Mahler measure and the Vol-Det Conjecture

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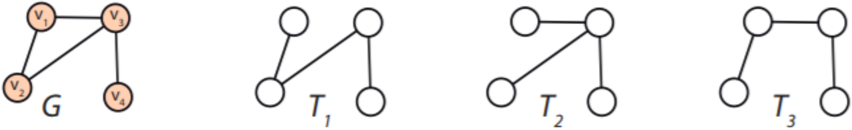
The Matrix-Tree Theorem

Let $\tau(G) = \#$ spanning trees of a graph G.



If G has vertices $\{v_1, \ldots, v_n\}$, then its Laplacian matrix $L(G) = D - A$, where $D_{jj} = \deg(v_j)$ and A is the adjacency matrix of G.

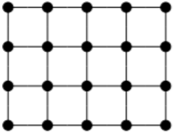
$$L(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Theorem (Kirchoff, 1847)

$$\tau(G) = \text{any cofactor of } L(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i$$

Temperley's bijection

Computing $\tau(G)$ is still surprisingly difficult.
Temperley discovered the general formula for the $m \times n$ grid in 1974.



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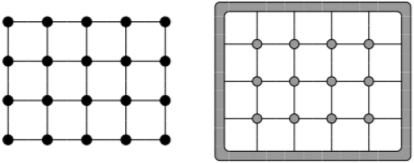

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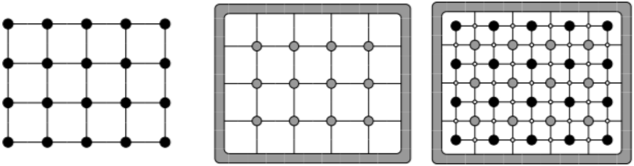

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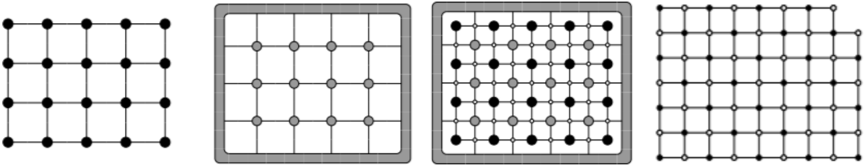

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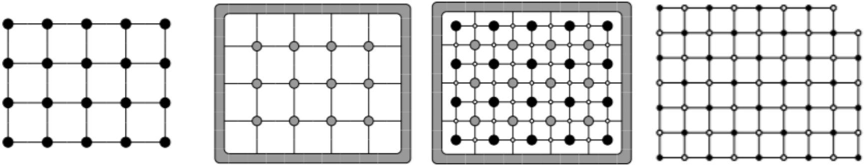

$$\tau(G_{m\times n}) = \prod_{j=1}^{m-1}\prod_{k=1}^{n-1}\left(4 - 2\cos\frac{\pi j}{m} - 2\cos\frac{\pi k}{n}\right)$$

For $m = 4,\ n = 5$, we get $\tau(G_{4\times 5}) = 4,140,081$

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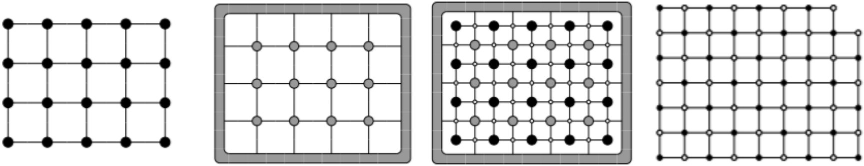

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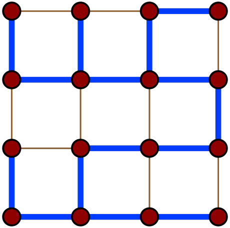
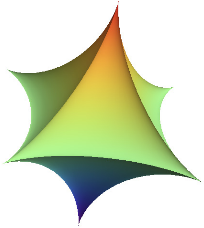
$$\lim_{m,n\to\infty}\frac{\pi \log\tau(G_{m\times n})}{m\cdot n} = \frac{2}{\pi}\int_0^\pi\int_0^\pi \log|2\cos\theta + 2i\cos\phi|\,d\theta\,d\phi = 4\mathrm{C},$$

where C is Catalan's constant, $\mathrm{C} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.916$

Growth and hyperbolic volume – Example 1





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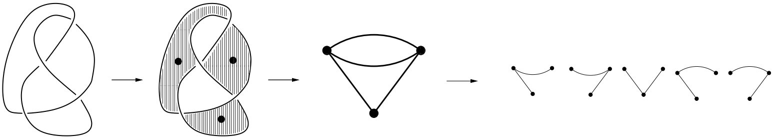
$$4\mathrm{C} = v_{\mathrm{oct}} \approx 3.6638$$

Knot determinant

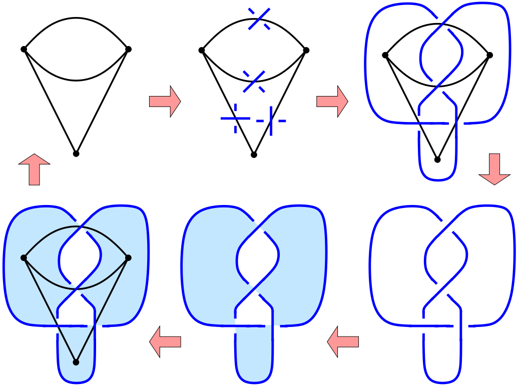
The knot determinant was one of the first computable knot invariants (computable = not of the form "minimize something over all diagrams")

$$\begin{aligned}
\det(K) = &\quad |\det(M + M^T)|, && M = \text{Seifert matrix} \\
= &\quad |H_1(\Sigma_2(K); \mathbb{Z})|, && \Sigma_2 = \text{2–fold branched cover of } K \\
= &\quad |V_K(-1)| = |\Delta_K(-1)|, && V_K,\, \Delta_K = \text{Jones, Alexander poly} \\
= &\quad \#\,\text{spanning trees } \tau(G_K), && G_K = \text{Tait graph of alternating } K
\end{aligned}$$



Alternating knots

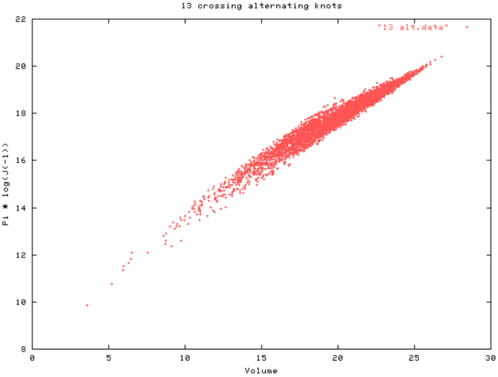
We can recover an alternating knot diagram (up to mirror image) from its Tait graph:



The other checkerboard coloring gives the planar dual of the Tait graph.

Determinant and hyperbolic volume

Dunfield (2000) suggested a relationship between $\det(K)$ and $Vol(S^3 - K)$:



Vol-Det Conjecture

Conjecture (Vol-Det Conjecture): For any alternating hyperbolic link K,

$$Vol(K) < 2\pi \log \det(K)$$

- Verified for all alternating knots ≤ 16 crossings.
- (Burton) Verified for 2-bridge links, alternating 3-braids.
- (Champanerkar-K-Purcell) 2π is sharp.

i.e., if $\alpha < 2\pi$ then there exist alternating hyperbolic knots K such that

$$\alpha \log \det(K) < Vol(K)$$

Remark Let K be a reduced alternating link diagram, and let K' be obtained by changing any proper subset of crossings of K.

- (Champanerkar-K-Purcell) $\det(K') < \det(K)$.
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Mahler measure

Mahler measure of polynomial $p(z)$ is defined as

$$m(p(z)) := \frac{1}{2\pi i}\int_{S^1} \log|p(z)|\,\frac{dz}{z} \overset{\text{Jensen}}{=} \sum_{\substack{\alpha_i \text{ roots of } p\\ |\alpha_i|\geq 1}} \log|\alpha_i|$$

2-variable Mahler measure:

$$\mathrm{m}(p(z,w)) := \frac{1}{(2\pi i)^2}\int_{S^1\times S^1} \log|p(z,w)|\,\frac{dz}{z}\frac{dw}{w}$$

2-variable Mahler measures are related to hyperbolic volume because often they can be expressed using dilogarithms.

Examples

$Vol($  $) = 2v_{\text{tet}} = 2.0298\ldots$

(Smyth) 1981 $Vol($  $) = 2\pi\,\mathrm{m}(1 + x + y) = \frac{3\sqrt{3}}{2} L(\chi_{-3}, 2)$

(Boyd) 2000 $Vol($  $) = \pi\,\mathrm{m}(A(L, M))$

$$= \pi\,\mathrm{m}(M^4 + L(1 - M^2 - 2M^4 - M^6 + M^8) - L^2 M^4)$$

(Kenyon) 2000 $Vol($  $) = \frac{2\pi}{5}\,\mathrm{m}(p(z, w))$

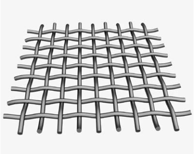
$$= \frac{2\pi}{5}\mathrm{m}\left(6 - w - \frac{1}{w} - z - \frac{1}{z} - \frac{w}{z} - \frac{z}{w}\right)$$

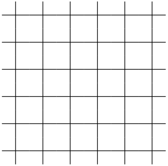
Biperiodic alternating links

Let $\mathcal{L}$ be an alternating link in $\mathbb{R}^2 \times I$ such that its projection graph $G(\mathcal{L})$ is a 4-valent, biperiodic tiling of the Euclidean plane.

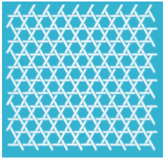
Examples:

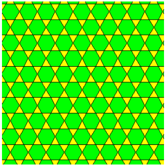
Square weave $\mathcal{W}$ and square lattice $G(\mathcal{W})$:





Triaxial link $\mathcal{Q}$ and trihexagonal lattice $G(\mathcal{Q})$:





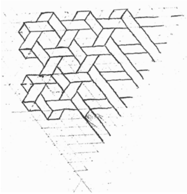
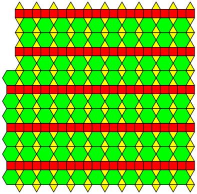
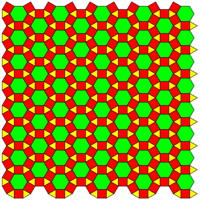
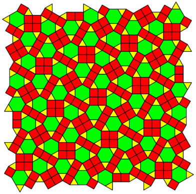
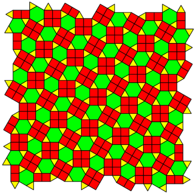
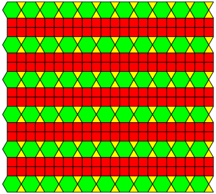
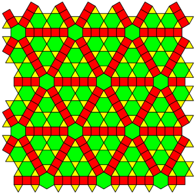


Figure above from Gauss's 1794 notebook

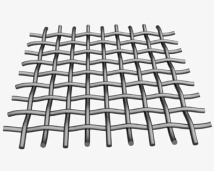
More examples of 4-valent semi-regular Euclidean tilings

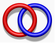







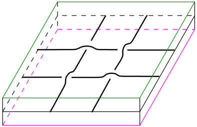
https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons

Geometry of the infinite square weave $\mathcal{W}$

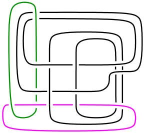
The $\mathbb{Z}^2$-quotient of $\mathbb{R}^3 - \mathcal{W}$ is a link complement in a thickened torus:
$$T^2 \times I - W.$$



$T^2 \times I \cong S^3 -$ , so it's also the complement of a link ℓ in S^3 with a Hopf sublink.



In this example, $S^3 - \ell$ has a complete hyperbolic structure with four regular ideal octahedra. 



Geometry of semi-regular biperiodic alternating links

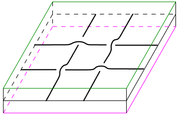
A biperiodic alternating link $\mathcal{L}$ is invariant under translations by a 2–dim lattice Λ, such that $L = \mathcal{L}/\Lambda$ is a link in $T^2 \times I$, with a toroidally alternating diagram on $T^2 \times 0$.

Theorem (Champanerkar-K-Purcell) If the projection graph $G(\mathcal{L})$ is a semi-regular Euclidean tiling, then $T^2 \times I - L$ is hyperbolic and decomposes into regular ideal tetrahedra and octahedra, with

$$Vol(T^2 \times I - L) = 10a\, v_{\text{tet}} + b\, v_{\text{oct}}$$

where $a = \#$hexagons, and $b = \#$squares in the fundamental domain.

Example: If $W = \mathcal{W}/\Lambda$, then
$Vol(T^2 \times I - W) = c(W) \cdot v_{\text{oct}}$ 

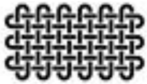
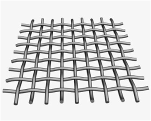


Diagrammatic convergence

$K_n \xrightarrow{\text{F}} \mathcal{L}$ denotes $\{K_n\}$ *Følner converges almost everywhere* to $\mathcal{L}$.

This means the alternating links K_n satisfy:

1. K_n contain increasing subsets of $\mathcal{L}$ which eventually exhaust $\mathcal{L}$: $\exists\ G_n \subset G(K_n)$ such that $G_n \subset G_{n+1}$, and $\bigcup G_n = G(\mathcal{L})$,
2. Følner condition for $G_n \subset G(\mathcal{L})$: $\displaystyle\lim_{n\to\infty} \frac{|\partial G_n|}{|G_n|} = 0$,
3. The K_n do not have too many other crossings: $\displaystyle\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1$.

 ,  ,  ,  , ... $\xrightarrow{\text{F}}$ 

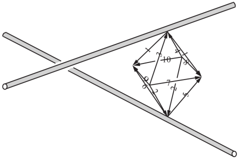
Geometrically maximal knots

Theorem (Champanerkar-K-Purcell) For hyperbolic alternating links K_n

$$K_n \xrightarrow{\mathrm{F}} \mathcal{W} \implies \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = v_{\mathrm{oct}} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}.$$

Geometrically maximal:

Can decompose $S^3 - K$ into octahedra, one octahedron at each crossing:



$$\implies \quad \frac{Vol(K)}{c(K)} < v_{\mathrm{oct}}$$

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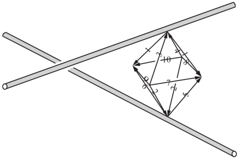
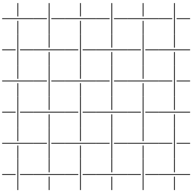


$$\implies \quad \frac{Vol(K)}{c(K)} < v_{\mathrm{oct}}$$

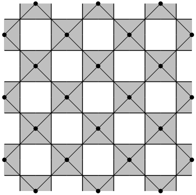
Question What analogous toroidal invariant is the limit for the determinant density?

Spanning tree entropy

Square weave $\mathcal{W}$



Tait graph $G_{\mathcal{W}}$ = square grid



Recall, spanning tree entropy of $G_{\mathcal{W}}$:
$G_n = n \times n$ square grid, #spanning trees $\tau(G_n)$, Catalan's $\mathrm{C} \approx 0.916$

$$\lim_{n\to\infty} \frac{\pi \log \tau(G_n)}{n^2} = 4\mathrm{C} = v_{\mathrm{oct}}$$

This is enough to establish the result for $\mathcal{W}$. But we want to compute

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$$

for $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$ for *any* biperiodic alternating link $\mathcal{L}$.

Spanning tree entropy

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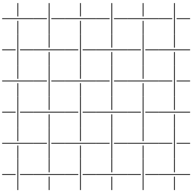

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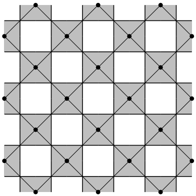

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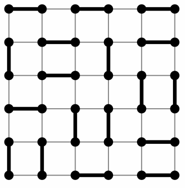
Dimers

A dimer covering of a graph G is a set of edges that covers every vertex exactly once, i.e. a perfect matching.



The dimer model is the study of the set of dimer coverings of G.
Let $Z(G) = \#$ dimer coverings of G.

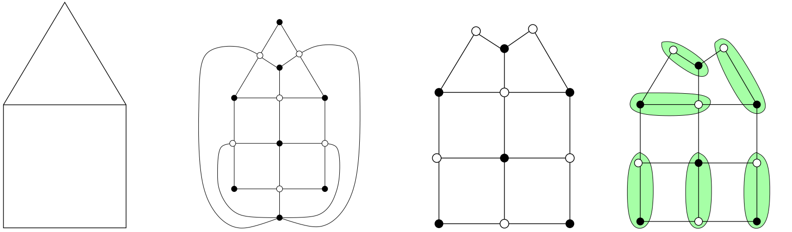
Theorem (Kasteleyn 1963) If G is a balanced bipartite planar graph,

$$Z(G) = det(K),$$

where K is a Kasteleyn matrix.

Dimers and spanning trees

For any finite plane graph G, overlay G and its dual G^*, delete a vertex of G and G^* (in the unbounded face) and delete all incident edges to get balanced bipartite overlaid graph G^b.

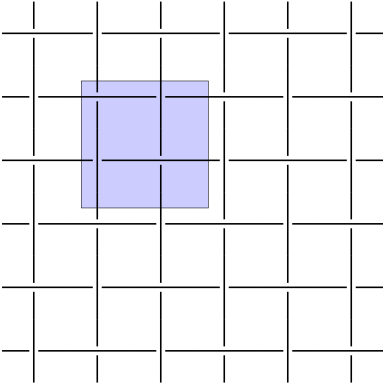


G $\quad$ $G \cup G^*$ $\quad$ G^b $\quad$ A dimer on G^b

Theorem (Burton-Pemantle '93, Propp '02) $\tau(G) = Z(G^b)$.

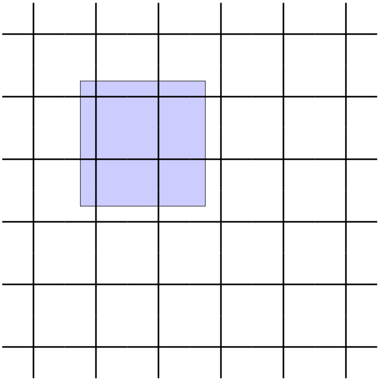
Biperiodic overlaid graph

Biperiodic alternating link $\mathcal{L} \to$ Biperiodic bipartite overlaid graph $G^b_{\mathcal{L}}$.



Biperiodic overlaid graph

Biperiodic alternating link $\mathcal{L} \to$ Biperiodic bipartite overlaid graph $G^b_{\mathcal{L}}$.



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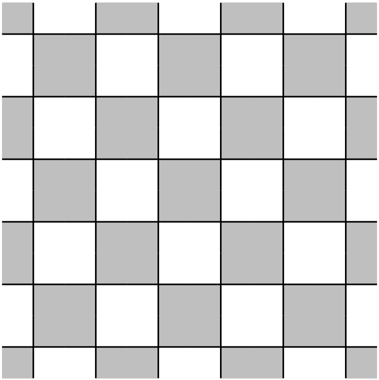

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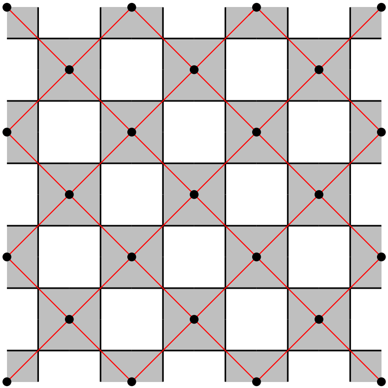

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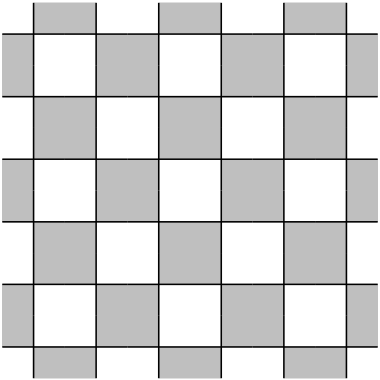

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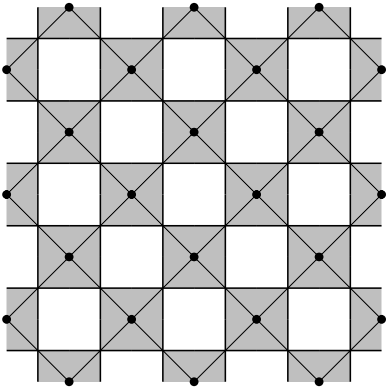

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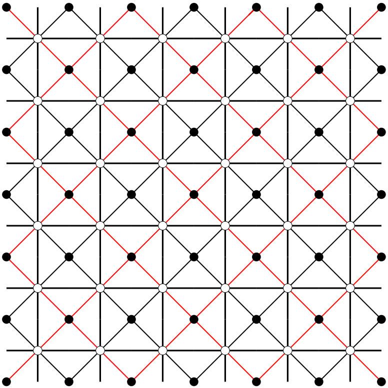

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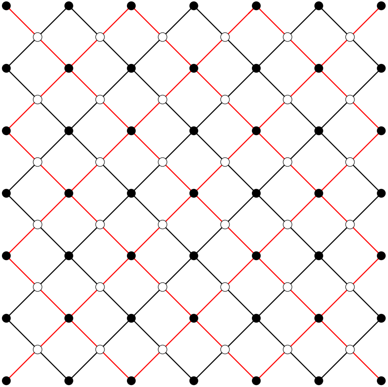

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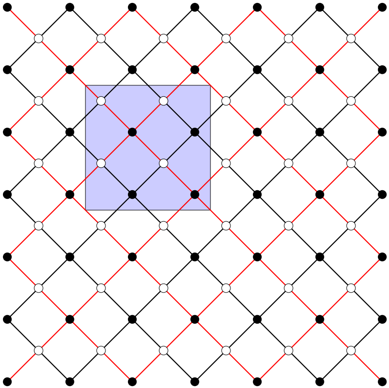


Kasteleyn matrix for toroidal dimer model

Let G^b be a finite balanced bipartite toroidal graph.

Kasteleyn matrix $K(z,w)$ for toroidal dimer model on G^b is defined by:

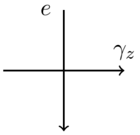
1. Choose signs on edges, such that each face with 0 mod 4 edges has an odd # of signs, called Kasteleyn weighting.
2. Choose a meridian and longitude basis on the torus, $\gamma_z,\ \gamma_w$. Orient each edge e from black to white. Let

$$\mu_e = z^{\gamma_z \cdot e}\ w^{\gamma_w \cdot e}$$

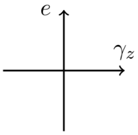
3. Order the black and white vertices.

Then $K(z,w)$ is the $|B| \times |W|$ adjacency matrix with entries $\pm\mu_e$.

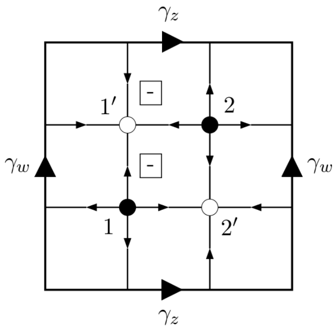
Kasteleyn matrix for toroidal dimer model



$\mu_e = \frac{1}{z}$



$\mu_e = z$



$$K(z,w) = \begin{bmatrix} -1 - 1/z & 1 + w \\ 1 + 1/w & 1 + z \end{bmatrix}$$

Toroidal dimer model

Let G^b be a biperiodic balanced bipartite planar graph, which is invariant under translations by 2-dim lattice Λ.

The characteristic polynomial of the toroidal dimer model on G^b is

$$p(z,w) = \det K(z,w).$$

Theorem (Kenyon-Okounkov-Sheffield, 2006)
If $G_n = G^b/n\Lambda$ is a toroidal exhaustion of G^b, then

$$\lim_{n\to\infty} \frac{\log Z(G_n)}{n^2} = \mathrm{m}(p(z,w)).$$

Note: This limit does not depend on the choices to get $K(z,w)$.

Determinant density convergence

Theorem (Champanerkar-K) Let $\mathcal{L}$ be any biperiodic alternating link, with toroidally alternating quotient link $L = \mathcal{L}/\Lambda$. Let $p(z,w)$ be the characteristic polynomial for the toroidal dimer model on $G^b_{\mathcal{L}}$.

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, \mathrm{m}(p(z,w))}{c(L)}.$$

Idea of proof: The following limits are equal:

1. Spanning tree model on the Tait graph $G_{\mathcal{L}}$,
   i.e. limit of spanning tree entropies of planar exhaustions of $G_{\mathcal{L}}$.
2. Toroidal dimer model on biperiodic overlaid graph $G^b_{\mathcal{L}}$,
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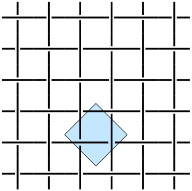
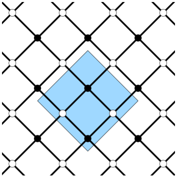
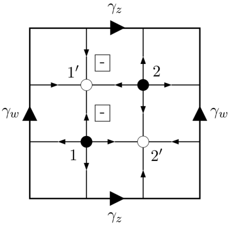
Square weave $\mathcal{W}$: Let's put it all together!



$\mathcal{W}$ & W



$G^b_{\mathcal{W}}$ & G^b_W



Kasteleyn weighting

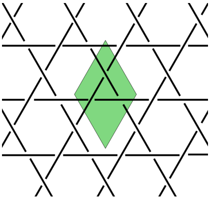
$$Vol(T^2 \times I - W) = 2\, v_{oct} = 7.32772\ldots$$

$$p(z,w) = -(4 + 1/w + w + 1/z + z)$$

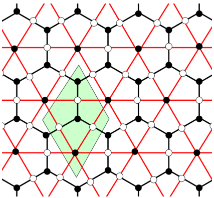
(Boyd 1998) $2\pi\, \mathrm{m}(p(z,w)) = 2\, v_{oct}$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, \mathrm{m}(p(z,w))}{c(W)} = v_{oct} = \frac{Vol(T^2 \times I - W)}{c(W)}$$

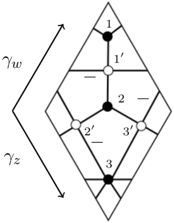
Triaxial link $\mathcal{Q}$



$\mathcal{Q}$ & Q



$G^b_{\mathcal{Q}}$ & G^b_Q



Kasteleyn weighting

$$Vol(T^2 \times I - Q) = 10\, v_{tet} = 10.14941\ldots$$

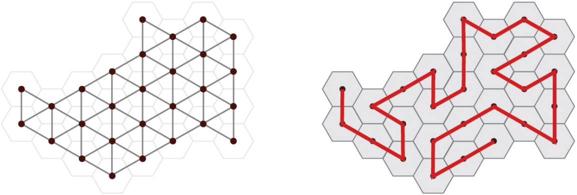
$$p(z,w) = 6 - w - 1/w - z - 1/z - w/z - z/w$$

(Boyd 1998) $2\pi\,\mathrm{m}(p(z,w)) = 10\, v_{tet}$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\,\mathrm{m}(p(z,w))}{c(Q)} = \frac{10 v_{\text{tet}}}{3} = \frac{Vol(T^2 \times I - Q)}{c(Q)}$$

Growth and hyperbolic volume – Example 2

We can use our results for the triaxial link to find the spanning tree entropy $T_\triangle$ for the regular triangular tiling, and $T_{\hexagon}$ for its dual hexagonal tiling:

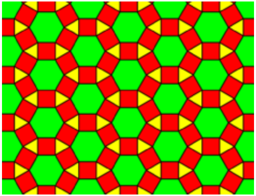


Each fundamental domain (3 crossings of Q) has 2 vertices on hexagons, and 1 vertex on the triangles, so we adjust accordingly:

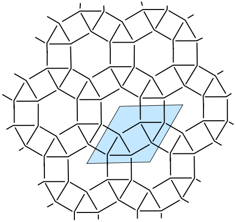
$$T_\triangle = \lim_{n\to\infty} \frac{2\pi \log \tau(G_n)}{v(G_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)/3} = 2\pi\, \mathrm{m}(p(z,w)) = 10\, v_{\mathrm{tet}}$$

$$T_{\hexagon} = \lim_{n\to\infty} \frac{2\pi \log \tau(G_n)}{v(G_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{2\, c(K_n)/3} = \frac{2\pi\, \mathrm{m}(p(z,w))}{2} = 5\, v_{\mathrm{tet}}$$

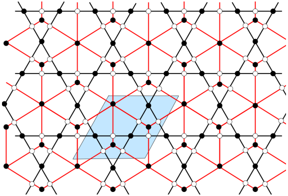
Rhombitrihexagonal link $\mathcal{R}$



$G(\mathcal{R})$



$\mathcal{R}$ & R



$G^b_{\mathcal{R}}$ & G^b_R

$$Vol(T^2 \times I - R) = 10\, v_{tet} + 3\, v_{oct} = 21.14100\ldots$$

$$p(z,w) = 6\,(6 - w - 1/w - z - 1/z - w/z - z/w)$$

$$2\pi\, \mathrm{m}(p(z,w)) = 10 v_{\text{tet}} + 2\pi \log 6 = 21.40737\ldots$$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, \mathrm{m}(p(z,w))}{c(R)} > \frac{Vol(T^2 \times I - R)}{c(R)}$$

Volume density convergence conjecture

Conjecture (Champanerkar-K-Purcell) Let $\mathcal{L}$ be any biperiodic alternating link, with toroidally alternating quotient link $L = \mathcal{L}/\Lambda$. For hyperbolic alternating links K_n

$$K_n \stackrel{\mathrm{F}}{\to} \mathcal{L} \implies \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = \frac{Vol(T^2 \times I - L)}{c(L)}.$$

We can prove this for the square weave $\mathcal{W}$ and the triaxial link $\mathcal{Q}$.

So in these two cases, if $K_n \stackrel{\mathrm{F}}{\to} \mathcal{L}$,

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, m(p(z,w))}{c(L)} = \frac{Vol(T^2 \times I - L)}{c(L)} = \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)}.$$

Mahler measure and the Vol-Det Conjecture

Vol-Det Conjecture: For any alternating hyperbolic link K,

$$Vol(K) < 2\pi \log \det(K).$$

Idea: Use biperiodic alternating links to obtain infinite families of links satisfying the Vol-Det Conjecture.

This is possible if for $K_n \xrightarrow{\text{F}} \mathcal{L}$, $\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} < \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$.

1. Prove using an exact Mahler measure computation that
$$Vol(T^2 \times I - L) < 2\pi\, \mathrm{m}(p(z,w)).$$
2. Use the geometry of $T^2 \times I - L$ to prove that
$$K_n \xrightarrow{\text{F}} \mathcal{L} \implies Vol(K_n) < 2\pi \log \det(K_n) \text{ for almost all } n.$$

e.g. Rhombitrihexagonal link $\mathcal{R}$.

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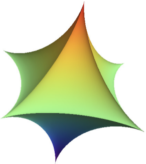
e.g. Rhombitrihexagonal link $\mathcal{R}$.

Bipyramid volume

Let B_n denote the hyperbolic regular ideal bipyramid whose link polygons at the two coning vertices are regular n–gons.

$$Vol(B_n) = n\left(\int_0^{2\pi/n} -\log|2\sin(\theta)|d\theta + 2\int_0^{\pi(n-2)/2n} -\log|2\sin(\theta)|d\theta\right).$$

e.g. B_4 = regular ideal octahedron



Theorem (Adams) $\quad Vol(B_n) < 2\pi\log(\frac{n}{2})$ and $Vol(B_n) \underset{n\to\infty}{\sim} 2\pi\log(\frac{n}{2})$.

Bipyramid volume

Let L be a link in $T^2 \times I$ with a toroidally alternating diagram on $T^2 \times 0$.

Define the bipyramid volume of L as

$$\mathrm{vol}^{\lozenge}(L) := \sum_{f \in \{\text{faces of } L\}} Vol\left(B_{\deg(f)}\right).$$

Theorem (Champanerkar-K-Purcell)

$$Vol(T^2 \times I - L) \leq \mathrm{vol}^{\lozenge}(L)$$

This is a sharp bound for volume, with equality for all semi-regular links.

Vol-Det Conjecture for infinite families of knots

Conjecture 1 (Champanerkar-K-Lalín) Let $\mathcal{L}$ be any hyperbolic biperiodic alternating link, with $L = \mathcal{L}/\Lambda$, $p(z, w)$ as above. Then

$$\mathrm{vol}^{\diamondsuit}(L) \leq 2\pi\, \mathrm{m}(p(z, w))$$

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$$Vol(T^2 \times I - L) \leq \mathrm{vol}^{\Diamond}(L) \leq 2\pi\,\mathrm{m}(p(z,w))$$

Theorem (Champanerkar-K-Lalín) For hyperbolic alternating links $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$,

$$\mathrm{vol}^{\Diamond}(L) < 2\pi\,\mathrm{m}(p(z,w)) \implies Vol(K_n) < 2\pi \log\det(K_n) \text{ for almost all } n.$$

Note: For any $\mathcal{L}$, the infinite families of knots or links satisfying the Vol-Det Conjecture include almost all K_n for *every* sequence $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$.

Vol-Det Conjecture for infinite families of knots

Theorem (Champanerkar-K-Lalín) For hyperbolic alternating links $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$,

$$\mathrm{vol}^{\lozenge}(L) < 2\pi\, \mathrm{m}(p(z,w)) \implies Vol(K_n) < 2\pi \log \det(K_n) \text{ for almost all } n.$$

Proof:

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{\mathrm{vol}^{\lozenge}(K_n)}{c(K_n)} = \frac{\mathrm{vol}^{\lozenge}(L)}{c(L)}$$

$$\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} \leq \lim_{n\to\infty} \frac{\mathrm{vol}^{\lozenge}(K_n)}{c(K_n)} = \frac{\mathrm{vol}^{\lozenge}(L)}{c(L)}$$

$$< \frac{2\pi\, \mathrm{m}(p(z,w))}{c(L)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$$

$$\implies \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} < \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} \qquad \square$$

Vol-Det Conjecture for infinite families of knots

Theorem (Champanerkar-K-Lalín) For hyperbolic alternating links $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$,

$$\mathrm{vol}^{\lozenge}(L) < 2\pi\, \mathrm{m}(p(z,w)) \implies Vol(K_n) < 2\pi \log \det(K_n) \text{ for almost all } n.$$

Proof:

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{\mathrm{vol}^{\lozenge}(K_n)}{c(K_n)} = \frac{\mathrm{vol}^{\lozenge}(L)}{c(L)}$$

$$\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} \leq \lim_{n\to\infty} \frac{\mathrm{vol}^{\lozenge}(K_n)}{c(K_n)} = \frac{\mathrm{vol}^{\lozenge}(L)}{c(L)}$$

$$< \frac{2\pi\, \mathrm{m}(p(z,w))}{c(L)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$$

$$\implies \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} < \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$$

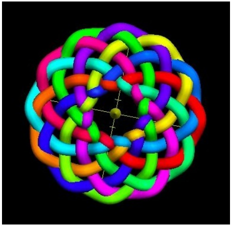
□

Remark

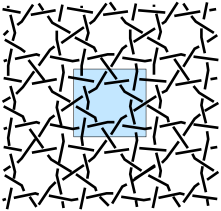
The proof above fails when $\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$.

e.g., the square weave $\mathcal{W}$, and the triaxial link $\mathcal{Q}$

We checked numerically for weaving knots $K_n \xrightarrow{\mathrm{F}} \mathcal{W}$ with hundreds of crossings that the Vol-Det Conjecture does hold.



A typical biperiodic alternating link



Faces of L:

1 octagon
4 pentagons
1 square
8 triangles

$$Vol((T^2 \times I) - L) \approx 47.644829$$

$$\mathrm{vol}^{\lozenge}(L) = Vol(B_8) + 4Vol(B_5) + v_{\mathrm{oct}} + 16v_{\mathrm{tet}} \approx 47.704628$$

$$p(z,w) = wz^2 + z^3 - 2wz + 104z^2 - 2z^3/w + w + 510z + 510z^2/w + z^3/w^2 - 2456z/w + 104z^2/w^2$$
$$+510/w + 1/z + 510z/w^2 + z^2/w^3 + 104/w^2 - 2/(wz) - 2z/w^3 + 1/w^3 + 1/(w^2z) + 104$$

Numerically, $2\pi\,\mathrm{m}(p(z,w)) \approx 47.9214$

So L satisfies Conjecture 1, and the inequality within a range of 0.6%,

$$Vol((T^2 \times I) - L) < \mathrm{vol}^{\lozenge}(L) < 2\pi\,\mathrm{m}(p(z,w)).$$

Exact Mahler measure $\mathrm{m}(p(x,y))$ for certain $p(x,y)$

$$\mathrm{m}(p(x,y)) = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |p(x,y)| \, \frac{dx}{x} \frac{dy}{y}$$

Consider $\mathbb{C}[x,y] = \mathbb{C}[x][y]$, so that for algebraic functions $y_j(x)$ of x,

$$p(x,y) = (y - y_1(x)) \cdots (y - y_d(x))$$

$$\mathrm{m}(p(x,y)) \overset{\text{Jensen}}{=} -\frac{1}{2\pi} \sum_{j=1}^{d} \int_{|x|=1, |y_j(x)| \geq 1} \eta(x, y_j)$$

where

$$\eta(x,y) := \log |x| d \arg y - \log |y| d \arg x$$

is a closed differential form, called the volume form.

Bloch-Wigner dilogarithm $D(z)$

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z)$$

where $\mathrm{Li}_2(z)$ is the classical dilogarithm.

1. $D(z)$ is continuous on $\widehat{\mathbb{C}}$, real analytic on $\mathbb{C} - \{0,1\}$.
2. $D(e^{i\theta}) = Л(\theta)$, where $Л(\theta)$ is the Lobachevsky function.
3. $D(z)$ satisfies the 5–term relation and other identities.
4. $D(z) = Vol(\triangle(z))$.
5. $dD(z) = \eta(z, 1-z)$.

So if $\eta(x, y)$ can be expressed in terms of $\eta(z, 1-z)$'s, then we can use Stokes' Theorem to evaluate $\mathrm{m}(p(x, y))$ exactly in terms of $D(z)$, and get hyperbolic volumes.

Exact Mahler measure $\mathrm{m}(p(x,y))$ for certain $p(x,y)$

Let $X = \{(x,y) \in \mathbb{C}^2 \mid p(x,y) = 0\}$. In $\mathbb{C}(\widehat{X})^* \wedge \mathbb{C}(\widehat{X})^*$, if we can write

$$(\star) \qquad x \wedge y = \sum_k \alpha_k (z_k \wedge (1 - z_k))$$

$$\implies \quad \eta(x,y) = \sum_k \alpha_k \; \eta(z_k, 1 - z_k) = \sum_k \alpha_k \; dD(z_k)$$

$$\mathrm{m}(p(x,y)) = -\frac{1}{2\pi} \sum_{j=1}^{d} \sum_k \alpha_k \; D(z_k)|_{\partial\{|x|=1, |y_j(x)| \geq 1\}}$$

A priori, we may not be able to solve $(\star)$.

(Champanerkar 2003) Can solve $(\star)$ for A–polynomial of any 1–cusped M^3.

If the curve has genus 0, then we can solve $(\star)$ by parametrizing the curve.

Example: Square weave polynomial (Boyd 1998)

Step 1: $$p(z,w) = -\left(4 + w + \frac{1}{w} + z + \frac{1}{z}\right)$$

$$\begin{aligned} -p(z/w, wz) =& 4 + wz + \frac{1}{wz} + \frac{z}{w} + \frac{w}{z} \\ =& \frac{1}{wz}(1 + iw + iz + wz)(1 - iw - iz + wz) \end{aligned}$$

$$\Longrightarrow \quad \mathrm{m}(p(z,w)) = \mathrm{m}(1 + iw + iz + wz) + \mathrm{m}(1 - iw - iz + wz)$$

$$\Longrightarrow \quad \mathrm{m}(p(z,w)) = 2\,\mathrm{m}(1 + iw + iz + wz)$$

Step 2: $1 + iw + iz + wz = 0 \implies z = \dfrac{1 + iw}{w + i}$.

$$(\star) \quad w \wedge z = w \wedge \frac{1 + iw}{i + w} = iw \wedge (1 + iw) - iw \wedge (1 - iw)$$
$$= (-iw) \wedge (1 - (-iw)) - (iw \wedge (1 - iw)).$$

If $w = e^{\mathrm{i}\theta}$, $|z| = \left|\dfrac{1 + iw}{w + i}\right| = \left|\cot\left(\dfrac{2\theta + \pi}{4}\right)\right| \implies |z| \geq 1$ iff $-\pi \leq \theta \leq 0$.
So we must integrate between $w = -1$ and $w = 1$.

$$\mathrm{m}(p(z, w)) = -\frac{1}{2\pi} \sum_{j=1}^{d} \sum_{k} \alpha_k \, D(z_k)|_{\partial\{|w|=1, |z_k(w)| \geq 1\}}$$

So we must evaluate $-\frac{1}{2\pi}(D(-iw) - D(iw))$ on the boundary $w|_{-1}^{1}$

$$2\pi \, \mathrm{m}(p(z, w)) = 2(-D(-i \cdot 1) + D(i \cdot 1) + D(-i \cdot (-1)) - D(i \cdot (-1)))$$
$$= 8 \, D(i) = 2 \, v_{\mathrm{oct}}.$$

Step 2: $\quad 1 + iw + iz + wz = 0 \implies z = \dfrac{1+iw}{w+i}.$

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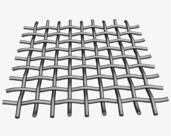
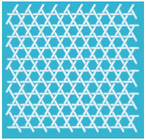
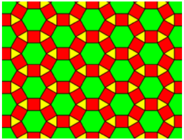
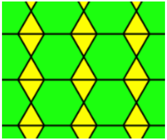
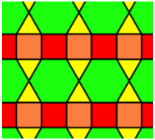
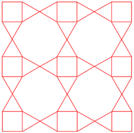
New examples

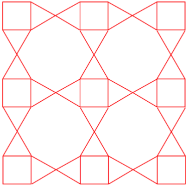
Conjecture 1. $Vol(T^2 \times I - L) \leq \mathrm{vol}^{\lozenge}(L) \leq 2\pi\, \mathrm{m}(p(z,w))$.

We used Boyd's computations to prove Conjecture 1 for the square weave, triaxial and rhombitrihexagonal links:

We use new exact Mahler measure computations to prove Conjecture 1 for more examples:



$$p(z,w) = -w^2z^2 + 6w^2z + 6wz^2 - w^2 + 28wz - z^2 + 6w + 6z - 1$$

$$\begin{aligned} 2\pi\,\mathrm{m}(p(z,w)) &= \arccos\left(-\tfrac{7}{9}\right)\log(17+12\sqrt{2}) + 8D(i) + 4D\left(\tfrac{\sqrt{7+4\sqrt{2}i}}{3}\right) - 4D\left(-\tfrac{\sqrt{7+4\sqrt{2}i}}{3}\right) \\ &\approx 19.771532321797992256575200922336735211 \end{aligned}$$

$$\begin{aligned} \mathrm{vol}^{\Diamond}(L) &\approx 19.6379 \\ Vol(T^2 \times I - L) &\approx 19.5597 \end{aligned}$$

So L satisfies Conjecture 1, and the inequality within a range of 0.4%,

$$Vol(T^2 \times I - L) < \mathrm{vol}^{\Diamond}(L) < 2\pi\,\mathrm{m}(p(z,w)).$$

Why 2π?

A tower of covers: $\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = M$.

For M^3, $H_1(M_n;\mathbb{Z})$ can have arbitrarily large torsion subgroups $TH_1(M_n)$.

Conjecture For closed or 1-cusped hyperbolic M^3, if $\bigcap_n \pi_1 M_n = \{1\}$ for a tower of regular covers M_n,

$$\lim_{n\to\infty} \frac{\log|TH_1(M_n)|}{Vol(M_n)} = \frac{1}{6\pi}$$

This is a special case of Lück's Approximation Conjecture in L^2–torsion theory: For closed or 1-cusped hyperbolic M^3, the analytic L^2–torsion of covering transformations of $\mathbb{H}^3$ is

$$\rho^{(2)}(M) = -\frac{1}{6\pi} Vol(M).$$

Recall, $\det(K) = |H_1(\Sigma_2(K))|$, homology of 2–fold branched cover of K.
$= |TH_1(X(K))|$, torsion of 2–fold cyclic cover of $S^3 - K$.

Let $X(L) =$ 2–fold cyclic cover of $T^2 \times I - L$, given by kernel of $\pi_1(T^2 \times I - L) \to \mathbb{Z}/2\mathbb{Z}$, with L meridians $\to 1$, Hopf link meridians $\to 0$.

Theorem (Champanerkar-K) Let $\mathcal{L}$ be any hyperbolic biperiodic alternating link, with $L_n = \mathcal{L}/(n\mathbb{Z} \times n\mathbb{Z})$, $p(z,w)$ for L_1 as above,

$$\lim_{n\to\infty} \frac{\log |TH_1(X(L_n))|}{Vol(X(L_n))} = \frac{\mathrm{m}(p(z,w))}{2\,Vol(T^2 \times I - L_1)}$$

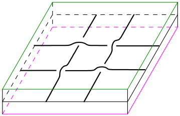
For the subsequence $n = 2^j$, we get a tower of covers with this limit:

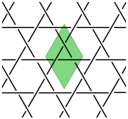
$$\cdots \to X(L_{2n}) \to X(L_n) \to \cdots \to X(L_1).$$

Note: Since $X(\mathcal{L})$ is a common cover, $\bigcap_n \pi_1 X(L_n) \neq \{1\}$.

Growth and hyperbolic volume – Example 3

$W_n = \mathcal{W}/(n\mathbb{Z} \times n\mathbb{Z})$
$c(W_n) = 4n^2$





$Q_n = \mathcal{Q}/(n\mathbb{Z} \times n\mathbb{Z})$
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Theorem (Champanerkar-K) For square weave $\mathcal{W}$ and triaxial link $\mathcal{Q}$,

$$\lim_{n\to\infty} \frac{\log |TH_1(X(W_n))|}{Vol(X(W_n))} = \lim_{n\to\infty} \frac{\log |TH_1(X(Q_n))|}{Vol(X(Q_n))} = \frac{1}{4\pi}$$

As far as we know, these are the first examples of non-cyclic towers of covers of hyperbolic 3–manifolds whose exponential homological torsion growth can be computed exactly in terms of volume growth.

Question Can $1/4\pi$ be explained in terms of L^2–torsion of covering transformations of $X(\mathcal{W})$ and of $X(\mathcal{Q})$?

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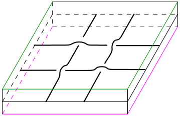

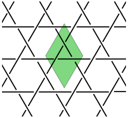

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Theorem $\lim_{n\to\infty} \frac{\log|TH_1(X(L_n))|}{Vol(X(L_n))} = \frac{\mathrm{m}(p(z,w))}{2\,Vol(T^2\times I - L_1)}.$

Conjecture 1 $Vol(T^2\times I - L) \leq \mathrm{vol}^{\Diamond}(L) \leq 2\pi\,\mathrm{m}(p(z,w)).$

Together, these imply that for any hyperbolic biperiodic alternating link $\mathcal{L}$,

Conjecture 2 $\lim_{n\to\infty} \frac{\log|TH_1(X(L_n))|}{Vol(X(L_n))} \geq \frac{1}{4\pi},$

with equality for the square weave and the triaxial link.

Example: For the Rhombitrihexagonal link $\mathcal{R}$,

$$\lim_{n\to\infty} \frac{\log|TH_1(X(R_n))|}{Vol(X(R_n))} = \frac{1}{4\pi}\left(\frac{10\,v_{\mathrm{tet}} + 2\pi\,\log(6)}{10\,v_{\mathrm{tet}} + 3\,v_{\mathrm{oct}}}\right) \approx \frac{1.0126}{4\pi}$$

and similarly for other examples whose $\mathrm{m}(p(z,w))$ we computed exactly.


Thank You