$ multiplying J in (1.1)-(1.2) were replaced by another positive factor $\frac{1}{\mu(n)}$. Then (4.10) would become

(4.11) $$Z_n = \int \exp(-nF_n(Y))\, dP_n(\sqrt{\mu(n)}\,Y).$$

If $\frac{n}{\mu(n)} \to \text{const} > 0$, then the only changes in our limit results are changes of scale (unless const = 1). If $\frac{n}{\mu(n)} \to \infty$, then it is impossible to define the specific free energy for the system and there are no limit theorems. If $\frac{n}{\mu(n)} \to 0$, then $(\mu(n))^{-1} \ln Z_n \to -\inf I$ over $\mathscr{Y}$, there is exactly one non-degenerate minimum point at $Y^* = 0$, and the only limit theorems are $\frac{S_n}{\mu(n)} \xrightarrow{\mathscr{D}} 0$ and $\frac{S_n}{\sqrt{\mu(n)}} \xrightarrow{\mathscr{D}}$ Gaussian. Thus, the choice $\mu(n) = n$ is essentially the only interesting one.

Proof. We use Fourier analysis for real-valued functions on the group Z/nZ. Given $X_1, \ldots, X_n$ real, we define numbers $\left\{x(p);\ |p| \leqslant \frac{n-1}{2}\right\}$ by

$$x(p) := \frac{1}{n} \sum_{j=1}^{n} X_j B_p\left(\frac{j}{n}\right), \qquad |p| \leqslant \frac{n-1}{2}.$$

Then

(4.12) $$X_j = \sum_{|p| \leqslant \frac{n-1}{2}} x(p) B_p\left(\frac{j}{n}\right).$$

The quadratic form in the exponent of (1.2) can be rewritten as

(4.13) $$\frac{1}{n} \sum_{j,l=1}^{n} J\left(\frac{l}{n} - \frac{j}{n}\right) X_l X_j = n \sum_{|p| \leqslant \frac{n-1}{2}} \hat{J}_n(p)(x(p))^2.$$

We substitute this into (1.2), then introduce new variables $\{y(p): |p| \leqslant \leqslant \frac{n-1}{2}\}$:

$$\prod_{|p| \leqslant \frac{n-1}{2}} \exp\left(\frac{1}{2} n\hat{J}(p)(x(p))^2\right) =$$

$$= \prod_{|p| \leqslant \frac{n-1}{2}} \left\{(2\pi \hat{J}_n(p))^{-\frac{1}{2}} \int_{R^n} \exp\left(\sqrt{n}x(p)y(p) - \right.\right.$$

$$\left.\left. - \frac{(y(p))^2}{2\hat{J}_n(p)}\right) dy(p)\right\} =$$

(4.14)

$$= \prod (2\pi \hat{J}_n(p))^{-\frac{1}{2}} \int_{R^n} \exp\left(\sum \sqrt{n}x(p)y(p)\right) \times$$

$$\times \exp\left(-\sum \frac{(y(p))^2}{2\hat{J}_n(p)}\right) \prod dy(p).$$

Now use Parseval's identity

$$\sum_{|p| \leqslant \frac{n-1}{2}} x(p)y(p) = \frac{1}{n} \sum_{j=1}^{n} X_j Y_j, \tag{4.15}$$

where $Y_j := \sum_{0 \leqslant p \leqslant \frac{n-1}{2}} y(p) B_p\left(\frac{j}{n}\right)$, and do the integration in (1.2) with respect to $\prod d\rho(X_j)$. Identifying Y_j with $Y\left(\frac{j}{n}\right)$ for $Y \in \mathscr{T}_n$ and using (4.5), we find $Z(n) = \int \exp\left(-nF_n\left(\frac{Y}{\sqrt{n}}\right)\right) dP_n(Y)$, which implies (4.10).■

The next item is to relate the distribution of $\frac{S_n - nc}{n^{1-\gamma}}$ to the measure Q_n in (4.9). Given γ real, we define the probability measure $dQ_{n,\gamma}(Y) := := dQ_n(n^{-\gamma} Y)$; i.e.,

$$dQ_{n,\gamma}(Y) := \frac{\exp\left(-nF_n\left(\frac{Y}{n^{\gamma}}\right)\right) dP_n(n^{\frac{1}{2}-\gamma} Y)}{Z(n)}. \tag{4.16}$$

We used (4.10) to rewrite the denominator of Q_n.

Lemma 4.3. *Let constants* γ *and* c *be given. Define* $h_n\colon \mathscr{Y} \to \boldsymbol{R}$ *by*

$$(4.17) \qquad h_n(Y) := (\hat{J}_n(0))^{-1} \int_0^1 Y\,dt, \qquad Y \in \mathscr{Y},$$

and let $\bar{Y}$ *be any element of* $\mathscr{Y}$ *so that* $h_n(\bar{Y}) = c$. *Define*

$$(4.18) \qquad \sigma^2_{n,\gamma} := (\hat{J}_n(0) n^{1-2\gamma})^{-1}$$

and let $N_{n,\gamma}$ *be an* $N(0, \sigma^2_{n,\gamma})$ *random variable which is independendent of the spin random variables* $\{X_1^{(n)}, \ldots, X_n^{(n)}\}$. *Then for any real number* s

$$(4.19) \qquad \mathrm{Prob}\left\{\frac{S_n - nc}{n^{1-\gamma}} + N_{n,\gamma} \leqslant s\right\} = Q_{n,\gamma}\{Y\colon h_n(Y - n^\gamma \bar{Y}) \leqslant s\};$$

i.e.,

$$(4.20) \qquad \frac{S_n - nc}{n^{1-\gamma}} + N_{n,\gamma} \quad \textit{is distributed by} \quad dQ_{n,\gamma}(\cdot + n^\gamma \bar{Y}) \circ h_n^{-1}.$$

Remark 4.4. For Theorem 2.3, we will pick $c = \gamma = 0$, $\bar{Y} = 0$; for Theorem 2.4, $\gamma = \frac{1}{2}$, $c = (\hat{J}_n(0))^{-1} \int_0^1 Y^*\,dt$, $\bar{Y} = Y^*$; for Theorem 2.5, $\gamma = \dfrac{1}{2k}$ and the same c and $\bar{Y}$ as for Theorem 1.4.

Proof. (4.19) is equivalent to

$$(4.21) \qquad \begin{aligned} &E\left\{\exp\left(r\,\frac{S_n - nc}{n^{1-\gamma}}\right)\right\} = \\ &= \exp\left(-\frac{1}{2} r^2 \sigma_n^2\right) \int \exp\left(r[h_n(Y) - n^\gamma c]\right) dQ_{n,\gamma}(Y), \end{aligned}$$

each r real, since h_n is linear, $h_n(\bar{Y}) = c$, N_n is independent of S_n, and $E \exp(rN_n) = \exp\left(\frac{1}{2} r^2 \sigma_n^2\right)$. It suffices to prove (4.21) for $c = 0$ since both sides contain the same multiplicative factor $\exp(-rn^\gamma c)$. We write $Z(n, H)$, $F_n(Y, H)$ for $Z(n)$, $F_n(Y)$. Then for r real

$$E\{\exp\left(r\frac{S_n}{n^{1-\gamma}}\right)\} = \frac{Z\left(n, H+\frac{r}{n^{1-\gamma}}\right)}{Z(n,H)} =$$

$$= \frac{\int \exp\left(-nF_n\left(Y, H+\frac{r}{n^{1-\gamma}}\right)\right) dP_n(\sqrt{n}\,Y)}{Z(n,H)} =$$

$$(4.22) \qquad = \frac{\int \exp\left(-nF_n\left(Y+\frac{r}{n^{1-\gamma}}, H\right)\right) dP_n(\sqrt{n}\,Y)}{Z(n,H)} =$$

$$= \frac{\int \exp(-nF_n(Y,H))\, dP_n\left(\sqrt{n}\left(Y-\frac{r}{n^{1-\gamma}}\right)\right)}{Z(n,H)} =$$

$$= \frac{\exp\left(\frac{-r^2\sigma_n^2}{2}\right)\int \exp(rh_n(n^\gamma Y))\exp(-nF_n(Y,H))\, dP_n(\sqrt{n}\,Y)}{Z(n,H)}.$$

Changing variables $Y \to n^{-\gamma}Y$, we obtain (4.21) with $c = 0$.∎

In order to state the limit results for $Q_{n,\gamma}$, we need more definitions. If Y^* is a non-degenerate minimum point of G, then one may prove that the operator $\mathscr{I}_{Y^*} := (\mathscr{K}_{Y^*})^{-1}$ defined in (3.29) is symmetric, positive-definite, and trace-class. Thus $\mathscr{I}_{Y^*}$ is the covariance operator of a mean zero Gaussian measure on $\mathscr{Y}$ (Varadhan [19], p. 72), which we call P_{Y^*}. Given T a trace class operator on $L^2[0,1]$ with eigenvalues $\{\mu_j\}$, we define

$$(4.23) \qquad \operatorname{Det}(I-T) := \prod_j (1-\mu_j).$$

$\operatorname{Det}(I-T)$ converges nicely and is called the characteristic determinant of T (Gohberg – Kreĭn [12], § 4.1). For Y^* a non-degenerate minimum point, the quantity

$$(4.24) \qquad A_{Y^*} := \left(\operatorname{Det}\left(I-\sqrt{\mathscr{I}}\,M_{Y^*}\sqrt{\mathscr{I}}\right)\right)^{-\frac{1}{2}}$$

arises. A_{Y^*} is well-defined since $\sqrt{\mathscr{I}}\,M_{Y^*}\sqrt{\mathscr{I}}$ is trace-class and $I-\sqrt{\mathscr{I}}\,M_{Y^*}\sqrt{\mathscr{I}}$ is positive-definite.

In the limit theorems, the two types of convergence which arise are weak convergence and vague convergence on $\mathscr{Y}$. The first was defined after (4.6). To define the second, we say that a function $\psi\colon \mathscr{Y} \to R$ vanishes uniformly at infinity if for any $\epsilon > 0$ there exists $D \in (0, \infty)$ so that $|\psi(Y)| < \epsilon$ whenever $\|Y\| > D$. Given probability measures $\{\xi_n;\ n = 1, 2, \ldots\}$ and a measure ξ on $\mathscr{Y}$, we say that the ξ_n tend vaguely to ξ, and write $\xi_n \xrightarrow{v} \xi$, if $\int \psi d\xi_n \to \int \psi d\xi$ for all $\psi \in C(\mathscr{Y})$ which vanish uniformly at infinity. For $Y \in \mathscr{Y}$, δ_Y denotes the point mass at Y.

Theorem 4.5. *Assume that G has finitely many minimum points, $Y_1^*, \ldots, Y_\alpha^*$, some $\alpha \geqslant 1$. If $\alpha = 1$, then*

$$(4.25)\qquad Q_n \Longrightarrow \delta_{Y_j^*}.$$

If $\alpha \geqslant 1$, assume either that each Y_j^* is non-degenerate or that H is a constant function and at least one Y_j^* is degenerate. For $j = 1, \ldots, \alpha$, let k_j and λ_j denote the type and strength, respectively of Y_j^* and define

$$(4.26)\qquad \bar{b}_j := \begin{cases} A_{Y_j^*} \quad \text{if} \quad k_j = 1, & \text{if} \quad k_j = 1, \\ \int_R \exp\left(-\dfrac{\lambda_j u^{2k_j}}{(2k_j)!}\right) du & \text{if} \quad k_j \geqslant 2. \end{cases}$$

Define the maximal type $\bar{k}$ by

$$(4.27)\qquad \bar{k} := \max\{k_j\colon j = 1, \ldots, \alpha\}$$

and let

$$(4.28)\qquad \tilde{b}_j := \begin{cases} 0 & \text{if} \quad k_j < \bar{k} \\ \dfrac{\bar{b}_j}{\sum\limits_{\{k_j = \bar{k}\}} \bar{b}_j} & \text{if} \quad k_j = \bar{k}. \end{cases}$$

Then

$$(4.29)\qquad Q_n \Longrightarrow \sum_{j=1}^{\alpha} b_j \delta_{Y_j^*}.$$

We note that if $\alpha = 1$, then (4.25) holds even if Y_1^* is a non-constant, degenerate minimum point.

The next two theorems give the fluctuations about the minimum points Y_j^*, first in the non-degenerate case, then in the degenerate case. We have stated them separately to emphasize the very different nature of the limits: Gaussian measures on all of $\mathscr{Y}$ versus measures concentrated on a one-dimensional subspace of $\mathscr{Y}$.

Theorem 4.6. *Assume that G has finitely many minimum points $Y_1^*, \ldots, Y_\alpha^*$, some $\alpha \geqslant 1$, and that each Y_j^* is non-degenerate. Then for each $j = 1, \ldots, \alpha$*

$$(4.30) \qquad Q_{n,\frac{1}{2}}(\cdot + \sqrt{n}\, Y_j^*) \begin{cases} \Longrightarrow P_{Y_1^*} & \text{if} \quad \alpha = 1, \\ \xrightarrow{\nu} \widetilde{b}_j P_{Y_j^*} & \text{if} \quad \alpha \geqslant 2. \end{cases}$$

In this case, $\widetilde{b}_j := \dfrac{A_{Y_j^*}}{\sum_{j=1}^{\alpha} A_{Y_j^*}}$.

Given a probability measure ξ on $\boldsymbol{R}$, $\xi^\#$ denotes the probability measure on $\mathscr{Y}$ defined by

$$(4.31) \qquad \int_{\mathscr{Y}} \psi\, d\xi^\# := \int_{\boldsymbol{R}} \psi(u1)\, d\xi(u), \qquad \psi \in C(\mathscr{Y}).$$

Given $k \geqslant 2$ an integer and $\lambda > 0$, we define $\xi_{k,\lambda}$ to be the probability measure on $\boldsymbol{R}$

$$(4.32) \qquad d\xi_{k,\lambda}(u) := \frac{\exp\left(-\dfrac{\lambda}{(2k)!}\, u^{2k}\right) du}{\int_{\boldsymbol{R}} \exp\left(-\dfrac{\lambda}{(2k)!}\, u^{2k}\right) du}$$

Theorem 4.7. *Assume that H is a constant function. Assume that G has finitely many minimum points $Y_1^*, \ldots, Y_\alpha^*$, $\alpha \geqslant 1$, such that $\alpha_1 \geqslant 1$ of them are degenerate and the rest non-degenerate. Let k_j, λ_j, b_j ($j = 1, \ldots, \alpha$) and $\bar{k}$ be as in Theorem 4.5. Then for each $j = 1, \ldots, \alpha$,*

$$(4.33) \qquad Q_{n,\frac{1}{2k_j}}(\cdot + n^{\frac{1}{2k_j}} Y_j^*) \xrightarrow{\nu} \widetilde{b}_j \xi^\#_{k_j,\lambda_j} \quad \text{if} \quad k_j = \bar{k}.$$

If $\alpha = 1$ *or, more generally, if there exists a unique index* j *(say* $j = 1$*) so that* $k_j = \bar{k}$, *then* (4.33) *can be strengthened to*

$$(4.34)\qquad Q_{n,\frac{1}{2k_1}}(\cdot + n^{\frac{1}{2k}} Y_1^*) \Longrightarrow \xi^{\#}_{k_1,\lambda_1}.$$

The next theorem is obtained from a result which arises in the proofs of Theorems 4.5-4.7. Setting $\psi \equiv 1$ in (4.36) gives the first term in the asymptotic expansion of the integral (4.10) which represents $Z(n)$. To simplify the statement, we assume that G has a unique minimum point Y^*. We define

$$(4.35)\qquad \widetilde{A} := \Big[\prod_{p \neq 0} (1 - \hat{J}(p)) \Big]^{-\frac{1}{2}};$$

the infinite product converges and is non-zero because of our assumptions on J and (3.4).

Theorem 4.8. *Assume that* G *has a unique minimum point* Y^*. *Assume that* Y^* *is non-degenerate or that* H *is a constant function and* $Y^* = y^*$, *a constant, is degenerate. Let* k *and* λ *denote the type and strength, respectively, of* Y^*. *Then as* $n \to \infty$,

$$e^{nG(Y^*)} \int \psi(Y) \exp(-nF_n(Y))\, dP_n(\sqrt{n}Y) =$$

$$(4.36)\qquad = \begin{cases} \psi(Y^*) A_{Y^*} + o(1), & \text{if} \quad k = 1, \\ n^{\frac{1}{2} - \frac{1}{2k}} \Big(\dfrac{\widetilde{A}}{\sqrt{2\pi}} \psi(y^*) \displaystyle\int_R \exp\Big(-\Big(\frac{\lambda}{(2k)!} u^{2k}\Big)\Big) du + o(1) \Big), \\ & \text{if} \quad k \geqslant 2, \end{cases}$$

for all $\psi \in C(\mathscr{Y})$.

We end this section by discussing the asymptotics of the random field $\{S_{n,W};\ W \in \mathscr{Y}\}$, where $S_{n,W} := \sum_{j=1}^{n} W\big(\frac{j}{n}\big) X_j^{(n)}$. To simplify the statements, we assume that G has a unique minimum point Y^*. We have proved the theorem under the hypothesis that $W \in \mathscr{T}_m$, some $m \geqslant 1$ ($\mathscr{T}_m$ is defined in (4.1)). We have do doubt that by an approximation argument one can extend the theorem to cover all $W \in \mathscr{D}(\mathscr{I}^{-1})$.

Theorem 4.9. *Let G have a unique minimum point Y^* and pick $W \in \mathscr{T}_m$, some $m \geqslant 1$. Then*

$$(4.37)\qquad \frac{S_{n,W}}{n} \xrightarrow{\mathscr{D}} (\mathscr{I}^{-1} W, Y^*).$$

Assume further that Y^* is non-degenerate and that W is not identically zero. Define

$$(4.38)\qquad \sigma^2_{Y^*,W} := (\mathscr{I}_{Y^*} \mathscr{I}^{-1} W, \mathscr{I}^{-1} W) - (\mathscr{I}^{-1} W, W).$$

Then $\sigma^2_{Y^*,W} > 0$ and

$$(4.39)\qquad \frac{S_{n,W} - n(\mathscr{I}^{-1} W, Y^*)}{\sqrt{n}} \xrightarrow{\mathscr{D}} \frac{1}{\sqrt{2\pi\sigma^2_{Y^*,W}}} \exp\left(-\frac{u^2}{2\sigma^2_{Y^*,W}}\right) du.$$

Now assume that H is a constant function and $Y^* = y^*$, a constant, is degenerate of type $k \geqslant 2$. Then

$$\frac{S_{n,W} - ny^*(1, W)}{n^{1-\frac{1}{2k}}} \xrightarrow{\mathscr{D}}$$

$$(4.40)\qquad \xrightarrow{\mathscr{D}} \begin{cases} \delta_0 & \text{if} \quad (1, W) = 0, \\ \dfrac{\exp\left(-\lambda_W \dfrac{u^{2k}}{(2k)!}\right) du}{\int_R \exp\left(-\lambda_W \dfrac{u^{2k}}{(2k)!}\right) du} & \text{if} \quad (1, W) \neq 0, \end{cases}$$

where $\lambda_W := \dfrac{\lambda}{(1, W)^{2k}}$, λ being the strength of Y^*.

The limits (4.37) and (4.40) are consistent. If $Y^* = y^*$, a constant, then the right hand side of (4.37) is $y^*(1, W)$. We point out two interesting features of (4.37). Applying (4.37) repeatedly with $W = B_p$, $p \in Z$, we are able to reconstruct Y^*. Indeed, for each $W = B_p$, (4.37) gives us the Fourier coefficient (B_p, Y^*). Also, if we take a sequence of W's to approximate the characteristic function of an interval Φ of $[0, 1]$, (4.37) suggests that

$$(4.41)\qquad \frac{1}{n}\sum_{\frac{j}{n}\in\Phi} X_j^{(n)} \xrightarrow{\mathscr{D}} \delta_{\int_\Phi \mathscr{I}^{-1}Y^*},$$

i.e., in the thermodynamic limit, (1.1)-(1.2) is described by the non-random field

$$(4.42)\qquad X(t) := (\mathscr{I}^{-1}Y^*)(t) = \varphi'(Y^*(t) + H(t)), \qquad t\in[0,1].$$

This should be compared with Ellis – Newman [8], Theorem 8. We say a word concerning the proof of Theorem 4.9. For $n \geqslant m$, Lemma 4.3 is valid with $S_{n,W}$ replacing S_n, $\sigma^2_{n,\gamma,W} := \dfrac{(\mathscr{I}_n^{-1}W, W)}{n^{1-2\gamma}}$ replacing $\sigma^2_{n,\gamma}$, $N_{n,\gamma,W} := N(0, \sigma^2_{n,\gamma,W})$ replacing $N_{n,\gamma}$, and $h_{n,W} := (\mathscr{I}_n^{-1}W, Y)$ replacing h_n. The proof is the same (we need $W\in\mathscr{T}_m$ with $m\leqslant n$ so that $\tau_n W = W$; this is used in the third equality of (4.22)). The theorem now follows from Theorems 4.5-4.7. To show $\sigma^2_{Y^*,W} > 0$, we use (3.32) with W replacing 1 and find

$$(4.43)\qquad \sigma^2_{Y^*,W} = \sum_{i=0}^{\infty} (\|\sqrt{\mathscr{M}}\,(\mathscr{I}\mathscr{M})^i W\|_2^2 + \|\sqrt{\mathscr{I}}\,\mathscr{M}(\mathscr{I}\mathscr{M})^i W\|_2^2) > 0.$$

5. PROOFS OF THEOREMS OF SECTION 2

We first prove Theorem 2.2, then Theorems 2.3-2.5. We give the more gereral statements of Theorems 2.4-2.5 without the assumption that there exists a unique minimum point. Finally, we prove the first two parts of Theorem 2.1, then prove (2.7) in the special case that H is a constant function.

Proof of Theorem 2.2. The first assertion follows from (5.15) (see below), which implies $G_{\text{c.w.}}(y) \to +\infty$ as $|y| \to +\infty$, and from the real analyticity of $G_{\text{c.w.}}$. The latter is a consequence of the real analyticity of φ.

Assuming $H = h$, we prove that if Y is a non-constant function, then $G(Y) > \min G$ over $\mathscr{Y}$. Then the second assertion in the theorem follows from the fact that G and $G_{\text{c.w.}}$ agree on constants. Define

$$(5.1)\qquad a := \min\{G(Y)\colon\ Y \text{ constant}\} =$$

$$= \min_{R} G_{\text{c.w.}} = \min \left\{\frac{y^2}{2} - \varphi(y+h) \colon y \text{ real}\right\}.$$

Thus, for any $Y \in \mathscr{Y}$ and $t \in [0, 1]$,

$$\frac{(Y(t))^2}{2} - \varphi(Y(t) + h) \geqslant a, \tag{5.2}$$

so that

$$\begin{aligned} &\frac{1}{2} \sum_{p \in Z} (y(p))^2 - \int_0^1 \varphi(Y(t) + h)\, dt = \\ &= \frac{1}{2} \int_0^1 Y^2\, dt - \int_0^1 \varphi(Y + h)\, dt \geqslant a. \end{aligned} \tag{5.3}$$

The first equality uses (3.9). Then

$$\begin{aligned} G(Y) &= \frac{1}{2} \sum_{p \in Z} (\hat{J}(p))^{-1} (y(p))^2 - \int_0^1 \varphi(Y + h)\, dt \geqslant \\ &\geqslant a + \frac{1}{2} \sum_{p \neq 0} (\hat{J}^{-1}(p) - 1) y(p)^2. \end{aligned} \tag{5.4}$$

If Y is non-constant, then $y(p)$ is non-zero for some $p \neq 0$. Hence $G(Y) > a$ since $\hat{J}^{-1}(p) > 1$ for all $p \neq 0$ (by (3.4)). This is what we wanted to show.

Now assume that G has a minimum point Y^* which is a constant y^*. We prove that H is a constant function. By (3.19),

$$y^* = (\mathscr{J}^{-1} Y^*)(t) = \varphi'(y^* + H(t)) \quad \text{for all } t \in [0, 1]. \tag{5.5}$$

Since φ' is real analytic, (5.5) implies $y^* + H(t)$ is a constant for all $t \in$ $\in [0, 1]$. Thus, H is a constant function.

The rest of Theorem 2.2 is taken from Ellis – Newman [6].∎

The proofs of Theorems 2.3-2.5 are based in part upon the following fact.

Lemma 5.1. *Suppose that for each* n, T_n *and* U_n *are independent random variables such that* $U_n \xrightarrow{\mathscr{D}} \xi$, *where* $\int \exp(iru)\, d\xi(u) \neq 0$ *for all*

$r \in R$. *Then* $T_n \to \mu$ *if and only if* $T_n + U_n \xrightarrow{\mathscr{D}} \mu * \xi$.

Proof. Weak convergence of measures on R is equivalent to pointwise convergence of their characteristic functions.∎

Proof of Theorem 2.3. We prove (2.10)-(2.11), where

$$(5.6) \qquad b_j := \sum_{\left\{i:\, \int_0^1 Y_i^* dt = \bar{y}_j\right\}} \widetilde{b}_i, \qquad j = 1, \dots, \theta;$$

the $\{\widetilde{b}_i\}$ are defined in (4.28). We use (4.20) with $c = \gamma = 0$ and (4.25), (4.29). As $n \to \infty$, $h_n(Y) \to h(Y) := \int_0^1 Y\, dt$, uniformly on compact subsets of $\mathscr{Y}$. By Billingsley [2], p. 34.

$$(5.7) \qquad Q_n \circ h_n^{-1} \Rightarrow Q \circ h^{-1},$$

where Q stands for the right hand side of (4.25) or (4.29). We are done since $N_{n,0} \longrightarrow \delta_0$ $(\sigma_{n,0} \to 0)$ and $N_{n,0}$ is independent of S_n.∎

Proof of Theorem 2.4. We prove (2.12), where

$$(5.8) \qquad \sigma^2_{Y^*} := (\lambda(Y^*))^{-1} - 1;$$

$\lambda(Y^*)$ is the strength of the non-degenerate minimum point Y^*, defined in Definition 3.3; $\sigma^2_{Y^*} > 0$ by Lemma 3.4. We use (4.20) with $c = (\hat{J}_n(0))^{-1} \int_0^1 Y^*\, dt$, $\gamma = \frac{1}{2}$, $\bar{Y} = Y^*$, together with the case $\alpha = 1$ of (4.30) with $Y_1^* = Y^*$. By Billingsley [2], p. 34,

$$(5.9) \qquad Q_{n,\frac{1}{2}}(\cdot + \sqrt{n}\, Y^*) \circ h_n^{-1} \Rightarrow P_{Y^*} \circ h^{-1},$$

where $h(Y) := \int_0^1 Y\, dt$, $Y \in \mathscr{Y}$. Given an interval Φ of R, $P_{Y^*} \circ h^{-1}(\Phi) = P_{Y^*}\left\{Y \mid \int_0^1 Y\, dt \in \Phi\right\}$. We claim that under P_{Y^*}, $\int_0^1 Y\, dt$ is distributed like an $N(0, (\lambda(Y^*))^{-1})$ random variable. Indeed, under P_{Y^*}, we may express

$$(5.10)\qquad Y = \sum_{j=1}^{\infty} N_j(0, \mu_j) S_j,$$

where $\{\mu_j;\ j = 1, 2, \ldots\}$, $\{S_j;\ j = 1, 2, \ldots\}$ denote the eigenvalues (all positive) and corresponding eigenvectors of $\mathscr{I}_{Y^*}$ and $\{N_j(0, \mu_j);\ j = 1, 2, \ldots\}$ are independent normal random variables with mean 0 and variances $\{\mu_j\}$. Thus,

$$(5.11)\qquad \int_0^1 Y\,dt \ \text{ is distributed like } \ N\Big(0, \sum_{j=1}^{\infty} \mu_j \Big(\int_0^1 S_j\,dt\Big)^2\Big);$$

but $(\lambda(Y^*))^{-1} := (\mathscr{I}_{Y^*}1, 1) = \sum_{j=1}^{\infty} \mu_j \Big(\int_0^1 S_j\,dt\Big)^2$. This proves the claim. Since $c := (\hat{J}_n(0))^{-1} \int_0^1 Y^*\,dt \to \int_0^1 Y^*\,dt$ and $N_{n,\gamma} \to N(0,1)$ in (4.20), we are done by Lemma 5.1.∎

Proof of Theorem 2.5. This follows in identical fashion from (4.20) with $c := (\hat{J}_n(0))^{-1} y^*$, $\gamma = \frac{1}{2k}$, $\bar{Y} = Y^*$, and from (4.34) with $k_1 = k$, $\lambda_1 = \lambda$, $Y_1^* = Y^*$.∎

We give the more general statements of Theorems 2.4-2.5 without the assumption that there exists a unique minimum point. Under the same hypotheses as Theorem 4.6 and with the same notation as Theorems 2.3, 2.4, 4.6, we have as a generalization of (2.12)

$$(5.12)\qquad \frac{S_n - n\int_0^1 Y_j^*\,dt}{\sqrt{n}} \begin{cases} \xrightarrow{\mathscr{D}} dN_{\sigma^2_{Y^*}} & \text{if } \alpha = 1, \\[2ex] \xrightarrow{v} \displaystyle\sum_{\{i:\ \int_0^1 Y_i^* dt = \int_0^1 Y_j^* dt\}} \widetilde{b}_i\, dN_{\sigma^2_{Y_i^*}} & \text{if } \alpha \geqslant 2, \end{cases}$$

for each $j = 1, \ldots, \alpha$. We emphasize that in general the second line is a sum of Gaussian measures which is not Gaussian unless that $\{\sigma^2_{Y_i^*}\}$ are all the same. If the numbers $\Big\{\int_0^1 Y_i^*\,dt\Big\}$ are all the same, then the second line of (5.12) can be strengthened to convergence in distribution.

Under the same hypotheses as Theorem 4.7 and with the same notation as Theorems 2.3, 2.5, 4.7, we have as a generalization of (2.13)

$$(5.13)\qquad \frac{S_n - ny_j^*}{n^{1-\frac{1}{2k_j}}} \begin{cases} \xrightarrow{\mathscr{D}} \delta_0 & \text{if } k_j < \bar{k}, \\ \xrightarrow{\nu} \tilde{b}_j \xi_{k_j, \lambda_j} & \text{if } k_j = \bar{k}, \end{cases}$$

for each $j = 1, \ldots, \alpha$. In (5.13), y_j^* is the value of the constant function Y_j^*. There is a strengthening of (5.13) which corresponds to (4.34). (The results (5.12)-(5.13) reduce to (2.2) if all the numbers $\{\int_0^1 Y_j^* \, dt;\ j = 1, \ldots, \alpha\}$ are distinct; then $\theta = \alpha$.) To prove (5.12)-(5.13), one uses Lemma 4.3 together with results from which Theorems 4.5-4.7 are proved; unlike the proofs of Theorems 2.4-2.5, Theorems 4.6-4.7 are not enough. The reason is that unlike weak convergence, vague convergence on $\mathscr{Y}$ does not in general carry over to vague convergence on R under a sequence of continuous maps (like h_n).

We now prove the first two parts of Theorem 2.1, then prove (2.7) in the case that H is a constant function.

Proof of first two parts of Theorem 2.1. The operator $\sqrt{\mathscr{I}}$ is defined after (3.10) and is invertible. The range of $\sqrt{\mathscr{I}}$ is the space $\mathscr{H}_{\mathscr{I}}$ defined in (3.12). We proved $\mathscr{H}_{\mathscr{I}} \subset \mathscr{Y}$ in (3.14).

We next show that $\inf G$ over $\mathscr{Y}$ is attained at some point of $\mathscr{Y}$. Given $\delta > 0$, we prove that there exists a constant B so that

$$(5.14)\qquad F(Y) \geqslant -\delta \| Y \|_2^2 - B \quad \text{for all} \quad Y \in \mathscr{Y}.$$

For any $\delta > 0$, x, y real, we have $yx \leqslant \frac{\delta}{2} y^2 + \frac{x^2}{2\delta^2}$. Thus

$$(5.15)\qquad \varphi(y) \leqslant \frac{\delta}{2} y^2 + \bar{B}, \quad \bar{B} := \ln \int \exp\left(\frac{x^2}{2\delta^2}\right) d\rho(x);$$

$\bar{B} < \infty$ because of (1.4). We have

$$(5.16)\qquad F(Y) = -\int_0^1 \varphi(Y + H)\, dt \geqslant -\frac{\delta}{2} \| Y + H \|_2^2 - \bar{B} \geqslant$$

$$\geqslant -\delta \| Y \|_2^2 - (\bar{B} + \delta \| H \|_2^2),$$

as claimed. Since $\| \mathscr{J}^{\frac{1}{2}} \|_{2,2} = 1$, we have

$$F(Y) \geqslant -\delta \| Y \|_{\mathscr{J}}^2 - B \quad \text{for all} \quad Y \in \mathscr{H}_{\mathscr{J}}. \tag{5.17}$$

It is clear that if G takes on its minimum over $\mathscr{Y}$, then it can only be at a point of $\mathscr{H}_{\mathscr{J}}$. On $\mathscr{H}_{\mathscr{J}}$, G is a bounded functional, and by (5.17), G is coercive (take $0 < \delta < 1$). One can easily show that G is weakly lower semicontinuous on $\mathscr{H}_{\mathscr{J}}$, so that we are done by Berger [1], p. 301.∎

Before we prove (2.7) in the special case that H is a constant function, we comment upon the proof in the general case, which will appear in Ellis – Rosen [10]. By Lemma 4.1, we must consider $n^{-1} \ln \int \exp(-nF_n(Y))\, dP_n(\sqrt{n}\, Y)$. If one replaces F_n by the functional F and P_n by the measure P (defined after (4.5)), then in Simon [16], § 18, or Donsker – Varadhan [5], § 6, it is shown that the resulting expression tends to $\inf G$ over $\mathscr{Y}$. In view of (4.6), (4.8), the statement (2.7) is then not surprising. We prove (2.7) by extending Simon's methods to the n-dependent case. In Varadhan [18], § 3, it is shown that (2.7) follows if certain asymptotic properties of the measures P_n are valid (see (3.1) in that paper). While we are able to verify these properties, we prefer the self-contained proof based on Simon [16].

Proof of (2.7) for constant H. To ease the notation, we take $H = 0$; $H = h$, a non-zero constant, is handled similarly. By Lemma 4.1 and Theorem 2.2, it suffices to prove

$$\begin{aligned} &\frac{1}{n} \ln \int \exp\left(-F_n\left(\frac{Y}{\sqrt{n}}\right)\right) dP_n(Y) \to \\ &\to -\inf\left\{\frac{y^2}{2} - \varphi(y) \colon y \text{ real}\right\} = \sup\left\{\varphi(y) - \frac{y^2}{2} \colon y \text{ real}\right\}. \end{aligned} \tag{5.18}$$

We prove this assuming that J satisfies the hypotheses of Section 3 except that $J \in C^\infty$ is replaced by

$$\hat{J}(p) \leqslant \frac{\text{const}}{|p|^{2+\epsilon}}, \quad \text{some } \epsilon > 0, \text{ all } p \in Z. \tag{5.19}$$

In E l l i s – R o s e n [10], we will prove (2.7) for general $H \in \mathscr{Y}$ assuming only $\sum_{p \in Z} p\hat{J}(p) < \infty$ instead of (5.19) and dropping (3.3). The latter will be used only to prove the probabilistic limit theorems (Theorems 2.3-2.5, 4.5-4.9).

We still work with odd n. The set $\mathscr{T}_n$, defined in (4.1), is isomorphic to R^n by the correspondence

$$(5.20)\qquad Y := \sum_{|p| \leqslant \frac{n-1}{2}} y(p) B_p \in \mathscr{T}_n \leftrightarrow \{y(p)\colon\ |p| \leqslant \frac{n-1}{2}\} \in R^n.$$

Because of this, we consider $\prod dy(p)$ to be a measure on $\mathscr{T}_n$. (Here and below, all products and sums over p are either for $|p| \leqslant \frac{n-1}{2}$, in which case the limits are omitted, or for $1 \leqslant |p| \leqslant \frac{n-1}{2}$, in which case we write $\prod'$ and $\sum'$.) We define

$$(5.21)\qquad b := \sup\left\{\varphi(y) - \frac{y^2}{2}\colon\ y \text{ real}\right\};$$

$b < \infty$ because of (5.15). We also define

$$(5.22)\qquad \begin{aligned} V_n(Y) &:= \prod (2\pi \hat{J}_n(p))^{-\frac{1}{2}} \exp\left(\sum_{j=1}^{n} \varphi\left(\frac{Y\left(\frac{j}{n}\right)}{\sqrt{n}}\right)\right) \times \\ &\times \exp\left(-\frac{1}{2} \sum \frac{(y(p))^2}{\hat{J}_n(p)}\right), \qquad Y \in \mathscr{T}_n. \end{aligned}$$

By (4.5), the left hand side of (5.18) can be written as

$$(5.23)\qquad \int_{\mathscr{T}_n} V_n(Y) \prod dy(p).$$

We prove (5.18) by proving

$$(5.24)\qquad \liminf_{n \to \infty} \frac{1}{n} \ln \int_{\mathscr{T}_n} V_n \prod dy(p) \geqslant b,$$

$$(5.25)\qquad \limsup_{n \to \infty} \frac{1}{n} \ln \int_{\mathscr{T}_n} V_n \prod dy(p) \leqslant b.$$

Proof of lower bound (5.24). Given $0<\delta<\frac{\epsilon}{2}$, $\nu>0$, $M>0$, we define the subset Θ_n of $\mathscr{T}_n$ by

$$\Theta_n := \Big\{Y\colon\ |y(0)|\leqslant M\sqrt{n};\ |y(p)|\leqslant \frac{\nu\sqrt{n}}{|p|^{1+\delta}},$$

(5.26)

$$1\leqslant |p|\leqslant \frac{n-1}{2}\Big\}.$$

We choose M so that there is some $y\in[-M,M]$ with $\varphi(y)-\frac{y^2}{2}=b$. For any $Y\in\Theta_n$, we have

(5.27) $$\Big\|\frac{Y}{\sqrt{n}}\Big\|\leqslant\sqrt{2}\ \sum\frac{|y(p)|}{\sqrt{n}}\leqslant C<\infty,$$

(5.28) $$\Big\|\frac{Y-y(0)}{\sqrt{n}}\Big\|\leqslant\sqrt{2}\sum{}'\ \frac{|y(p)|}{\sqrt{n}}\leqslant(\text{const})\,\nu,$$

estimates valid for all Θ_n. Let $\mu>0$ be given. By the uniform continuity of φ on $[-C,C]$ and by (5.28), we may pick ν so small that

$$\varphi\left(\frac{Y\left(\frac{j}{n}\right)}{\sqrt{n}}\right)\geqslant\varphi\Big(\frac{y(0)}{\sqrt{n}}\Big)-\mu,\quad\text{all}\quad n\geqslant 1,\ j=1,\ldots,n,\ Y\in\Theta_n.$$

Now

$$\int_{\mathscr{T}_n} V_n\ \Pi\, dy(p)\geqslant\int_{\Theta_n} V_n\ \Pi\, dy(p)\geqslant$$

$$\geqslant e^{-n\mu}\int_{|y(0)|\leqslant M\sqrt{n}}\exp\Big[n\varphi\Big(\frac{y(0)}{\sqrt{n}}\Big)-(\hat{J}_n^{-1}(0))\frac{(y(0))^2}{2}\Big]\times$$

(5.29) $$\times\frac{dy(0)}{\sqrt{2\pi\hat{J}_n(0)}}\ \Pi'\ (2\pi\hat{J}_n(p))^{-\frac{1}{2}}\int_{\Gamma_n}\exp\Big[-\sum{}'\ \hat{J}_n^{-1}(p)\,\frac{(y(p))^2}{2}\Big]\times$$

$$\times\ \Pi'\ dy(p).$$

where $\Gamma_n := \Big\{|y(p)|\leqslant\frac{\nu\sqrt{n}}{|p|^{1+\delta}},\ 1\leqslant|p|\leqslant\frac{n-1}{2}\Big\}$.

Here and below, we write $\hat{J}_n^{-1}(p)$ for $(\hat{J}_n(p))^{-1}$. The last integral in (5.29) can be written as

$$(5.30)\qquad \zeta_n\left(\left\{Y:\ |y(p)|\leqslant \frac{\nu\sqrt{n}}{|p|^{1+\delta}},\ 1\leqslant |p|\leqslant \frac{n-1}{2}\right\}\right),$$

where ζ_n is the probability measure on $\boldsymbol{R}^{n-1}$ defined by

$$(5.31)\qquad \zeta_n := \prod{}'(2\pi\hat{J}_n(p))^{-\frac{1}{2}} \exp\left[-\sum{}' \hat{J}_n^{-1}(p)\,\frac{(y(p))^2}{2}\right]\prod{}' dy(p).$$

Below, we prove

$$(5.32)\qquad \zeta_n\left(\left\{Y:\ |y(p)|\leqslant \frac{\nu\sqrt{n}}{|p|^{1+\delta}},\right.\right.$$
$$\left.\left. 1\leqslant |p|\leqslant \frac{n-1}{2}\right\}\right)\to 1\quad \text{as}\quad n\to\infty.$$

We return to (5.29). By (5.19), since $\hat{J}(0) = 1 < \hat{J}_n(0)$,

$$(5.33)\qquad 0 < 1-\hat{J}_n^{-1}(0)\leqslant \hat{J}_n(0)-\hat{J}(0) = \sum_{j\neq 0}\hat{J}(jn)\leqslant \frac{\text{const}}{n^{2+\epsilon}},$$

so that $\hat{J}_n^{-1}(0)\to 1$. By Laplace's method on $\boldsymbol{R}$ (Erdélyi [11], §2.4), we have

$$(5.34)\qquad \lim_{n\to\infty}\frac{1}{n}\ln\int_{|y(0)|\leqslant M}\exp\left[n\left\{\varphi(y(0))-\hat{J}_n^{-1}(0)\,\frac{(y(0))^2}{2}\right\}\right]dy(0) = b$$

and so by (5.32)

$$(5.35)\qquad \liminf_{n\to\infty}\frac{1}{n}\ln\int_{\Upsilon_n} V_n\prod dy(p)\geqslant b-\mu.$$

This gives (5.24) since μ is arbitrary. We prove (5.32). First note that by (3.3) and (5.19),

$$(5.36)\qquad \begin{aligned}&\frac{\hat{J}_n^{-1}(p)}{|p|^{2+2\delta}}\geqslant L^{-1}\frac{\hat{J}^{-1}(p)}{|p|^{2+2\delta}}\geqslant \text{const} > 0,\\ &\quad \text{all}\ \ n\geqslant 1,\ \ |p|\leqslant \frac{n-1}{2}.\end{aligned}$$

Thus,

(5.37)

$$\zeta_n\left(\left\{Y:\ |y(p)| \leqslant \frac{\nu\sqrt{n}}{|p|^{1+\delta}},\ 1 \leqslant p \leqslant \frac{n-1}{2}\right\}^C\right) \leqslant$$

$$\leqslant n \max_{1 \leqslant |p| \leqslant \frac{n-1}{2}} \int_{|y(p)| > \frac{\nu\sqrt{n}}{|p|^{1+\delta}}} e^{-\frac{\hat{J}_n^{-1}(p)(y(p))^2}{2}} (2\pi \hat{J}_n(p))^{-\frac{1}{2}}\, dy(p) \leqslant$$

$$\leqslant 2n \max_{1 \leqslant |p| \leqslant \frac{n-1}{2}} \left\{ \frac{|p|^{1+\delta}}{\nu\sqrt{n}} \frac{(\hat{J}_n(p))^{\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{n\nu^2}{2} \frac{\hat{J}_n^{-1}(p)}{|p|^{2+2\delta}}\right)\right\} \leqslant$$

$$\leqslant (\text{const})\sqrt{n}\,\exp\left(-(\text{const})n\right) \to 0,$$

where const stands for a positive constant. This proves (5.32) and completes the proof of (5.24).

Proof of upper bound (5.25)

Lemma 5.2.

(5.38) $$\hat{J}(p) < \hat{J}_n(p) < \hat{J}(p) + \frac{\text{const}}{n^{2+\epsilon}}, \qquad n \geqslant 1,\ \ |p| \leqslant \frac{n-1}{2};$$

(5.39) $$\frac{\hat{J}_n^{-1}(p) - 1}{|p|^{2+2\delta}} \geqslant \text{const} > 0 \ \ \textit{for} \ \ \delta < \frac{\epsilon}{2},$$

$$n \ \textit{sufficiently large}, \quad 1 \leqslant |p| \leqslant \frac{n-1}{2};$$

(5.40) $$\lim_{n\to\infty} \prod_{1 \leqslant |p| \leqslant \frac{n-1}{2}} (1 - \hat{J}_n(p))^{\frac{1}{2}} \ \ \textit{exists.}$$

Proof. We have $\hat{J}_n(p) = \hat{J}(p) + \sum_{j \neq 0} \hat{J}(p + jn)$. Using (5.19), one can show that the sum is bounded by $\frac{\text{const}}{n^{2+\epsilon}}$. This gives (5.38). For all $p \neq 0$ sufficiently small, (5.39) follows from (5.38) and (3.4). For large p, we have by (3.3) and (5.19), (5.41) $\quad \hat{J}_n^{-1}(p) - 1 > L^{-1}\hat{J}^{-1}(p) - 1 \geqslant \text{const}\,|p|^{2+2\delta}$

This gives (5.39). (5.40) holds since for all n sufficiently large and all $1 \leqslant |p| \leqslant \frac{n-1}{2}$, $1 - \hat{J}_n(p) \geqslant \text{const} > 0$ and since

(5.42) $$0 < \sum_{1 \leqslant |p| \leqslant \frac{n-1}{2}} \hat{J}_n(p) < \sum_{p \in Z} \hat{J}(p) < \infty. \blacksquare$$

In terms of a number $K > 0$ to be picked later, we define three sets which partition $\mathscr{T}_n$:

$$\Theta_n(1) := \{Y\colon\ |y(0)| \leqslant K\sqrt{n}\},$$

(5.43) $$\Theta_n(2) := \{Y\colon\ |y(0)| > K\sqrt{n}\} \cap \Lambda_n,$$

$$\Theta_n(3) := \{Y\colon\ |y(0)| > K\sqrt{n}\} \cap (\Lambda_n)^C,$$

where

(5.44) $$\Lambda_n := \left\{Y\colon\ \left|Y\left(\tfrac{j}{n}\right) - y(0)\right| \leqslant \frac{|y(0)|}{2},\quad \text{all } j = 1, \ldots, n\right\}.$$

We shall prove that for all n sufficiently large

(5.45) $$\int_{\Theta_n(i)} V_n \prod dy(p) \leqslant \begin{cases} (\text{const})\sqrt{n}\,e^{nb}, & \text{for } i = 1, \\ (\text{const})e^{nb}, & \text{for } i = 2, 3. \end{cases}$$

This gives (5.25).

Upper bound for $\Theta_n(1)$. We have

(5.46) $$\frac{1}{n}\sum_{j=1}^{n} \left(Y\left(\tfrac{j}{n}\right)\right)^2 = (y(p))^2 \quad \text{for} \quad Y \in \mathscr{T}_n.$$

Since $\varphi(y) \leqslant b + \frac{y^2}{2}$ for y real, we have by (5.46)

(5.47) $$\sum_{j=1}^{n} \varphi\left(\frac{Y\left(\frac{j}{n}\right)}{\sqrt{n}}\right) \leqslant nb + \frac{1}{2}\sum (y(p))^2 \quad \text{for} \quad Y \in \mathscr{T}_n.$$

Hence

(5.48)

$$\int_{\Theta_n(1)} V_n \prod dy(p) \leqslant$$

$$\leqslant e^{nb}(2\pi\hat{J}_n(0))^{-\frac{1}{2}} \int_{|y(0)|\leqslant K\sqrt{n}} \exp\left[\frac{1}{2}(1-J_n^{-1}(0))(y(0)^2\right] dy(0)\times$$

$$\times \prod{}' (2\pi\hat{J}_n(p))^{-\frac{1}{2}} \int_{R^{n-1}} \exp\left[-\frac{1}{2}\sum{}' (\hat{J}_n^{-1}(p)-1)(y(p))^2\right] dy(p) =$$

$$= e^{nb}\prod{}' (1-\hat{J}_n(p))^{-\frac{1}{2}} \times$$

$$\times (2\pi\hat{J}_n(0))^{-\frac{1}{2}} \int_{|y(0)|\leqslant K\sqrt{n}} \exp\left[\frac{1}{2}(1-\hat{J}_n^{-1}(0))(y(0))^2\right] dy(0).$$

This is valid for all n sufficiently large by (5.39). By (5.33), the integral involving $y(0)$ is $O(\sqrt{n})$. We obtain (5.45) for $i = 1$ by (5.40).

Upper bound for $\Theta_n(2)$. By (5.15), we can find $K > 0$ so that

$$\varphi(y) \leqslant \frac{y^2}{4} \quad \text{for} \quad |y| \geqslant \frac{K}{2}. \tag{5.49}$$

Provisionally, this value of K is used in the definitions of the set in (5.43). For $Y \in \Theta_n(2)$,

$$\left|\frac{Y\left(\frac{j}{n}\right)}{\sqrt{n}}\right| \geqslant \frac{|y(0)|}{2\sqrt{n}} > \frac{K}{2}, \tag{5.50}$$

so that by (5.49), (5.46)

$$\sum_{j=1}^{n} \varphi\left(\frac{Y\left(\frac{j}{n}\right)}{\sqrt{n}}\right) \leqslant \frac{1}{4}\sum (y(p))^2, \qquad Y \in \Theta_n(2). \tag{5.51}$$

Hence for all n sufficiently large

(5.52)

$$\int_{\Theta_n(2)} V_n \prod dy(p) \leqslant$$

$$\leqslant (2\pi\hat{J}_n(0))^{-\frac{1}{2}} \int_{|y(0)|>K\sqrt{n}} \exp\left[-\frac{1}{2}\left(\hat{J}_n^{-1}(0)-\frac{1}{2}\right)(y(0))^2\right] dy(0) \times$$

$$\times \prod{}' (2\pi\hat{J}_n(p))^{-\frac{1}{2}} \times$$

$$\times \int_{R^{n-1}} \exp\left[-\frac{1}{2}\sum{}' \left(\hat{J}_n^{-1}(p)-\frac{1}{2}\right)(y(p))^2\right] \prod{}' dy(p) \leqslant$$

$$\leqslant \prod{}' \left(1-\frac{\hat{J}_n(p)}{2}\right)^{-\frac{1}{2}} (2\pi\hat{J}_n(0))^{-\frac{1}{2}} \int_{|y(0)|>K\sqrt{n}} e^{-\frac{(y(0))^2}{8}} dy(0) \leqslant$$

$$\leqslant (\mathrm{const})n^{-\frac{1}{2}} \exp\left(-\frac{nK^2}{8}\right).$$

By increasing K, if necessary, we obtain (5.45) for $i = 2$; this increased K is used in (5.43).

Upper bound for $\Theta_n(3)$. For all n sufficiently large, we define the probability measure ξ_n on R^{n-1} by

(5.53)

$$\xi_n := \prod{}' \left[\frac{\hat{J}_n^{-1}(p)-1}{2\pi}\right]^{\frac{1}{2}} \times$$

$$\times \exp\left[-\frac{1}{2}\sum{}' (\hat{J}_n^{-1}(p)-1)(y(p))^2\right] \prod{}' dy(p).$$

For real $y(0)$ fixed, we define the set

(5.54) $$\Omega_n(y(0)) := \Lambda_n^C \cap \left\{Y\colon \int_0^1 Y\,dt = y(0)\right\},$$

where Λ_n is defined in (5.44). We prove below that for all n sufficiently large there exist positive constants C_1, C_2 so that

$$(5.55) \qquad \xi_n(\Omega_n(y(0)) \leqslant C_1\sqrt{n}\, e^{-\frac{C_2(y(0))^2}{2}}, \quad \text{for } |y(0)| > K\sqrt{n}.$$

Using (5.47), then (5.55), we have for all n sufficiently large

(5.56)

$$\int_{\Theta_n(3)} V_n \prod dy(p) \leqslant$$

$$\leqslant e^{nb} \prod (2\pi \hat{J}_n(p))^{-\frac{1}{2}} \int_{|y(0)|>K\sqrt{n}} \exp\left[\frac{1}{2}(1-\hat{J}_n^{-1}(0))(y(0))^2\right] \times$$

$$\times \left\{ \int_{\Omega_n(y(0))} \exp\left[-\frac{1}{2}\sum{}' (\hat{J}_n^{-1}(p)-1)(y(p))^2\right] \prod{}' dy(p)\right\} dy(0) =$$

$$= e^{nb} \prod{}' (1-\hat{J}_n(p))^{-\frac{1}{2}} (2\pi \hat{J}_n(0))^{-\frac{1}{2}} \times$$

$$\times \int_{|y(0)|>K\sqrt{n}} \exp\left[\frac{1}{2}(1-\hat{J}_n^{-1}(0))(y(0))^2\right] \xi_n(\Omega_n(y(0)))\, dy(0) \leqslant$$

$$\leqslant e^{nb} (\text{const})\sqrt{n} \int_{|y(0)|>K\sqrt{n}} \exp\left[-\frac{1}{2}(C_2(1-\hat{J}_n^{-1}(0))) \times\right.$$

$$\left.\times (y(0))^2\right] dy(0) \leqslant e^{nb}(\text{const})\sqrt{n} \exp\left(-\frac{nC_2K^2}{4}\right).$$

This gives (5.45) for $i = 3$. We prove (5.55). Pick $\nu > 0$ so small that $\nu \sum_{p\neq 0} |p|^{-(1+\delta)} \leqslant \frac{1}{2}$, where $0 < \delta < \frac{\epsilon}{2}$. If $|y(p)| \leqslant \frac{\nu |y(0)|}{|p|^{1+\delta}}$, for $1 \leqslant |p| \leqslant \frac{n-1}{2}$, then $\| Y - y(0)\| \leqslant \frac{|y(0)|}{2}$, where $Y := \sum y(p) B_p$.

Thus,

$$(5.57) \qquad \begin{aligned} &\xi_n(\Omega_n(y(0))) \leqslant \\ &\leqslant \xi_n\left(\left\{Y\colon |y(p)| \leqslant \nu \frac{|y(0)|}{|p|^{1+\delta}},\ 1 \leqslant |p| \leqslant \frac{n-1}{2}\right\}^C\right). \end{aligned}$$

For any number $a > 0$, one may show by the same proof used in (5.37) that

$$\xi_n\left(\left\{Y\colon |y(p)| \leqslant \frac{a}{|p|^{1+\delta}},\quad 1 \leqslant |p| \leqslant \frac{n-1}{2}\right\}^C\right) \leqslant \tag{5.58}$$
$$\leqslant (\text{const})\,\frac{n}{a}\exp\left(-\,\text{const})a^2\right),$$

with $\text{const} > 0$. Thus

$$\xi_n(\Omega_n(y(0))) \leqslant (\text{const})\,\frac{n}{|y(0)|}\exp\left(-\,(\text{const})(y(0))^2\right) \leqslant \tag{5.59}$$
$$\leqslant (\text{const})\sqrt{n}\,\exp\left(-\,(\text{const})(y(0))^2\right) \quad \text{for} \quad |y(0)| > K\sqrt{n}.$$

This gives (5.55) and completes the proof of the upper bound (5.25).∎

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