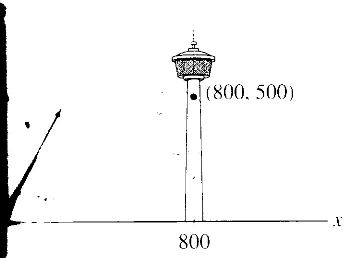

3 Trajectory of the bullet.

EXAMPLE 4 A bullet is fired from the ground at an angle of 60° above the horizontal. What initial speed v_0 must the bullet have in order to hit a point 500 ft high on a tower located 800 ft away (ignoring air resistance)?

Solution Place the gun at the origin, and let $\mathbf{r}(t)$ be the position vector of the bullet (Figure 3).

***Step 1.* Use Newton's Law.**

Gravity exerts a downward force of magnitude mg, where m is the mass of the bullet and $g = 32$ ft/s^2. In vector form,

$$\mathbf{F} = \langle 0, -gm\rangle = m\langle 0, -g\rangle$$

In this case, Newton's Second Law $\mathbf{F} = m\mathbf{r}''(t)$ reduces to $\mathbf{r}''(t) = \langle 0, -g\rangle$. We determine $\mathbf{r}(t)$ by integrating twice:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -32\rangle\,du = \langle 0, -32t\rangle + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u)\,du = \int_0^t \left(\langle 0, -32u\rangle + \mathbf{v}_0\right)du = \langle 0, -16t^2\rangle + t\mathbf{v}_0 + \mathbf{r}_0$$

Here, $\mathbf{r}_0$ is the initial position and $\mathbf{v}_0$ is the initial velocity. By our choice of coordinates, $\mathbf{r}_0 = \mathbf{0}$. The initial velocity $\mathbf{v}_0$ has unknown length v_0, but we know that it points in the direction of the unit vector $\langle \cos 60°, \sin 60°\rangle$. Therefore,

$$\mathbf{v}_0 = v_0\langle \cos 60°, \sin 60°\rangle = v_0\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

$$\mathbf{r}(t) = \langle 0, -16t^2\rangle + tv_0\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

***Step 2.* Solve for $\mathbf{v}_0$.**

The position vector of the point on the tower is $\langle 800, 500\rangle$, so the bullet will hit a point on the tower 500 ft high if there exists a time t such that

$$\mathbf{r}(t) = \langle 0, -16t^2\rangle + tv_0\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \langle 800, 500\rangle$$

Equating components, we obtain the equations

$$\frac{1}{2}tv_0 = 800, \qquad -16t^2 + \frac{\sqrt{3}}{2}tv_0 = 500$$

The first equation yields $t = \dfrac{1{,}600}{v_0}$. Now substitute in the second equation and solve:

$$-16\left(\frac{1{,}600}{v_0}\right)^2 + \frac{\sqrt{3}}{2}\left(\frac{1{,}600}{v_0}\right)v_0 = 500$$

$$\left(\frac{1{,}600}{v_0}\right)^2 = \frac{800\sqrt{3} - 500}{16}$$

$$v_0^2 = \frac{16(1{,}600)^2}{800\sqrt{3} - 500}$$

$$v_0 = \frac{6{,}400}{\sqrt{800\sqrt{3} - 500}} \approx 215 \text{ ft/s}$$