2.5 The Pythagorean theorem

The Pythagorean theorem is about areas, and indeed Euclid proves it immediately after he has developed the theory of area for parallelograms and triangles in Book I of the *Elements*. First let us recall the statement of the theorem.

Pythagorean theorem. *For any right-angled triangle, the sum of the squares on the two shorter sides equals the square on the hypotenuse.*

We follow Euclid's proof, in which he divides the square on the hypotenuse into the two rectangles shown in Figure 2.13. He then shows that the light gray square equals the light gray rectangle and that the dark gray square equals the dark gray rectangle, so the sum of the light and dark squares is the square on the hypotenuse, as required.

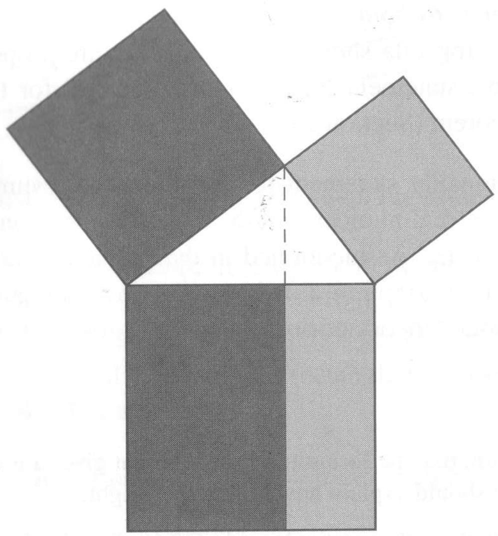

Figure 2.13: Dividing the square for Euclid's proof

First we show equality for the light gray regions in Figure 2.13, and in fact we show that *half* of the light gray square equals half of the light gray rectangle. We start with a light gray triangle that is obviously half of the light gray square, and we successively replace it with triangles of the same base or height, ending with a triangle that is obviously half of the light gray rectangle (Figure 2.14).

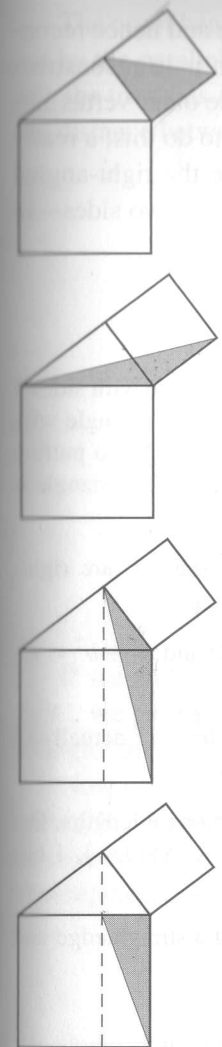

Figure 2.14: Changing the triangle without changing its area

The same argument applies to the dark gray regions, and thus, the Pythagorean theorem is proved. □

Figure 2.13 suggests a natural way to construct a square equal in area to a given rectangle. Given the light gray rectangle, say, the problem is to reconstruct the rest of Figure 2.13.