# Exponential decay in the mapping class group

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$S$ closed orientable surface genus $\geqslant 2$

Mapping class group $\mathrm{MCG}(S) = G = \mathrm{Homeo}^+(S)/\mathrm{isotopy}$

Thm: (Nielsen-Thurston Classification) Elements of $G$ are:

- periodic, $g^n = 1$
- reducible: $g$ fixes a disjoint collection of simple closed curves
- pseudo-Anosov (pA): everything else

Random walks on $G$:

Let $\mu$ probability distribution on $G$ with finite support, then a random walk of length $n$ is

$$w_n = s_1 s_2 \dots s_n,$$

where the $s_i$ are independent identically $\mu$-distributed random variables

Thm [Rivin][Kowalski]:

$$\mathbb{P}(w_n \text{ is pA}) \text{ is } 1 - O(c^n), \quad c < 1$$

Uses: action on homology $G \to Sp(2g, \mathbb{Z})$

Def: Torelli subgroup $T$ is $\ker\{G \to Sp(2g, \mathbb{Z})\}$

Thm [Malestein-Souto][Lubotzky-Meiri]:

$$\mathbb{P}(\text{random walk on } T \text{ is pA }) \text{ is } 1 - O(c^n)$$

Uses: action on homology of double covers

Thm [M]:

$$\mathbb{P}(w_n \text{ on } H < G \text{ is pA }) \text{ is } 1 - O(c^n)$$

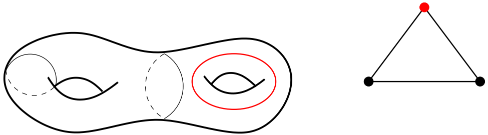
where $H = \langle \text{supp } \mu \rangle$ is a non-elementary subgroup of $G$

Uses: action of $G$ on the curve complex $\mathcal{C}(S)$

The mapping class group acts on the complex of curves $\mathcal{C}(S)$.

The complex of curves is a simplicial complex.

- vertices: isotopy classes of essential simple closed curves.
- simplices: spanned by disjoint simple closed curves.



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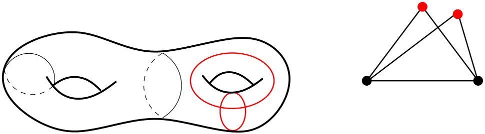

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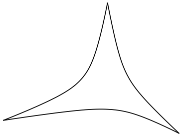
[Masur-Minsky] the complex of curves is $\delta$-hyperbolic.

Recall a metric space is $\delta$-hyperbolic if every geodesic triangle is $\delta$-thin, i.e. any side is contained in a $\delta$-neighbourhood of the other two.



Examples: hyperbolic space, trees, the complex of curves $\mathcal{C}(S)$.

Isometries of $\delta$-hyperbolic spaces are

- elliptic, fix a point in the interior (periodic, reducible)
- parabolic (none of these)
- hyperbolic (pA)

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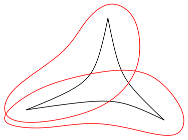

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Choose basepoint in $x_0 \in \mathcal{C}(S)$

Curve complex orbit metric $(G, d)$ is:

$$d(g, h) = d_{\mathcal{C}(S)}(g(x_0), h(x_0))$$

$g$ is pA $\Leftrightarrow$ $g$ acts as a hyperbolic isometry on $\mathcal{C}(S)$

$\Leftrightarrow \tau(g) > 0$, $\tau(g) = \lim_{n\to\infty} \frac{1}{n} d(1, g^n)$

$H < G$ non-elementary $\Leftrightarrow g, h \in H$ pA with distinct fixed points in Gromov boundary $\partial\mathcal{C}(S)$

Thm [M]:

$$\mathbb{P}(\tau(w_n) \leqslant B) \text{ is } O(c^n)$$

for $H = \langle \text{supp } \mu \rangle$ non-elementary

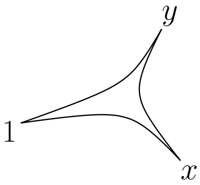
recall $w_n = s_1 s_2 \dots s_n$, so $w_n$ distributed as $n$-fold convolution $\mu_n = \mu \star \mu \star \cdots \star \mu$

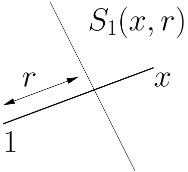
Thm [Kaimanovich-Masur]: A random walk on $G$ converges almost surely to a point in the Gromov boundary $\partial \mathcal{C}(S)$

This gives a hitting measure $\nu = \lim_{n \to \infty} \mu_n$

Shadow sets: $S_1(x, r) = \{y \in G \mid (x \cdot y)_1 \geqslant r\}$

Gromov product: $(x \cdot y)_1 = \frac{1}{2}(d(1, x) + d(1, y) - d(x, y))$




Estimates:

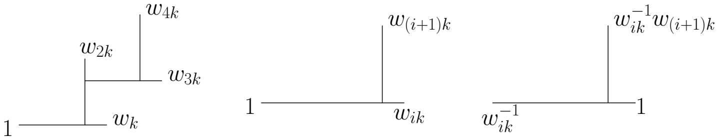
- $\nu(S_1(x, r)) \leqslant c^r$
- $\mu_n(S_1(x, r)) \leqslant Kc^r$

$K, c$ independent of $x, r$

Lemma: (Linear progress) There is $L > 0, c < 1$ such that

$$\mathbb{P}(d(1, w_n) \leqslant Ln) \text{ is } O(c^n)$$

$$d(1, w_{kn}) = d(1, w_k) + d(w_n, w_{2k}) + \cdots + d(w_{(n-1)k}, w_{nk})$$
$$-B_1 \qquad\qquad\qquad -B_n$$




$$\mathbb{P}(B_i \geqslant r) = \mu_n(S_1(w_{in}^{-1}, r)) \leqslant Kc^r$$

## Concentration of measure

$X_i$ independent identically distributed

$$\mathbb{P}(|\sum X_i - n\mathbb{E}(X_i)| > tn) \text{ is } O(c^n)$$

[Bernstein] $X_i$ finite support

[Chernoff-Hoeffding] $X_i$ exponential decay

Furthermore $c \to 0$ as $t \to \infty$

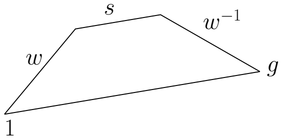
Distribution of pAs ($\tau(w_n) \leqslant B$)

Relative conjugacy bounds ( cf [Masur-Minsky][Tao] )

$a, b$ conjugate, then there is $w$ such that $a = wbw^{-1}$ with

$$d(1, w) \leqslant K(d(1, a) + d(1, b))$$

If $g$ conjugate to short word $s$, then $g$ close to $wsw^{-1}$, quasigeodesic, so $(g, g^{-1})$ close to diagonal in $G \times G$




estimate: $\mathbb{P}((w_n, w_n^{-1}) \in N_t(\Delta)) \leqslant Kc^n$