Math 233 Fall 2021 Sample Exam 1

Problem 1. Let $\vec{\mathbf{u}} = (4, 4, 5)$ and $\vec{\mathbf{v}} = (2, -1, 1)$.

(a) Find a unit vector in the direction of $\vec{\mathbf{v}}$.

(b) Find $||\text{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}||$.

(c) Express $\vec{\mathbf{u}}$ as the sum $\vec{\mathbf{m}} + \vec{\mathbf{n}}$, where $\vec{\mathbf{m}} = \vec{\mathbf{u}}_{||}$ is parallel to $\vec{\mathbf{v}}$, and $\vec{\mathbf{n}} = \vec{\mathbf{u}}_{\perp}$ is orthogonal to $\vec{\mathbf{v}}$.

Problem 2. Consider three points $P(2, -1, 0)$, $Q(0, -2, 1)$ and $R(3, 0, -1)$.

(a) Find a parametric equation of the line through Q and R.

(b) Find the equation of the plane passing through P, Q, and R.

(c) Find the area of triangle $\triangle PQR$.

Problem 3.

(a) Find the angle between the planes $x - y = 3$ and $-y + z = 1$.
(Hint: The angle between the planes is the angle between their normal vectors.)

(b) Find the equation of the plane that passes through the point $(1, 2, -1)$ and is perpendicular to the line $\ell(t) = (2 + 3t,\ -t,\ 4 + t)$.

(c) Find the equation of a plane containing the line $\ell(t) = (2 + 3t,\ -t,\ 4 + t)$ and passing through the point $P(0, 2, -1)$.

(d) Find the line of intersection between the planes $z = 2x - 4y + 2$ and $x - y - 2z = 4$.

Problem 4. For each equation below, describe the corresponding surface S in $\mathbb{R}^3$, sketch its three traces and then sketch S.

(a) ________________________________ $x^2 + 4y^2 + 4z^2 = 16$

(b) ________________________________ $4x^2 + y^2 + 4z^2 = 16$

(c) ________________________________ $z = 9x^2 + 4y^2$

(d) ________________________________ $z = 9x^2 - 4y^2$

(e) ________________________________ $9x^2 + 4y^2 = 2z^2 + 72$

(f) ________________________________ $9x^2 + 4z^2 = 2y^2 - 72$

(g) ________________________________ $9x^2 + 4y^2 = 2z^2$

(h) ________________________________ $9x^2 - 4y^2 = 72$

Problem 5. Sketch the level sets of the function $f(x, y, z) = x^2 - y^2 - z^2$.

Problem 6. The position of a particle is $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + e^{-t}\,\mathbf{k}$.

(a) Show that the speed of the particle at time t is $e^t + e^{-t}$.

(b) Find the total distance travelled by the particle for $1 \le t \le 3$.

Problem 7. A string in the shape of a helix has a height of 15 cm and makes three full rotations over a circle of radius 4 cm.

(a) Find a parametrization $\mathbf{r}(t)$ for the string.

(b) Compute the length of the string.

Problem 8. Show that if position $\vec{\mathbf{r}}(t)$ satisfies $||\vec{\mathbf{r}}(t)|| = c$, then velocity $\vec{\mathbf{v}}(t)$ is orthogonal to $\vec{\mathbf{r}}(t)$.

Problem 9.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^4 + y^4}$ does not exist.

(b) Let $h(x, y) = x \sin(x + 2y)$. Verify Clairaut's Theorem: $h_{xy} = h_{yx}$.

Problem 10.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{|x| + |y|}$ does not exist.

(b) Let $h(x, z) = e^{xz - x^2 z^3}$. Compute $h_z(2, 1)$.