Sample problems for Linear Algebra, Spring 2016, Chapters 6–7

Problem 1. Find a 2×2 matrix A that has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Problem 2. If possible, diagonalize the following matrices:

$$A = \begin{bmatrix} 5 & 0 & 0 \\ -4 & 3 & 0 \\ 1 & -3 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 3. Compute A^{100} for $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$.

Problem 4.

(a) Prove that if A is diagonalizable, then A^T is diagonalizable.

(b) Prove that if the eigenvectors of A form an orthogonal basis (A is "orthogonally diagonalizable"), then A is symmetric.

(c) Suppose that A is a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $B = [\mathbf{v}_1 \ \mathbf{v}_2]$. Prove that $\det(B) \neq 0$.

Problem 5. Let T_1, T_2, T_3 be linear transformations given by $T_i(\mathbf{x}) = A_i\mathbf{x}$ as follows. For each T_i, determine whether it is one-to-one, onto, both or neither.

$$A_1 = \begin{bmatrix} 4 & -1 \\ -2 & 2 \\ 0 & 3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -12 & -8 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

Problem 6. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation.

(a) Prove that if T is one-to-one, then T is onto.

(b) Prove that if T is onto, then T is one-to-one.

Problem 7. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

(a) Prove that if $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly independent, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b) Give an example such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, but $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are not linearly independent.

Problem 8. Describe all linear transformations $T : \mathbf{R} \to \mathbf{R}$.

Problem 9. Let $T : V \to W$ be a linear transformation. Prove $\ker(T)$ is subspace of V.

Problem 10. Answer the numerical WebWork questions about linear transformations!