On Sequences Generic in the Sense of Magidor

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Abstract

The main result of this paper is a combinatorial characterization of Magidor-generic sequences. Using this characterization, I show that the critical sequences of certain iterations are Magidor-generic over the target model. I then employ these results in order to analyze which other Magidor sequences exist in a Magidor extension. One result in this direction is that if we temporarily identify Magidor sequences with their ranges, then Magidor sequences are *maximal*, in the sense that they contain any other Magidor sequence that is present in their forcing extension, even if the other sequence is generic for a different Magidor forcing. A stronger result holds if both sequences come from the same forcing: I show that a Magidor sequence is almost *unique* in its forcing extension, in the sense that any other sequence generic for the same forcing which is present in the same forcing extension coincides with the original sequence at all but finitely many coordinates, and at all limit coordinates. Further, I ask the question: If $d \in \mathrm{V}[c]$, where c and d are Magidor-generic over V, then which Magidor forcing can d be generic for? It turns out that it must essentially be a collapsed version of the Magidor forcing for which c was generic. I treat several related questions as well. Finally, I introduce a special case of Magidor forcing which I call minimal Magidor forcing. This approach simplifies the forcing, and I prove that it doesn't restrict the class of possible Magidor sequences. I.e., if c is generic for a Magidor forcing over V, then it is generic for a minimal Magidor forcing over V.

1 Introduction

Magidor forcing was introduced in [Mag78] with the purpose of collapsing the cofinality of a measurable cardinal to an uncountable regular cardinal, without collapsing cardinals. It was known before how to collapse the cofinality of a measurable cardinal to ω, by Příkrý forcing (see [Pří70]), a method that is by now commonplace in set theory. When looking at the definitions of these forcing notions, it is obvious that they are related in some way, and I am investigating the similarities more closely in this work.

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There are three results on Příkrý forcing that go together very well, that I am particularly interested in, and that were wanting analogs in the context of Magidor forcing. The first of these is due to Mathias ([Mat73]). It says that the ω-sequences added by Příkrý forcing with respect to a normal measure U on κ are exactly those whose range is almost contained in any $A \in U$. This is what I call the Characterization Theorem. The second result, which was observed by Solovay, is that the sequence of critical points resulting from iterating U ω many times satisfies this criterion, over the ω-th model of the iteration. So the sequence of critical points is Příkrý-generic over the direct limit model. The third result is that if c is Příkrý-generic over V and $d \in \mathrm{V}[c]$ is also Příkrý-generic, then $\operatorname{ran}(d)$ is almost contained in $\operatorname{ran}(c)$. This is what I call Maximality.

These three results, the combinatorial characterization, the genericity of the critical sequence, and the maximality of Příkrý sequences, are closely connected: the genericity of the critical sequence follows immediately from the combinatorial characterization, and the maximality of Příkrý sequences can be shown quite elegantly using the fact that the critical sequence is Příkrý-generic. I have previously proven analogs of these facts for a class of generalizations of Příkrý forcing, which includes what is sometimes referred to as diagonal Příkrý forcing (see [Git10, Section 1.3]) – this was done in [Fuc05].

I prove analogs of these results for Magidor forcing in the present paper. I give some background and basic results on Magidor forcing in Section 2. Most of the results in that section are known from [Mag78], but I introduce some new concepts.

In Section 3, I develop the technical tools to prove the main result, the Characterization Theorem, utilizing some machinery which was introduced by Magidor in [Mag78].

In Section 4, I state and prove the Characterization Theorem. To understand this characterization, I have to give some context first. The starting point is a sequence $\langle U_\gamma \mid \gamma < \alpha \rangle$ of normal ultrafilters on a measurable cardinal κ which is increasing in the Mitchell order. For $\gamma < \delta$, a function f^δ_γ is fixed which witnesses that U_γ is below U_δ in the Mitchell order, i.e., so that $U_\gamma = [f^\delta_\gamma]_{U_\delta}$. Magidor forcing will add a function $g : \alpha \longrightarrow \kappa$ so that (in case α is a limit ordinal) the range of g is cofinal in κ, without collapsing cardinals. I show that essentially, g is characterized by two combinatorial properties. Firstly, whenever $\langle X_\gamma \mid \gamma < \alpha \rangle$ is a sequence in the ground model such that for all $\gamma < \alpha$, $X_\gamma \in U_\gamma$, then for sufficiently large ξ, $g(\xi) \in X_\xi$. Secondly, whenever $\lambda < \alpha$ is a limit ordinal and $\langle Y_\gamma \mid \gamma < \lambda \rangle$ is a sequence in the ground model such that for all $\gamma < \lambda$, $Y_\gamma \in f^\lambda_\gamma(g(\lambda))$, then for sufficiently large $\xi < \lambda$, $g(\xi) \in Y_\xi$. The precise statement can be found in Theorem 4.4.

It is then fairly straightforward to deduce that the critical sequence of an adequate iteration satisfies this characterization, and is hence Magidor-generic over the limit model. This is done in section 5. It was known that the critical sequence of such an iteration is Magidor-generic over the final model, as shown by Dehornoy in [Deh83], without using a combinatorial characterization of Magidor genericity, and using a tree version of Magidor forcing. It turns out, though, that in order to prove the maximality of Magidor sequences, I need more flexibility when iterating - basically, given a condition p in Magidor forcing, I have to iterate in such a way that if $\pi : \mathrm{V} \longrightarrow M$ is the resulting embedding, $\pi(p)$ belongs to the generic filter associated to the critical sequence.

This leads to the method of “iterating along a condition”, which I use in Section 6. I develop this method in the proof of the Maximality Theorem, Theorem 6.1. A strong form of the theorem says that the range of any Magidor sequence d in a given Magidor-generic extension $\mathrm{V}[c]$ is almost contained in the range of c - even if d comes from a *different* Magidor forcing.

In Section 7 on uniqueness, I improve the result of the Maximality Theorem in the case that $d \in \mathrm{V}[c]$ and both c and d are generic for the *same* Magidor forcing over V. The result is that in this situation, for almost all ξ, and for all limit ξ, $c(\xi) = d(\xi)$. So not only does the range of c almost contain every Magidor sequence $d \in \mathrm{V}[c]$ which is generic for the same forcing, but in fact, $c(\xi) = d(\xi)$ for almost all ξ. In particular, $\mathrm{V}[c] = \mathrm{V}[d]$. This is Theorem 7.5 and Corollary 7.6. There are also some miscellaneous results in this section that use an argument introduced there. The theme is the question what other Příkrý-type or Magidor sequences can exist in $\mathrm{V}[c]$, where c is a Magidor sequence. I show that a Magidor extension contains no Příkrý sequence. I also analyze what can be said in the general case that $d \in \mathrm{V}[c]$, where both d and c are Magidor generic, but possibly for different Magidor forcings, about pointwise equality between c and d, and about the relationship between the sequences of ultrafilters and representing functions used in the two forcings. The most general result here is Lemma 7.11. I also introduce collapses of Magidor forcings and show in Theorem 7.16 that the Magidor sequences present in $\mathrm{V}[c]$ are (modulo finite) exactly those that are generic for a collapse of the Magidor forcing giving rise to the sequence c.

In Section 8, I introduce a very natural version of Magidor forcing, which I call minimal Magidor forcing. It simplifies the combinatorics of Magidor forcing, since it allows to work with functions g_γ instead of f^δ_γ. The main result of this section is that any Magidor sequence is generic for a minimal Magidor forcing. This result is Theorem 8.2.

I would like to than Ronald Jensen for discussing Příkrý forcing and Magidor forcing with me many years ago. I would also like to thank the unknown referee for reading an earlier version of this paper very carefully, and for making very useful suggestions.

2 Basics on Magidor Forcing

Let κ be measurable of Mitchell order α, and let $\vec{U} = \langle U_\gamma \mid \gamma < \alpha \rangle$ be a sequence of normal ultrafilters on κ, increasing in the Mitchell order $\lhd$. For $\mu < \nu < \alpha$, let $f^\nu_\mu : \kappa \longrightarrow \mathrm{V}$ be a function representing U_μ in the ultrapower of V by U_ν, i.e.,

$$U_\mu = [f^\nu_\mu]_{U_\nu}.$$

Magidor [Mag78] created a notion of forcing $\mathbb{M} = \mathbb{M}(\langle U_\gamma \mid \gamma < \alpha\rangle, \langle f^\nu_\mu \mid \mu < \nu < \alpha\rangle)$ in this setting, with regular α, which, if α is a limit ordinal, adds an α-sequence cofinal in κ without collapsing cardinals. I will write $\vec{f}$ for the sequence $\langle f^\nu_\mu \mid \mu < \nu < \alpha\rangle$ of functions. Also, I will say that the forcing has *length* α and is *based at* κ. I will not require in general that α is a limit ordinal (even though originally the point of the forcing was to change the cofinality of κ to α, so α was assumed to be a regular cardinal).

In order to define the forcing $\mathbb{M}$, Magidor first noted that for $0 < \gamma < \alpha$, the following two sets belong to U_γ.

$$\begin{array}{rcl} A_\gamma & = & \{\delta < \kappa \mid \forall \mu < \nu < \gamma \quad f^\gamma_\mu(\delta) \lhd f^\gamma_\nu(\delta) \text{ are normal ultrafilters on } \delta\} \\ B_\gamma & = & \{\delta \in A_\gamma \mid \forall \mu < \nu < \gamma \quad [f^\nu_\mu \restriction \delta]_{f^\gamma_\nu(\delta)} = f^\gamma_\mu(\delta)\} \end{array}$$

Of course, $B_0 := \{\delta < \kappa \mid \delta \text{ is inaccessible}\} \in U_0$ also.

Definition 2.1. For $a \in [\alpha]^{<\omega}$, define functions $l_a : \alpha \longrightarrow \alpha \cup \{-1\}$ and $r_a : \alpha \longrightarrow \alpha + 1$ by

$$\begin{array}{rcl} l_a(\gamma) & = & \begin{cases} \max(a \cap \gamma) & \text{if } a \cap \gamma \neq \emptyset \\ -1 & \text{otherwise.} \end{cases} \\ r_a(\gamma) & = & \min((a \cup \{\alpha\}) \setminus (\gamma + 1)). \end{array}$$

I will define the forcing $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f}, \tilde{\alpha})$, the Magidor forcing wrt. $\vec{U}, \vec{f}$ *above* $\tilde{\alpha}$, for $\mathrm{lh}(\vec{U}) = \alpha \leq \tilde{\alpha} < \kappa$ as follows. The usual Magidor forcing $\mathbb{M}(\vec{U}, \vec{f})$ will be $\mathbb{M}(\vec{U}, \vec{f}, \alpha)$, i.e., Magidor forcing above α. I will say that the *length* of $\mathbb{M}(\vec{U}, \vec{f})$ is α, i.e., the domain of $\vec{U}$, and I will say that the forcing is *at* κ, the measurable cardinal on which the normal ultrafilters from $\vec{U}$ live.

Conditions are pairs of functions $\langle g, G\rangle$ such that

1. $\mathrm{dom}(g)$ is a finite subset of α, and $\mathrm{dom}(G) = \alpha \setminus \mathrm{dom}(g)$,
2. For all $\gamma \in \mathrm{dom}(g)$, $g(\gamma) \in B_\gamma$, $g(\gamma) > \tilde{\alpha}$, and g is strictly increasing,
3. For all $\gamma \in \mathrm{dom}(G)$, if $\theta := r_{\mathrm{dom}(g)}(\gamma) < \alpha$, then $G(\gamma) \in f^\theta_\gamma(g(\theta))$, and if $\theta = \alpha$, then $G(\gamma) \in U_\gamma$,
4. If $\gamma < \delta < \alpha$, $\gamma \in \mathrm{dom}(g)$ and $\delta \in \mathrm{dom}(G)$, then $g(\gamma) \cap G(\delta) = \emptyset$.

The ordering on $\mathbb{M}$ is defined by saying that $\langle g', G'\rangle \leq \langle g, G\rangle$ iff

1. $g \subseteq g'$,
2. For all $\gamma \in \mathrm{dom}(G')$, $G'(\gamma) \subseteq G(\gamma)$,
3. For all $\gamma \in \mathrm{dom}(g') \setminus \mathrm{dom}(g)$, $g'(\gamma) \in G(\gamma)$.

The reason for introducing the Magidor forcing above $\tilde{\alpha}$ is purely technical, and will only be relevant in the proof of the Characterization Theorem. Note that the only occurrence of $\tilde{\alpha}$ is in point 2. of the definition of what a condition is. I will regard $\vec{U}$, $\vec{f}$, $\tilde{\alpha}$ and $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f}, \tilde{\alpha})$ as fixed through section 4.

Lemma 2.2. *Suppose $\gamma < \alpha$, $\delta < \kappa$, and $\langle g, G\rangle \in \mathbb{M}$ is a condition with $\gamma \notin a := \mathrm{dom}(g)$ such that there is an extension $\langle g', G'\rangle \le \langle g, G\rangle$ with $g' = g \cup \{\langle \gamma, \delta\rangle\}$. Then there is a weakest condition $\langle f, F\rangle = \langle g, G\rangle_{\langle \gamma, \delta\rangle}$ with these properties (i.e., $\langle f, F\rangle \le \langle g, G\rangle$, $f(\gamma) = \delta$ and $\langle g', G'\rangle \le \langle f, F\rangle$). This condition is defined by*

$$\begin{array}{rcl} f & = & g \cup \{\langle \gamma, \delta\rangle\}, \\ \mathrm{dom}(F) & = & \alpha \setminus (a \cup \{\gamma\}), \\ F(\xi) & = & \begin{cases} G(\xi) \cap \delta & \text{if } r_{a\cup\{\gamma\}}(\xi) = \gamma, \\ G(\xi) \setminus (\delta + 1) & \text{if } l_{a\cup\{\gamma\}}(\xi) = \gamma, \\ G(\xi) & \text{if } \xi \in \alpha \setminus (a \cup \{\gamma\}) \text{ and the above cases fail.} \end{cases} \end{array}$$

Lemma 2.3. *Let $\langle g, G\rangle$ be a condition, let $\gamma \in \alpha \setminus \mathrm{dom}(g)$, and let $\delta < \kappa$. Then there is a condition $\langle g \cup \{\langle \gamma, \delta\rangle\}, G'\rangle \le \langle g, G\rangle$ iff $\delta \in G(\gamma)$ and for all $\xi < \gamma$ with $r_{\mathrm{dom}(g)}(\xi) = \gamma$, $G(\xi) \cap \delta \in f^{\gamma}_{\xi}(\delta)$.*

Lemma 2.4. *Suppose $\alpha_0 < \ldots < \alpha_{n-1} < \alpha$, $\delta_0 < \ldots < \delta_{n-1} < \kappa$, and $\langle g, G\rangle \in \mathbb{M}$ is a condition with $\gamma_0, \ldots, \gamma_{n-1} \notin a := \mathrm{dom}(g)$ such that there is an extension $\langle g', G'\rangle \le \langle g, G\rangle$ with $g' = g \cup h$, where $h = \{\langle \gamma_i, \delta_i\rangle \mid i < n\}$. Then there is a weakest condition with these properties, which I denote by $\langle g, G\rangle_{\langle \gamma_0, \delta_0\rangle, \ldots, \langle \gamma_{n-1}, \delta_{n-1}\rangle}$ or $\langle g, G\rangle_h$.*

Proof. The condition in question can be obtained by applying the definition given in the previous lemma n times. More precisely, define $\langle \langle g_i, G_i\rangle \mid i < n\rangle$ by setting $\langle g_0, G_0\rangle = \langle g, G\rangle_{\langle \alpha_0, \delta_0\rangle}$, and $\langle g_{i+1}, G_{i+1}\rangle = \langle g_i, G_i\rangle_{\langle \alpha_{i+1}, \delta_{i+1}\rangle}$, for $i + 1 < n$. Then $\langle g, G\rangle_h = \langle g_{n-1}, G_{n-1}\rangle$. □

The proof of [Mag78, Lemma 3.2] actually shows the following, even though it doesn't state it that way.

Lemma 2.5. *Let $\langle g, H\rangle \in \mathbb{M}$, and let $\gamma \in \alpha \setminus \mathrm{dom}(g)$. Then there is a condition $\langle g, H^{+\gamma}\rangle \le \langle g, H\rangle$ such that for all $\rho \in \alpha \setminus (\mathrm{dom}(g) \cup \{\gamma\})$, $H^{+\gamma}(\rho) = H(\rho)$, and such that for every $\xi \in H^{+\gamma}(\gamma)$, there is a condition of the form $\langle g \cup \{\langle \gamma, \xi\rangle\}, Z\rangle$ extending $\langle g, H^{+\gamma}\rangle$ – i.e., equivalently, $\langle g, H^{+\gamma}\rangle_{\langle \gamma, \xi\rangle} \in \mathbb{M}$.*

Proof. The proof is in [Mag78], but I would like to review the definition of $H^{+\gamma}$. Clearly, it suffices to define $H^{+\gamma}(\gamma)$. Let $a = \mathrm{dom}(g)$, $\beta = r_a(\gamma)$, $\mu = l_a(\gamma)$. Then $H^{+\gamma}(\gamma) = H(\gamma) \cap \bigcap_{\mu < \eta < \gamma} D_\eta$, where in case $\beta < \alpha$,

$$D_\eta = \{\delta < g(\beta) \mid H(\eta) \cap \delta \in f^{\gamma}_{\eta}(\delta)\},$$

for $\mu < \eta < \gamma$, and in case $\beta = \alpha$,

$$D_\eta = \{\delta < \kappa \mid H(\eta) \cap \delta \in f^{\gamma}_{\eta}(\delta)\}.$$

In the more involved case $\beta < \alpha$, the point here is that $D_\eta \in f^\beta_\gamma(\theta)$, where $\theta = g(\beta)$. This follows since $\theta \in B_\beta$, because this implies that $f^\beta_\eta(\theta) = [f^\gamma_\eta \restriction \theta]_{f^\beta_\gamma(\theta)}$. Since $H(\eta) \in f^\beta_\eta(\theta)$, this means that $\{\delta < \theta \mid H(\eta) \cap \delta \in f^\gamma_\eta(\delta)\} = D_\eta \in f^\beta_\gamma(\theta)$.

So $H^{+\gamma}(\gamma)$ depends only on $H\restriction(\gamma+1)$ and g. $\square$

Definition 2.6. A condition $\langle g, G\rangle \in \mathbb{M}$ is *pruned* if for any $\alpha_0 < \ldots < \alpha_{n-1} < \alpha$ with $\alpha_i \notin \mathrm{dom}(g)$, and any $\delta_0 \in G(\alpha_0)$, ..., $\delta_{n-1} \in G(\alpha_{n-1})$ with $\delta_0 < \delta_1 < \ldots < \delta_{n-1}$, there is a condition

$$\langle g \cup \{\langle \alpha_0, \delta_0\rangle, \ldots, \langle \alpha_{n-1}, \delta_{n-1}\rangle\}, H\rangle \leq \langle g, G\rangle.$$

Lemma 2.7. *A condition $\langle g, G\rangle \in \mathbb{M}$ is pruned iff for any $\alpha_0 < \alpha$ with $\alpha_0 \notin \mathrm{dom}(g)$ and any $\delta_0 \in G(\alpha_0)$, there is a condition $\langle g \cup \{\langle \alpha_0, \delta_0\rangle\}, H\rangle \leq \langle g, G\rangle$. This, in turn, is equivalent to saying that $\langle g, G\rangle_{\langle \alpha_0, \delta_0\rangle} \in \mathbb{M}$.*

Proof. For the purpose of this proof, let's say that $\langle g, G\rangle$ is *weakly pruned* if for any $\alpha_0 < \alpha$ with $\alpha_0 \notin \mathrm{dom}(g)$ and any $\delta_0 \in G(\alpha_0)$, $\langle g, G\rangle_{\langle \alpha_0, \delta_0\rangle} \in \mathbb{M}$. Clearly, if $\langle g, G\rangle$ is pruned, then it is weakly pruned. For the converse, assume that $\langle g, G\rangle$ is weakly pruned. To show it is pruned, let $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha$ be given, so that $\alpha_i \notin \mathrm{dom}(g)$, and let $\delta_0 < \delta_1 < \ldots < \delta_{n-1}$ be such that $\delta_i \in G(\alpha_1)$. Let $h = \{\langle \alpha_i, \delta_i\rangle \mid i < n\}$. Since $\langle g, G\rangle$ is weakly pruned, we know that $\langle g \cup \{\langle \alpha_i, \delta_i\rangle\}, H_{\alpha_i}\rangle := \langle g, G\rangle_{\langle \alpha_i, \delta_i\rangle} \in \mathbb{M}$, for all $i < n$. Let's use the definition of $\langle g \cup h, H\rangle := \langle g, G\rangle_h$, even though we haven't shown that this is a condition yet. To see that it is a condition, note that it is clear that $g \cup h$ is strictly monotonous, since if $\mu < \nu$ and $\mu, \nu \in \mathrm{dom}(g \cup h)$, then either both μ and ν are in the domain of g, in which case $g(\mu) < g(\nu)$ since $\langle g, G\rangle$ is a condition, or both are in the domain of h, in which case $h(\mu) < h(\nu)$ by assumption, or one is in the domain of g and the other is in the domain of h - say ξ is the one that is in the domain of h, then $(g \cup h)(\mu) < (g \cup h)(\nu)$ because $\langle g, G\rangle_{\langle \xi, h(\xi)\rangle} \in \mathbb{M}$. The only other point that needs to be verified is that for all $\xi \in \alpha \setminus \mathrm{dom}(g \cup h)$, $H(\xi)$ has measure one with respect to the appropriate normal ultrafilter. To see this, fix such a ξ. If $\xi > \alpha_{n-1}$, then $H(\xi) = H_{\alpha_{n-1}}(\xi)$, and the appropriate ultrafilter is either U_ξ or $f_\xi^{r_{\mathrm{dom}(g)}(\xi)}(g(r_{\mathrm{dom}(g)}(\xi)))$. If $\alpha_i < \xi < \alpha_{i+1}$, then the relevant ultrafilter in $\langle g, H\rangle$ is the same as in $\langle g, H_{\alpha_{i+1}}\rangle$, and $H(\xi) = H_{\alpha_{i+1}}(\xi) \setminus (\delta_i + 1)$, which clearly belongs to that ultrafilter. If $\xi < \alpha_0$, then $H(\xi) = H_{\alpha_0}(\xi)$, which again belongs to the right ultrafilter. $\square$

Lemma 2.8. *Any condition $\langle g, G\rangle \in \mathbb{M}$ has an extension $\langle g, G'\rangle$ which is pruned.*

Proof. By recursion on $\gamma < \alpha$, define a sequence $\langle G_\gamma \mid \gamma < \alpha\rangle$, so that $\langle g, G_\gamma\rangle \in \mathbb{M}$, as follows: If $0 \notin \mathrm{dom}(g)$, then $\langle g, G_0\rangle = \langle g, G^{+0}\rangle$ (the condition from Lemma 2.5). Otherwise, if $0 \in \mathrm{dom}(g)$, then $G_0 = G$.

If $\langle G_\gamma \mid \gamma < \xi\rangle$ is already defined, then define, for $\mu \notin \mathrm{dom}(g)$: $\bar{G}_\xi(\mu) = G_\mu(\mu)$, for $\mu < \xi$, and let $\bar{G}_\xi(\mu) = G(\mu)$, for $\xi \leq \mu \leq \alpha$. Inductively, $\langle g, \bar{G}_\xi\rangle \in \mathbb{M}$. Now, if $\xi \in \mathrm{dom}(g)$, then let $G_\xi = \bar{G}_\xi$, and if $\xi \notin \mathrm{dom}(g)$, then let $\langle g, G_\xi\rangle = \langle g, \bar{G}^{+\xi}_\xi\rangle$.

Finally, let $G'(\xi) = G_\xi(\xi)$, for $\xi \in \alpha \setminus \mathrm{dom}(g)$.

To see that $\langle g, G' \rangle$ is as desired, first note that it is easy to see that it is a valid condition. To see that it is pruned, it suffices by Lemma 2.7 to show that if $\alpha_0 \in \alpha \setminus \mathrm{dom}(g)$ and $\delta_0 \in G'(\alpha_0)$, then $\langle g, G' \rangle_{\langle \alpha_0, \delta_0 \rangle} \in \mathbb{M}$. To see this, note that $G' \restriction (\alpha_0 + 1) = G_{\alpha_0} \restriction (\alpha_0 + 1)$ and $G' \restriction \alpha_0 = \bar{G}_{\alpha_0}$. Since $\langle g, G_{\alpha_0} \rangle = \langle g, \bar{G}^{+\alpha_0}_{\alpha_0} \rangle$, it follows that for any $\delta \in G'(\alpha_0) = G_{\alpha_0}(\alpha_0)$ and any $\xi < \alpha_0$ with $r_{\mathrm{dom}(g)}(\xi) = \alpha_0$, $G'(\xi) \cap \delta = G_{\alpha_0}(\xi) \cap \delta \in f^{\alpha_0}_\xi(\delta)$. This suffices, by Lemma 2.3. In particular, this is true for δ. □

Definition 2.9. For a condition $\langle g, G \rangle \in \mathbb{M}$ and an ordinal $\beta < \alpha$, let

$$\begin{aligned} \langle g, G \rangle_\beta &= \langle g \restriction (\beta + 1), G \restriction (\beta + 1) \rangle, \\ \langle g, G \rangle^\beta &= \langle g \restriction (\alpha \setminus (\beta + 1)), G \restriction (\alpha \setminus (\beta + 1)) \rangle. \end{aligned}$$

If $\langle g, G \rangle$ and $\langle h, H \rangle$ are conditions with $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(h)$ and $g(\beta) = h(\beta)$, then let $\langle g, G \rangle_\beta {}^\frown \langle h, H \rangle^\beta$ be the unique condition $\langle i, I \rangle$ with $\langle i, I \rangle_\beta = \langle g, G \rangle_\beta$ and $\langle i, I \rangle^\beta = \langle h, H \rangle^\beta$ (it is easy to see that there is such a condition).

The main combinatorial result from [Mag78] I shall use is the following diagonalization lemma.

Lemma 2.10 (Diagonalization, [Mag78, Lemma 4.2])**.** *Let $\langle g, G \rangle \in \mathbb{M}$, $\gamma \in \alpha \setminus \mathrm{dom}(g)$. Let $\rho = r_{\mathrm{dom}(g)}(\gamma)$, $\eta = g(\rho)$, $Z = f^\rho_\gamma(\eta)$ (where as usual, $g(\alpha)$ is understood to be κ and $f^\alpha_\gamma(\kappa)$ is understood to be U_γ). Let $A \in Z$, and for every $\xi \in A$, let $\langle g \cup \{\langle \gamma, \xi \rangle\}, H^\xi \rangle \leq \langle g, G \rangle$.*

Then there exists a condition $\langle g, H \rangle \leq \langle g, G \rangle$ such that for every $\langle j, J \rangle \leq \langle g, H \rangle$ with $\gamma \in \mathrm{dom}(j)$, it follows that $\langle j, J \rangle \leq \langle g \cup \{\langle \gamma, j(\gamma) \rangle\}, H^{j(\gamma)} \rangle$.

If, moreover, $\beta \in \gamma \cap \mathrm{dom}(g)$ is such that for every $\xi \in A$, $H^\xi \restriction \beta = G \restriction \beta$, then $\langle g, H \rangle$ can be chosen so that $\langle g, H \rangle_\beta = \langle g, G \rangle_\beta$.

One main fact from [Mag78] about Magidor forcing that I shall need is the following.

Fact 2.11. *If G is $\mathbb{M}$-generic, then letting $c = \bigcup \{s \mid \exists T \quad \langle s, T \rangle \in G\}$, it follows that $\mathrm{V}[G] = \mathrm{V}[c]$, and $\mathrm{V}[c]$ has no new subsets of $c(0)$.*

3 Capturing dense open sets

In this section, I develop the technical tools needed for the proof of the Characterization Theorem 4.4 in the next section. Let Δ be a dense open subset of $\mathbb{M}$, fixed for this section.

Lemma 3.1. *Let $\langle g, G \rangle \in \mathbb{M}$, $a = \mathrm{dom}(g)$, and let $\alpha_0 < \ldots < \alpha_{n-1} < \alpha$, $\alpha_i \notin a$. Then there exists a condition $\langle g, H \rangle \leq \langle g, G \rangle$ such that*

(∗) *If there exists a $\langle j, J \rangle \leq \langle g, H \rangle$ such that $\langle j, J \rangle \in \Delta$ and $\mathrm{dom}(j) \setminus \mathrm{dom}(g) = \{\alpha_0, \ldots, \alpha_{n-1}\}$, then for all $\langle s, B \rangle \leq \langle g, H \rangle$ with $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq (\mathrm{dom}(s) \setminus \mathrm{dom}(g))$, it follows that $\langle s, B \rangle \in \Delta$.*

Proof. It may be assumed that $\langle g, G\rangle$ is pruned, for otherwise, one could instead work with a pruned $\langle g, G'\rangle \leq \langle g, G\rangle$.

I prove the claim by induction on n.

In the case $n = 0$, no αs are given. If there is an extension $\langle g, J\rangle \leq \langle g, G\rangle$ with $\langle g, J\rangle \in \Delta$, then let $\langle g, H\rangle$ be such an extension of $\langle g, G\rangle$, and we are done, since then, any extension $\langle s, B\rangle \leq \langle g, H\rangle$ belongs to Δ, since Δ is open. If there is no such extension of $\langle g, G\rangle$, then we can let $\langle g, H\rangle = \langle g, G\rangle$. This choice makes $(*)$ vacuously true.

Now suppose the claim has been proven for $n - 1$. For $\xi \in G(\alpha_0)$, let

$$\langle g^\xi, G^\xi\rangle = \langle g, G\rangle_{\langle \alpha_0, \xi\rangle}.$$

Apply the inductive assumption to $\langle g^\xi, G^\xi\rangle$, in order to get a condition $\langle g^\xi, H^\xi\rangle \leq \langle g^\xi, G^\xi\rangle$ with

$(*_\xi)$ if there is an $\langle i, I\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\langle i, I\rangle \in \Delta$ and $\mathrm{dom}(i) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$, then for all $\langle s, B\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\{\alpha_1, \ldots, \alpha_{n-1}\} \subseteq \mathrm{dom}(s)$, it follows that $\langle s, B\rangle \in \Delta$.

Let $Z = f_{\alpha_0}^{r_a(\alpha_0)}(g(r_a(\alpha_0)))$. Let C be the set of all $\xi \in G(\alpha_0)$ such that there is a condition $\langle s, B\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\mathrm{dom}(s) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\langle s, B\rangle \in \Delta$. If $C \in Z$, then let $A = C$, and otherwise, let $A = G(\alpha_0) \setminus C \in Z$.

Now apply the Diagonalization Lemma to $\langle g, G\rangle$ and $\langle \langle g^\xi, H^\xi\rangle \mid \xi \in A\rangle$. Let the resulting condition be $\langle g, H\rangle$. I claim $\langle g, H\rangle$ satisfies $(*)$. To see this, let $\langle j, J\rangle \leq \langle g, H\rangle$ be such that $\mathrm{dom}(j) \setminus \mathrm{dom}(g) = \{\alpha_0, \ldots, \alpha_{n-1}\}$ and $\langle j, J\rangle \in \Delta$. Let $\xi = j(\alpha_0)$. Then $\langle j, J\rangle \leq \langle g^\xi, H^\xi\rangle$, by the choice of $\langle g, H\rangle$ according to the Diagonalization Lemma. Moreover, $\mathrm{dom}(j) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\langle j, J\rangle \in \Delta$. So $\xi \in C$, which means that $A = C$.

Now let $\langle s, B\rangle \leq \langle g, H\rangle$ be any condition with $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq \mathrm{dom}(s)$. I have to show that $\langle s, B\rangle \in \Delta$. Let $\xi' = s(\alpha_0)$. It follows that $\langle s, B\rangle \leq \langle g^{\xi'}, H^{\xi'}\rangle$ and $\xi' \in A = C$. Since $\xi' \in C$, there is a condition $\langle i, I\rangle \leq \langle g^{\xi'}, H^{\xi'}\rangle$ with $\mathrm{dom}(i) \setminus \mathrm{dom}(g^{\xi'}) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\langle i, I\rangle \in \Delta$. By $(*_{\xi'})$, it follows that $\langle s, B\rangle \in \Delta$. $\square$

The construction of the previous lemma can be modified so as to preserve initial segments of the conditions involved.

Lemma 3.2. *Let $\langle g, G\rangle \in \mathbb{M}$, $a = \mathrm{dom}(g)$, and let $\beta < \alpha_0 < \ldots < \alpha_{n-1} < \alpha$, $\alpha_i \notin a$, $\beta \in \mathrm{dom}(g)$. Then there exists a condition $\langle g, H\rangle \leq \langle g, G\rangle$ with $\langle g, H\rangle_\beta = \langle g, G\rangle_\beta$ such that*

$(*)$ *If there exists a $\langle j, J\rangle \leq \langle g, H\rangle$ such that $\langle j, J\rangle \in \Delta$, $\mathrm{dom}(j) \setminus \mathrm{dom}(g) = \{\alpha_0, \ldots, \alpha_{n-1}\}$ and $\langle j, J\rangle_\beta = \langle g, G\rangle_\beta$, then for all $\langle s, B\rangle \leq \langle g, H\rangle$ with $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq (\mathrm{dom}(s) \setminus \mathrm{dom}(g))$ it follows that $\langle s, B\rangle \in \Delta$.*

Proof. We may assume that $\langle g, G\rangle$ is pruned above β (meaning that for all $\gamma \in \alpha \setminus (\beta + 1)$ and for all $\xi \in G(\gamma)$, $\langle g, G\rangle_{\langle \gamma, \xi\rangle} \in \mathbb{M}$). The proof is again by induction on n.

In the case $n = 0$, if there is an extension $\langle g, J\rangle \leq \langle g, G\rangle$ with $\langle g, J\rangle \in \Delta$ and $\langle g, J\rangle_\beta = \langle g, G\rangle_\beta$, let $\langle g, H\rangle$ be such an extension of $\langle g, G\rangle$, and we are done. Otherwise setting $\langle g, H\rangle = \langle g, G\rangle$ works as before.

Now suppose the claim has been proven for $n - 1$. For $\xi \in G(\alpha_0)$, let $\langle g^\xi, G^\xi\rangle = \langle g, G\rangle_{\langle \alpha_0, \xi\rangle}$. Apply the inductive assumption to $\langle g^\xi, G^\xi\rangle$, noting that $\langle g^\xi, G^\xi\rangle_\beta = \langle g, G\rangle_\beta$ (because $\beta < \alpha_0$ and $\beta \in \mathrm{dom}(g)$), in order to get a condition $\langle g^\xi, H^\xi\rangle \leq \langle g^\xi, G^\xi\rangle$ with $\langle g^\xi, H^\xi\rangle_\beta = \langle g^\xi, G^\xi\rangle_\beta = \langle g, G\rangle_\beta$, such that

$(*_\xi)$ if there is an $\langle i, I\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\langle i, I\rangle \in \Delta$, $\mathrm{dom}(i) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\langle i, I\rangle_\beta = \langle g, G\rangle_\beta$, then for all $\langle s, B\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\{\alpha_1, \ldots, \alpha_{n-1}\} \subseteq \mathrm{dom}(s)$ it follows that $\langle s, B\rangle \in \Delta$.

Let Z be the same ultrafilter as before, and let C be the set of all $\xi \in G(\alpha_0)$ such that there is a condition $\langle s, B\rangle \leq \langle g^\xi, H^\xi\rangle$ with $\mathrm{dom}(s) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $\langle s, B\rangle_\beta = \langle g, G\rangle_\beta$, and $\langle s, B\rangle \in \Delta$. If $C \in Z$, then let $A = C$, and otherwise, let $A = G(\alpha_0) \setminus C \in Z$.

Apply the Diagonalization Lemma (the "moreover" part) to $\langle g, G\rangle$, β and $\langle \langle g^\xi, H^\xi\rangle \mid \xi \in A\rangle$. To see that the resulting condition, $\langle g, H\rangle$, satisfies $(*)$, let $\langle j, J\rangle \leq \langle g, H\rangle$ be such that $\mathrm{dom}(j) \setminus \mathrm{dom}(g) = \{\alpha_0, \ldots, \alpha_{n-1}\}$, $\langle j, J\rangle_\beta = \langle g, G\rangle_\beta$ and $\langle j, J\rangle \in \Delta$. Let $\xi = j(\alpha_0)$. Then $\langle j, J\rangle \leq \langle g^\xi, H^\xi\rangle$, $\mathrm{dom}(j) \setminus \mathrm{dom}(g^\xi) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\langle j, J\rangle \in \Delta$. So $\xi \in C$, which means that $A = C$.

Now let $\langle s, B\rangle \leq \langle g, H\rangle$ be any condition with $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq \mathrm{dom}(s)$, and let $\xi' = s(\alpha_0)$. It follows that $\langle s, B\rangle \leq \langle g^{\xi'}, H^{\xi'}\rangle$ and $\xi' \in A = C$. Since $\xi' \in C$, there is a condition $\langle i, I\rangle \leq \langle g^{\xi'}, H^{\xi'}\rangle$ with $\mathrm{dom}(i) \setminus \mathrm{dom}(g^{\xi'}) = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $\langle i, I\rangle_\beta = \langle g, G\rangle_\beta$ and $\langle i, I\rangle \in \Delta$. By $(*_{\xi'})$, it follows that $\langle s, B\rangle \in \Delta$. $\square$

It is easy to get the following "global" versions of the previous two lemmas.

Lemma 3.3. *Let $\langle g, G\rangle \in \mathbb{M}$. Then there exists a condition $\langle g, H\rangle \leq \langle g, G\rangle$ such that the following holds: If $\langle s, B\rangle$ is an extension of $\langle g, H\rangle$ that belongs to Δ, then for any $\langle s', B'\rangle \leq \langle g, H\rangle$ with $\mathrm{dom}(s) \subseteq \mathrm{dom}(s')$, it follows that $\langle s', B'\rangle \in \Delta$.*

Proof. For any sequence $\vec{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\rangle$ with $\alpha_0 < \ldots < \alpha_{n-1} < \alpha$ and $\{\alpha_0, \ldots, \alpha_{n-1}\} \cap \mathrm{dom}(g) = \emptyset$, let $\langle g, H^{\vec{\alpha}}\rangle$ satisfy $(*)$ of Lemma 3.1. There are $\alpha^{<\omega}$ many such sequences, and all the ultrafilters $f^\nu_\mu(\delta)$, for $\delta \in B_\gamma$, $\mu < \nu < \gamma$, are δ-complete, and $\delta > \alpha$. So we can define a common extension $\langle g, H\rangle$ of all the $\langle g, H^{\vec{\alpha}}\rangle$, by setting, for $\gamma \in \alpha \setminus \mathrm{dom}(g)$,

$$H(\gamma) = \bigcap_{\vec{\alpha}} H^{\vec{\alpha}}(\gamma).$$

Now if $\langle s, B\rangle \leq \langle g, H\rangle$, and $\langle s, B\rangle \in \Delta$, then let $\{\alpha_0, \ldots, \alpha_{n-1}\} = \mathrm{dom}(s) \setminus \mathrm{dom}(g)$. Let $\langle s', B'\rangle \leq \langle g, H\rangle$ with $\mathrm{dom}(s') = \mathrm{dom}(s)$ - so $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq \mathrm{dom}(s') \setminus \mathrm{dom}(g)$. Then $\langle s, B\rangle \leq \langle g, H^{\vec{\alpha}}\rangle$, so by $(*)$ with respect to $\vec{\alpha}$, it follows that $\langle s', B'\rangle \in \Delta$. $\square$

Lemma 3.4. *Let $\langle g, G\rangle \in \mathbb{M}$, and let $\beta \in \mathrm{dom}(g)$. Then there exists a condition $\langle g, H\rangle \leq \langle g, G\rangle$ with $\langle g, H\rangle_\beta = \langle g, G\rangle_\beta$ such that the following holds: If there exists a $\langle j, J\rangle \leq \langle g, H\rangle$ such that $\langle j, J\rangle \in \Delta$ and $\langle j, J\rangle_\beta = \langle g, G\rangle_\beta$, then for all $\langle s, B\rangle \leq \langle g, H\rangle$ with $\mathrm{dom}(j) \subseteq \mathrm{dom}(s)$, it follows that $\langle s, B\rangle \in \Delta$.*

For future reference, let's denote the condition $\langle g, H\rangle$ by $\langle g, G\rangle^{+,\beta}$.

Proof. As in the previous proof, for a given finite increasing sequence $\vec{\alpha}$, let $\langle g, H^{\vec{\alpha}}\rangle \leq \langle g, G\rangle$ satisfy $(*)$ of Lemma 3.2, and let $\langle g, H\rangle$ be a common extension of these. Suppose that $\langle j, J\rangle \leq \langle g, H\rangle$ is as in the lemma. Let $\mathrm{dom}(j)\setminus\mathrm{dom}(g) = \{\vec{\alpha}\}$. Then $\langle j, J\rangle \leq \langle g, H^{\vec{\alpha}}\rangle$ and $\langle j, J\rangle_\beta = \langle g, H^{\vec{\alpha}}\rangle$. Now if $\langle s, B\rangle \leq \langle g, H\rangle$, then $\langle s, B\rangle \leq \langle g, H^{\vec{\alpha}}\rangle$, and if in addition, $\mathrm{dom}(j) \subseteq \mathrm{dom}(s)$, then this means that $\{\vec{\alpha}\} \subseteq \mathrm{dom}(s)$, and so, it follows from $(*)$ of Lemma 3.2 with respect to $\vec{\alpha}$ that $\langle s, B\rangle \in \Delta$. $\square$

Theorem 3.5. *Let $\langle g, G\rangle \in \mathbb{M}$, and let $\beta \in \mathrm{dom}(g)$. Then there is a condition $\langle g, H\rangle \leq \langle g, G\rangle$ with $\langle g, H\rangle_\beta = \langle g, G\rangle_\beta$, such that the following holds:*

If $\langle i, I\rangle \leq \langle g, H\rangle$ and $\langle i, I\rangle \in \Delta$, then for any condition $\langle j, J\rangle \leq \langle i, I\rangle_\beta{}^\frown\langle g, H\rangle^\beta$ with $\mathrm{dom}(i) \subseteq \mathrm{dom}(j)$, it follows that $\langle j, J\rangle \in \Delta$.

Denote the condition $\langle g, H\rangle$ by $\langle g, G\rangle^{*,\beta}$.

Proof. Let $S = \{\langle i, I\rangle_\beta \mid \langle i, I\rangle \leq \langle g, G\rangle\}$, and note that S has cardinality $2^{g(\beta)}$. For $q = \langle t, T\rangle \in S$, let

$$\langle t \cup g, H'_q\rangle = (q^\frown\langle g, G\rangle^\beta)^{+,\beta}.$$

Note that the latter is always a condition, because $\beta \in \mathrm{dom}(g)$. Let $\langle g, H\rangle$ be a common extension of the conditions $\langle g, G\rangle_\beta{}^\frown\langle g, H'_q\rangle^\beta$, for $q \in S$. This is possible, because $2^{g(\beta)}$ is less than the completeness of the ultrafilters to which $H'_q(\gamma)$ belongs, for $\gamma > \beta$.

To see that $\langle g, H\rangle$ is as wished, let $\langle i, I\rangle \leq \langle g, H\rangle$, $\langle i, I\rangle \in \Delta$. Assume that $\langle j, J\rangle \leq \langle i, I\rangle_\beta{}^\frown\langle g, H\rangle^\beta$, with $\mathrm{dom}(i) \subseteq \mathrm{dom}(j)$. Let $q = \langle i, I\rangle_\beta$, so $q \in S$. Then

$$\langle i, I\rangle_\beta{}^\frown\langle g, H\rangle^\beta \leq (q^\frown\langle g, G\rangle^\beta)^{+,\beta}.$$

So, since $\langle j, J\rangle \leq \langle i, I\rangle_\beta{}^\frown\langle g, H\rangle^\beta$, it follows that $\langle j, J\rangle \leq (q^\frown\langle g, G\rangle^\beta)^{+,\beta}$. It now follows from the properties of $(q^\frown\langle g, G\rangle^\beta)^{+,\beta}$ (see Lemma 3.4) that $\langle j, J\rangle \in \Delta$, since $\langle i, I\rangle \leq (q^\frown\langle g, G\rangle^\beta)^{+,\beta}$. $\square$

4 The characterization

Definition 4.1. For $\beta < \alpha$ and $\delta \in B_\beta$, let

$$\begin{aligned}
\mathbb{M}_{\langle\beta,\delta\rangle} &= \{\langle g, G\rangle_\beta \mid \langle g, G\rangle \in \mathbb{M} \text{ and } g(\beta) = \delta\},\\
\mathbb{M}^-_{\langle\beta,\delta\rangle} &= \{\langle g{\restriction}\beta, G{\restriction}\beta\rangle \mid \langle g, G\rangle \in \mathbb{M}_{\langle\beta,\delta\rangle}\}.
\end{aligned}$$

The natural partial ordering of $\mathbb{M}^{-}_{\langle\beta,\delta\rangle}$ is defined by saying that $\langle g', G'\rangle \leq \langle g, G\rangle$ if $g \subseteq g'$, $G'(\xi) \subseteq G(\xi)$ for $\xi \notin \mathrm{dom}(g')$, and $g'(\xi) \in G(\xi)$, for $\xi \in \mathrm{dom}(g') \setminus \mathrm{dom}(g)$. Equivalently, this is the case iff $\langle g' \cup \{\langle\beta,\delta\rangle\}, G'\rangle^\frown\langle\emptyset, X\rangle \leq \langle g \cup \{\langle\beta,\delta\rangle\}, G\rangle^\frown\langle\emptyset, X\rangle$, where $X : \alpha \setminus (\beta+1) \longrightarrow \mathcal{P}(\kappa)$ is defined by $X(\xi) = \kappa \setminus (\delta+1)$.

Note that since $\delta \in B_\beta$, it follows that δ is a measurable cardinal of Mitchell order at least β. In fact, set

$$W_\gamma = f^\beta_\gamma(\delta), \text{ for } \gamma < \beta.$$

Then, since $\delta \in A_\beta$, it follows for $\mu < \nu < \beta$, W_μ and W_ν are normal ultrafilters on δ with $W_\mu \lhd W_\nu$.

Define, moreover, for $\mu < \nu < \beta$: $g^\nu_\mu = f^\nu_\mu{\restriction}\delta$. Then, since $\beta \in B_\beta$, it follows by unraveling the definitions that $W_\mu = [g^\nu_\mu]_{W_\nu}$, since this says precisely that $f^\beta_\mu(\delta) = [f^\nu_\mu{\restriction}\delta]_{f^\beta_\nu(\delta)}$, and the latter is true because $\delta \in B_\beta$.

So the ultrafilters $\langle W_\gamma \mid \gamma < \beta\rangle$ on δ and the functions $\langle g^\nu_\mu \mid \mu < \nu < \beta\rangle$ reflect the situation at κ, and they can be used to define a “smaller” version of Magidor forcing. Since $g^\nu_\mu = f^\nu_\mu{\restriction}\delta$, it follows that if $\bar{A}_\gamma$, $\bar{B}_\gamma$ are defined from these sequences as A_γ and B_γ were defined from $\langle U_\mu \mid \mu < \alpha\rangle$ and $\langle f^\nu_\mu \mid \mu < \nu < \alpha\rangle$, then

$$\bar{A}_\gamma = A_\gamma \cap \delta, \ \bar{B}_\gamma = B_\gamma \cap \delta,$$

for $\gamma < \beta$. It is now easy to see that the following lemma holds.

Lemma 4.2. *In the notation introduced above, $\mathbb{M}^{-}_{\langle\beta,\delta\rangle}$, equipped with the natural ordering, is the same as the Magidor forcing $\mathbb{M}(\langle W_\gamma \mid \gamma < \beta\rangle, \langle g^\nu_\mu \mid \mu < \nu < \beta\rangle, \tilde{\alpha})$, i.e, the Magidor forcing with respect to $\vec{W}$ and $\vec{g}$* above $\tilde{\alpha}$.

Definition 4.3. Let $c : \alpha \longrightarrow \kappa$ be a strictly increasing sequence such that for every $\gamma < \alpha$, $c(\gamma) \in B_\gamma$. A filter $F_c \subseteq \mathbb{M}$ can be associated to c by setting

$$F_c = \{\langle g, G\rangle \in \mathbb{M} \mid g \subseteq c \text{ and for all } i \in \alpha \setminus \mathrm{dom}(g), \langle g, G\rangle_{\langle i, c(i)\rangle} \leq \langle g, G\rangle\}.$$

If $\mathrm{V} \subseteq W$ is an inner model with $\mathbb{M} \in \mathrm{V}$, then $c \in W$ is said to be $\mathbb{M}$-generic over V iff F_c is.

Theorem 4.4 (Characterization)**.** *Let* V *be an inner model of W, and let $\mathbb{M} = \mathbb{M}(\langle U_\gamma \mid \gamma < \alpha\rangle, \langle f^\nu_\mu \mid \mu < \nu < \alpha\rangle, \tilde{\alpha}) \in \mathrm{V}$. Then c in W is $\mathbb{M}$-generic over* V *iff c is a strictly increasing sequence in $\prod_{\gamma<\alpha}(B_\gamma \setminus (\tilde{\alpha}+1))$ such that*

1. *For every function $X \in \mathrm{V} \cap \prod_{\gamma<\alpha} U_\gamma$, there is a $\zeta < \alpha$ such that for all $\xi < \alpha$ with $\xi > \zeta$, $c(\xi) \in X(\xi)$.*
2. *For every $\beta < \alpha$, and for every function $X \in \mathrm{V} \cap \prod_{\gamma<\beta} f^\beta_\gamma(c(\beta))$, there is a $\zeta < \alpha$ such that for all $\xi < \beta$ with $\xi > \zeta$, $c(\xi) \in X(\xi)$.*

Remark 4.5. Note that the second condition above is trivially satisfied for successor ordinals $\beta < \alpha$, as ζ can then be chosen to be $\beta - 1$. Note also that if we stick with the notation $f^{\alpha}_{\gamma}(\kappa) = U_\gamma$ and $c(\alpha) = \kappa$, then both conditions can be expressed in one by saying: For every $\beta \le \alpha$, and for every function $X \in \prod_{\gamma<\beta} f^{\beta}_{\gamma}(c(\beta))$, there is a $\zeta < \alpha$ such that for all $\xi < \beta$ with $\xi > \zeta$, $c(\xi) \in X(\xi)$.

Proof. For the easy direction, assume that c is generic. In particular, $c \in \prod_{\gamma<\alpha}(B_\gamma \setminus \tilde{\alpha} + 1)$ (so we wouldn't have had to make that assumption). We are left to show that conditions 1. and 2. hold. For 1., let $X \in \prod_{\gamma<\alpha} U_\gamma$ be given, and assume that α is a limit ordinal, since otherwise, there is nothing to show. Let Δ be the set of conditions $\langle g, H\rangle \in \mathbb{M}$ such that there is a $\zeta < \alpha$ with $\mathrm{dom}(g) \subseteq \zeta$ and for all $\xi \in (\zeta, \alpha)$, $H(\xi) \subseteq X(\xi)$. Obviously, Δ is dense in $\mathbb{M}$, so let $\langle g, H\rangle \in \Delta \cap F_c$. Let ζ be such that $\mathrm{dom}(g) \subseteq \zeta$ and for all $\xi \in (\zeta, \alpha)$, $H(\xi) \subseteq X(\xi)$ (such a ζ exists by definition of Δ). Then, for $\xi \in (\zeta, \alpha)$, it follows that $c(\xi) \in G(\xi) \subseteq X(\xi)$, by the definition of F_c. For 2., let $\beta < \alpha$ be a limit ordinal, and let $X \in \prod_{\gamma<\beta} f^{\beta}_{\gamma}(c(\beta))$ be given. Let Δ be the set of conditions $\langle g, H\rangle$ in $\mathbb{M}$ such that $\beta \in \mathrm{dom}(g)$ and either $g(\beta) = c(\beta)$ and for all $\xi \in (l_{\mathrm{dom}(g)}(\beta), \beta)$, $H(\xi) \subseteq X(\xi)$, or such that $g(\beta) \neq c(\beta)$. This is a dense set, and letting $\langle g, H\rangle$ be in the intersection of F_c with Δ, $\langle g, H\rangle$ clearly has to be of the first type and it follows as before that for large enough $\xi < \beta$, $c(\xi) \in H(\xi) \subseteq X(\xi)$.

Let's now assume that $c \in \prod_{\gamma<\alpha}(B_\gamma \setminus \tilde{\alpha} + 1)$ satisfies 1. and 2., and show that F_c is generic. Assume the contrary. Let κ be minimal such that there is a Magidor forcing $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$ for which the claim fails and each U_γ is a normal ultrafilter on κ. Let α be the length of the sequence $\vec{U}$.

Let $\Delta \subseteq \mathbb{M}$ be open dense, $\Delta \in \mathrm{V}$. I have to find a condition $\langle g, G\rangle \in \Delta$ such that $g \subseteq c$ and for all $i \in \alpha \setminus \mathrm{dom}(g)$, $\langle g, G\rangle_{\langle i, c(i)\rangle} \le \langle g, G\rangle$. This shows then that F_c is generic, so that $\mathbb{M}$ is not a counterexample after all.

Argue in V for awhile. Let $\mathrm{const}_\kappa : \mathrm{On} \longrightarrow \{\kappa\}$ be the constant function that maps every ordinal to κ. Let $\langle \emptyset, C\rangle \in \mathbb{M}$ be the condition $\langle \emptyset, \mathrm{const}_\kappa{\restriction}\alpha\rangle$.

Apply Lemma 3.3 to the condition $\langle \emptyset, C\rangle$ and Δ, resulting in a condition $\langle \emptyset, \tilde{G}_\emptyset\rangle$ such that if $\langle s, B\rangle \le \langle \emptyset, \tilde{G}_\emptyset\rangle$ and $\langle s, B\rangle \in \Delta$, then any $\langle s', B'\rangle \le \langle \emptyset, \tilde{G}_\emptyset\rangle$ with $\mathrm{dom}(s) \subseteq \mathrm{dom}(s')$ will belong to Δ. By strengthening $\langle \emptyset, \tilde{G}_\emptyset\rangle$ if necessary, it may be assumed to be pruned.

Now, let N be the set of strictly increasing functions $g : a \longrightarrow \kappa$, with $a \in [\alpha]^{<\omega}$, such that there is a condition of the form $\langle g, G\rangle \in \mathbb{M}$. Equivalently, $g \in N$ iff $\langle \emptyset, \mathrm{const}_\kappa{\restriction}\alpha\rangle_g \in \mathbb{M}$. For $g \in N$, let

$$\langle g, G_g\rangle = \langle \emptyset, \mathrm{const}_\kappa{\restriction}\alpha\rangle_g.$$

For $g \in N$ with $g \neq \emptyset$, let $\beta = \max(\mathrm{dom}(g))$, and set

$$\langle g, \tilde{G}_g\rangle = \langle g, G_g\rangle^{*,\beta},$$

as defined with respect to Δ. Note that if $\emptyset \neq \bar{g}$ is a proper initial segment of g, that is, $\bar{\beta} := \max(\mathrm{dom}(\bar{g})) < \max(\mathrm{dom}(g))$ and $\bar{g} = g{\restriction}(\bar{\beta} + 1)$, then $\langle \bar{g}, \tilde{G}_{\bar{g}}\rangle_{\bar{\beta}} = (\langle \emptyset, \mathrm{const}_\kappa{\restriction}\alpha\rangle_{\bar{g}})_{\bar{\beta}} = (\langle \emptyset, \mathrm{const}_\kappa{\restriction}\alpha\rangle_g)_{\bar{\beta}} = \langle g, \tilde{G}_g\rangle_{\bar{\beta}}$.

Define a sequence $X \in \prod_{\gamma<\alpha} U_\gamma$ as follows. For $\gamma < \alpha$ and $\delta < \kappa$, let

$$R_{\gamma,\delta} = \{g \in N \mid \mathrm{dom}(g) \subseteq \gamma \text{ and } (g = \emptyset, \text{ or else } g(\max(\mathrm{dom}(g))) = \delta)\}.$$

Set:

$$X(\gamma) = \underset{\delta<\kappa}{\triangle} \left(\bigcap_{g \in R_{\gamma,\delta}} \tilde{G}_g(\gamma) \right)$$

Since for fixed δ, the size of $R_{\gamma,\delta}$ is $\delta^{<\omega} < \kappa$, it follows that $\bigcap_{g\in R_{\gamma,\delta}} \tilde{G}_g(\gamma) \in U_\gamma$, so that the diagonal intersection over all $\delta < \kappa$ of these sets is also in U_γ. We may assume $\langle \emptyset, X\rangle$ is pruned, by shrinking if necessary.

Now, by condition 1., let $\zeta < \alpha$ be such that for all $\xi \in (\zeta, \alpha)$, it follows that $c(\xi) \in X(\xi)$.

We have seen that $\mathbb{M}^-_{\langle \zeta, c(\zeta)\rangle}$ can be viewed as a Magidor forcing above $\tilde{\alpha}$ whose sequence of measures is on $c(\zeta) < \kappa$. So by the minimality assumption, it follows that the theorem is true for $\bar{\mathbb{M}} := \mathbb{M}_{\langle \zeta, c(\zeta)\rangle}$. Moreover, by 2., $c{\restriction}\zeta$ satisfies 1. and 2. for that forcing - both conditions relativize down properly. So $c{\restriction}\zeta$ is $\bar{\mathbb{M}}$-generic over V.

Let $\nu = c(\zeta)$. Define a condition $\langle \{\langle \zeta, \nu\rangle\}, M\rangle$, where for $\gamma < \zeta$, $M(\gamma) = \nu$, and for $\zeta < \gamma < \alpha$, $M(\gamma) = X(\gamma) \setminus (\nu + 1)$. Set

$$\bar{\Delta} = \{\langle u{\restriction}\zeta, V{\restriction}\zeta\rangle \mid \langle u, V\rangle \in \Delta,\ \langle u, V\rangle \leq \langle \{\langle \zeta, \nu\rangle\}, M\rangle\}.$$

Then $\bar{\Delta}$ is dense in $\bar{\mathbb{M}}$: Let $\langle s, B\rangle \in \bar{\mathbb{M}}$. Let $\langle s', B'\rangle \in \mathbb{M}$ be defined by: $s' = s \cup \{\langle \zeta, \nu\rangle\}$, $B'{\restriction}(\zeta + 1) = B$ and $B'(\gamma) = X(\gamma) \setminus (\nu + 1)$, for $\zeta < \gamma < \kappa$. Let $\langle u, V\rangle \leq \langle s', B'\rangle$, $\langle u, V\rangle \in \Delta$. Then $\langle u, V\rangle$ witnesses that $\langle u{\restriction}\zeta, V{\restriction}\zeta\rangle \in \bar{\Delta}$, showing $\bar{\Delta}$ is dense, since $\langle u{\restriction}\zeta, V{\restriction}\zeta\rangle \leq \langle s, B\rangle$ in $\bar{\mathbb{M}}$. Clearly, $\bar{\Delta}$ is also open in $\bar{\mathbb{M}}$.

Since $c{\restriction}\zeta$ is $\bar{\mathbb{M}}$-generic, let $\langle i, I\rangle \in \bar{\Delta}$ be in the filter associated to $c{\restriction}\zeta$ with respect to $\bar{\mathbb{M}}$. So $i \subseteq c$ and for all $\xi \in \zeta \setminus \mathrm{dom}(i)$, $\langle i, I\rangle_{\langle \xi, c(\xi)\rangle} \leq \langle i, I\rangle$ in $\bar{\mathbb{M}}$. Let $\langle u, V\rangle \in \Delta$ witness that $\langle i, I\rangle$ is in $\bar{\Delta}$, so $u{\restriction}\zeta = i$, $V{\restriction}(\zeta + 1) = I$, and $\langle u, V\rangle \leq \langle \{\langle \zeta, \nu\rangle\}, M\rangle$.

Let $g = u{\restriction}(\zeta + 1)$. Then $g \in R_{\zeta+1,\nu}$. I claim that

$$\langle u, V\rangle \leq \langle g, \tilde{G}_g\rangle.$$

To see this, first note that

$$\langle \{\langle \zeta, \nu\rangle\}, M\rangle_{g{\restriction}\zeta} \leq \langle g, \tilde{G}_g\rangle$$

since up to and including ζ, these conditions are the same, and for $\zeta < \gamma < \alpha$, $M(\gamma) \subseteq \tilde{G}_g(\gamma)$. For if $\xi \in M(\gamma)$, then by definition, $\xi \in X(\gamma) \setminus (\nu + 1)$. $g \in R_{\zeta+1,\nu}$, and $\nu < \xi$, so by definition of $X(\gamma)$, $\xi \in \tilde{G}_g(\gamma)$. Moreover,

$$\langle u, V\rangle \leq \langle \{\langle \zeta, \nu\rangle\}, M\rangle_{g{\restriction}\zeta}$$

since $\langle u, V\rangle \leq \langle \{\langle \zeta, \nu\rangle\}, M\rangle$ and up to ζ, the condition on the right is the weakest condition which has $u{\restriction}(\zeta+1)$ in the first coordinate. So, putting the two previous displayed facts together, it follows that $\langle u, V\rangle \leq \langle g, \tilde{G}_g\rangle$, as claimed.

Now, remember that $\langle g, \tilde{G}_g \rangle = \langle g, G_g \rangle^{*,\zeta}$. So we have $\langle u, V \rangle \le \langle g, G_g \rangle^{*,\zeta}$ and $\langle u, V \rangle \in \Delta$. By the defining property of $\langle g, G_g \rangle^{*,\zeta}$, it follows that $\langle j, J \rangle \in \Delta$, for any condition $\langle j, J \rangle \le \langle u, V \rangle_\zeta {}^\frown \langle g, \tilde{G}_g \rangle^\zeta$ with $\mathrm{dom}(u) \subseteq \mathrm{dom}(j)$. Since $\langle u, V \rangle_\zeta {}^\frown \langle \{\langle \zeta, \nu \rangle\}, M \rangle^\zeta \le \langle u, V \rangle_\zeta {}^\frown \langle g, \tilde{G}_g \rangle^\zeta$, (as for all $\gamma > \zeta$ with $\gamma < \alpha$, it follows that $X(\gamma) \setminus (\nu + 1) \subseteq \tilde{G}_g(\gamma)$), the same conclusion can be drawn for any $\langle j, J \rangle \le \langle u, V \rangle_\zeta {}^\frown \langle \{\langle \zeta, \nu \rangle\}, M \rangle^\zeta$.

So let $a = \mathrm{dom}(u) \setminus (\zeta + 1)$. Let $\langle j, J \rangle = (\langle u, V \rangle_\zeta {}^\frown \langle \{\langle \zeta, \nu \rangle\}, M \rangle^\zeta)_{c \restriction a}$. Then $\langle j, J \rangle \le \langle u, V \rangle_\zeta {}^\frown \langle \{\langle \zeta, \nu \rangle\}, M \rangle^\zeta$, so $\langle j, J \rangle \in \Delta$, and $\langle j, J \rangle$ belongs to F_c, the filter associated to c. Note that $\langle j, J \rangle_{\langle \gamma, c(\gamma) \rangle} \le \langle j, J \rangle$, for all $\gamma \in \alpha \setminus \mathrm{dom}(j)$. This works for $\gamma < \zeta$, since $\langle j \restriction \zeta, J \restriction \zeta \rangle$ belongs to the filter associated to $c \restriction \zeta$, and it works for $\gamma > \zeta$, since $c(\gamma) \in X(\gamma)$ by choice of ζ, and since $\langle \emptyset, X \rangle$ is pruned. $\square$

The following corollary parallels the fact that restricting Příkrý sequences to unbounded subsets of ω results in Příkrý sequences.

Corollary 4.6. *If c is $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f}, \tilde{\alpha})$-generic over* V*, and d is an increasing function in $\prod_{\gamma < \alpha} (B_\gamma \setminus (\alpha + 1))$, where $\alpha = \mathrm{dom}(\vec{U})$, such that for all but finitely many $\gamma < \alpha$, and for all limit γ, $c(\gamma) = d(\gamma)$. Then d is also $\mathbb{M}$-generic over* V.

Proof. This follows immediately from Theorem 4.4, in particular the remark following its statement. $\square$

It is well-known that if c is a Příkrý-sequence for the Příkrý forcing with respect to a normal ultrafilter U on κ, then for a subset $X \subseteq \kappa$, X belongs to U if and only if c is almost contained in X. In particular, the ultrafilter U, and hence the forcing, can be recovered from the Příkrý sequence c. It is natural to ask to what extent this can be generalized to Magidor forcing. To what extent does a Magidor sequence determine its forcing (in particular, the sequences $\vec{U}$ and $\vec{f}$ on which it depends)?

Lemma 4.7. *Let c be $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$-generic, where α, the length of $\mathbb{M}$, is a limit ordinal. Then the sequence $\vec{U}$ is unique on a tail. That is, if c is also generic for $\mathbb{M}' = \mathbb{M}(\vec{U}', \vec{f}')$, then for sufficiently large $\gamma < \alpha$, $U_\gamma = U'_\gamma$.*

Proof. Suppose c is both $\mathbb{M}$- and $\mathbb{M}'$-generic over V. Clearly, both forcings must have the same height, say κ. For $\gamma < \alpha$, pick $X_\gamma \in U_\gamma \setminus U'_\gamma$ if $U_\gamma \neq U'_\gamma$, and otherwise, let $X_\gamma = \kappa$. By the Characterization Theorem, it follows that for large enough $\gamma < \alpha$, $c(\gamma) \in X_\gamma$, as c is generic for $\mathbb{M}$. But similarly, let $Y_\gamma = X_\gamma$ if $X_\gamma = \kappa$, and let $Y_\gamma = \kappa \setminus X_\gamma$ otherwise. Then it follows that for large enough γ, $c(\gamma) \in Y_\gamma$, since c is generic for $\mathbb{M}'$. So for large enough γ, $c(\gamma) \in X_\gamma \cap Y_\gamma$, which implies that $U_\gamma = U'_\gamma$, since X_γ and Y_γ are disjoint if $U_\gamma \neq U'_\gamma$. $\square$

Much more could be said about the sequences $\vec{f}$ and $\vec{f}'$, but I want to look at this more closely a little later, in Section 7. In particular, Lemma 7.11 provides information in a more general setting.

5 Iterations and genericity of the critical sequence

The iteration associated to $\vec{U}$ is the sequence of models and embeddings $\langle\langle M_\gamma \mid \gamma \leq \alpha\rangle, \langle\pi_{i,j} \mid i < j \leq \alpha\rangle\rangle$ defined by recursion as follows. $M_0 = \mathrm{V}$. If $\langle\langle M_\gamma \mid \gamma \leq \beta\rangle, \langle\pi_{i,j} \mid i < j \leq \beta\rangle\rangle$ has been defined already, then let $\vec{U}^\beta = \pi_{0,\beta}(\vec{U})$, and let

$$\pi_{\beta,\beta+1} : M_\beta \longrightarrow_{U^\beta_\beta} M_{\beta+1},$$

and set $\pi_{i,\beta+1} = \pi_{\beta,\beta+1} \circ \pi_{i,\beta}$. If λ is a limit ordinal and $\langle\langle M_\gamma \mid \gamma < \lambda\rangle, \langle\pi_{i,j} \mid i < j < \lambda\rangle\rangle$ have been defined already, then let

$$\langle M_\lambda, \langle\pi_{i,\lambda} \mid i < \lambda\rangle\rangle = \mathrm{dir\ lim}\langle\langle M_\gamma \mid \gamma < \lambda\rangle, \langle\pi_{i,j} \mid i < j < \lambda\rangle\rangle.$$

Let $\kappa_i = \mathrm{crit}(\pi_{i,i+1})$, and $\kappa_\alpha = \pi_{0,\alpha}(\kappa)$. Let $U_\mu = [f^\nu_\mu]_{U_\nu}$, for $\mu < \nu < \alpha$.

The following result was first shown in [Deh83], using a somewhat different approach to iterated ultrapowers and to Magidor forcing. The proof I present uses the characterization of Magidor genericity, Theorem 4.4.

Theorem 5.1. *The sequence $\langle\kappa_i \mid i < \alpha\rangle$ is $\mathbb{M}(\vec{U}^\alpha, \pi_{0,\alpha}(\vec{f}))$-generic over M_α.*

Proof. Let $\mathbb{M}' = \mathbb{M}(\vec{U}^\alpha, \pi_{0,\alpha}(\vec{f}))$, and note that $\mathbb{M}' = \pi_{0,\alpha}(\mathbb{M}(\vec{U}, \vec{f}))$. For $\gamma < \alpha$, let $B^{\mathbb{M}'}_\gamma$ be defined in M with respect to $\vec{U}^\alpha$ and $\pi_{0,\alpha}(\vec{f})$ just like B_γ was defined in V with respect to $\vec{U}$ and $\vec{f}$. Clearly then, $B^{\mathbb{M}'}_\gamma = \pi_{0,\alpha}(B_\gamma)$. It is now not hard to see that $\kappa_\gamma \in B^{\mathbb{M}'}_\gamma$. This is because $B_\gamma \in U_\gamma$, so that $\pi_{0,\gamma}(B_\gamma) \in U^\gamma_\gamma$. Since $\pi_{\gamma,\gamma+1} : M_\gamma \longrightarrow_{U^\gamma_\gamma} M_{\gamma+1}$, it follows that $\kappa_\gamma \in \pi_{0,\gamma+1}(B_\gamma)$, and since $\mathrm{crit}(\pi_{\gamma+1,\alpha}) > \kappa_\gamma$, this implies that $\kappa_\gamma \in \pi_{0,\alpha}(B_\gamma) = B^{\mathbb{M}'}_\gamma$.

Thus, $\vec{\kappa}$ is a strictly increasing sequence in $\prod_{\gamma<\alpha} B^{\mathbb{M}'}_\gamma \setminus (\alpha+1)$, so all that is left to do is to verify conditions 1. and 2. of Theorem 4.4.

To verify 1., let $\vec{X} = X \in M_\alpha \cap \prod_{\gamma<\alpha} U^\alpha_\gamma$ be given, and suppose α is a limit ordinal (if not, then there is nothing to show). So M_α is the direct limit of the previous models. It follows that $\kappa_\xi \in X_\xi$ whenever $\vec{X} \in \mathrm{ran}(\pi_{\xi,\alpha})$. To see this, for such ξ, let $\vec{X} = \pi_{\xi,\alpha}(\vec{\bar{X}})$. Then, clearly, $\vec{\bar{X}} \in \prod_{\gamma<\alpha} U^\xi_\gamma$. Since $\pi_{\xi,\xi+1} : M_\xi \longrightarrow_{U^\xi_\xi} M_{\xi+1}$, it follows that $\kappa_\xi \in \pi_{\xi,\xi+1}(\bar{X}_\xi)$, and so, $\kappa_\xi = \pi_{\xi+1,\alpha}(\kappa_\xi) \in \pi_{\xi+1,\alpha}(\pi_{\xi,\xi+1}(\bar{X}_\xi)) = X_\xi$. Since $\vec{X} \in \mathrm{ran}(\pi_{\xi,\alpha})$ for sufficiently large $\xi < \alpha$, this shows that 1. is satisfied.

To see that 2. is satisfied, I need two little observations on $\pi_{0,\alpha}(\vec{f})$.

(1) *For all $\mu < \nu < \alpha$, $\pi_{0,\alpha}(f^\nu_\mu)\restriction\kappa_\nu = \pi_{0,\nu}(f^\nu_\mu)$.*

Proof of (1). The domain of the function on the right hand side is $\pi_{0,\nu}(\mathrm{dom}(f^\nu_\mu)) = \kappa_\nu$. So the functions on the left and the right have the same domain κ_ν. For $\xi < \kappa_\nu$, $\pi_{0,\alpha}(f^\nu_\mu)(\xi) = \pi_{\nu,\alpha}(\pi_{0,\nu}(f^\nu_\mu))(\xi) = \pi_{\nu,\alpha}(\pi_{0,\nu}(f^\nu_\mu)(\xi))$, since $\xi < \kappa_\nu = \mathrm{crit}(\pi_{\nu,\alpha})$. Since $\pi_{0,\nu}(f^\nu_\mu)(\xi) \in \mathcal{P}(\mathcal{P}(\xi))$ and $\xi < \kappa_\nu$, it follows that $\pi_{\nu,\alpha}(\pi_{0,\nu}(f^\nu_\mu)(\xi)) = \pi_{0,\nu}(f^\nu_\mu)(\xi)$, which completes the proof. $\Box_{(1)}$

(2) *For all $\mu < \nu < \alpha$, $\pi_{0,\alpha}(f^\nu_\mu)(\kappa_\nu) = U^\nu_\mu$.*

Proof of (2). Since $[f^\nu_\mu]_{U_\nu} = U_\mu$, it follows that in M_ν, it is true that $[\pi_{0,\nu}(f^\nu_\mu)]_{U^\nu_\nu} = U^\nu_\mu$. Since $\pi_{\nu,\nu+1} : M_\nu \longrightarrow_{U^\nu_\nu} M_{\nu+1}$, it follows that $[\pi_{0,\nu}(f^\nu_\mu)]_{U^\nu_\nu} = \pi_{0,\nu+1}(f^\nu_\mu)(\kappa_\nu)$. So

$$\pi_{0,\nu+1}(f^\nu_\mu)(\kappa_\nu) = U^\nu_\mu.$$

Since $U^\nu_\mu \in \mathcal{P}(\mathcal{P}(\kappa_\nu))$ and $\kappa_\nu < \kappa_{\nu+1} = \mathrm{crit}(\pi_{\nu+1,\alpha})$, we can apply $\pi_{\nu+1,\alpha}$ to both sides of this equality to get the desired identity. $\square_{(2)}$

So, to verify 2., let $\beta < \alpha$ be a limit ordinal. Let $\vec{X} \in M_\alpha \cap \prod_{\gamma<\beta} \pi_{0,\alpha}(f^\beta_\gamma)(\kappa_\beta)$. This means by (2) that $\vec{X} \in \prod_{\gamma<\beta} U^\beta_\gamma$. Since $\mathrm{V}^{M_\beta}_{\kappa_\beta+1} = \mathrm{V}^{M_\alpha}_{\kappa_\beta+1}$, it follows that $\vec{X} \in M_\beta$. But now the same argument that established 1., with α replaced by β, also shows that for $\xi < \beta$ with $\vec{X} \in \mathrm{ran}(\pi_{\xi,\beta})$, it follows that $\kappa_\xi \in X_\xi$. $\square$

6 Maximality

Theorem 6.1 (Maximality)**.** *Let* $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$*, and let* c *be* $\mathbb{M}$*-generic over* V*. If* $d \in \mathrm{V}[c]$ *is* $\mathbb{M}$*-generic over* V*, then* $\mathrm{ran}(d)$ *is almost contained in* $\mathrm{ran}(c)$*, i.e.,* $\mathrm{ran}(d) \setminus \mathrm{ran}(c)$ *is finite.*

Proof. Suppose the theorem failed. Then let $\dot{d}$ be an $\mathbb{M}$-name and $p = \langle g, G\rangle \in \mathbb{M}$ be a pruned condition such that p forces over $M_0 = \mathrm{V}$ that $\dot{d}$ is V-generic and that $\mathrm{ran}(\dot{d}) \setminus \mathrm{ran}(\Gamma)$ is infinite, where Γ is a canonical name for the generic sequence added by $\mathbb{M}$. Let α be the length of $\mathbb{M}$.

Main case 1: $g = \emptyset$.

Let $\langle\langle M_\gamma \mid \gamma \le \alpha\rangle, \langle \pi_{i,j} \mid i \le j \le \alpha\rangle\rangle$ be the iteration corresponding to $\mathbb{M}$, and let $\vec{\kappa}$ be the sequence of critical points. Let $\pi = \pi_{0,\alpha}$. So $p' := \pi(p) = \langle \emptyset, G'\rangle$ forces over M_α with respect to $\mathbb{M}' = \pi(\mathbb{M})$ that $\dot{d}' := \pi(\dot{d})$ is M_α-generic for $\mathbb{M}'$ and that $\dot{d}' \setminus \Gamma'$ is infinite, where $\Gamma' = \pi(\Gamma)$ is a canonical name for the generic sequence added by $\mathbb{M}'$. The proof of Theorem 5.1 shows that for all $\gamma < \alpha$, $\kappa_\gamma \in G'(\gamma)$, since $G' \in \mathrm{ran}(\pi)$. So p' belongs to the M_α-generic filter associated to $\vec{\kappa}$, and hence, letting d' be the interpretation of $\dot{d}'$ by that filter, it follows that $d' \in M_\alpha[\vec{\kappa}]$ is generic for $\mathbb{M}'$ over M_α, and that $\mathrm{ran}(d') \setminus \{\vec{\kappa}\}$ is infinite. Let a consist of the first ω many $\xi < \alpha$ with $d'(\xi) \notin \{\vec{\kappa}\}$. For each $\gamma \in a$, let $f_\gamma : \kappa^{m_\gamma} \longrightarrow \kappa$ be such that

$$d'(\gamma) = \pi(f_\gamma)(g_\gamma),$$

for some $g_\gamma \in [\{\vec{\kappa}\}]^{m_\gamma}$ with $\max(g_\gamma) < d'(\gamma)$ (since $d'(\gamma)$ does not belong to $\{\vec{\kappa}\}$) and $m_\gamma < \omega$. So $d'(\gamma)$ belongs to the set

$$Z_\gamma = \{\mu < \kappa_\alpha \mid \exists g \in {}^{<\omega}\mu \quad \mu = \pi(f_\gamma)(g)\}.$$

It is obvious that $Z_\gamma \cap \rho$ is not stationary, for any regular uncountable ρ, for otherwise $Z_\gamma \cap \rho$ would have to have a stationary subset on which the regressive function selecting the minimal witness is constant, by Fodor's lemma, which would mean the stationary subset could only have one member. In particular, $Z_\gamma \cap \rho$ cannot belong to any normal ultrafilter on ρ.

Let $\bar{Z}_\gamma = \{\mu < \kappa \mid \exists g \in {}^{<\omega}\mu \quad \mu = f_\gamma(g)\}$. Then $\pi(\langle \bar{Z}_\gamma \mid \gamma \in a\rangle) = \langle Z_\gamma \mid \gamma \in a\rangle \in M_\alpha$, and hence, letting $\kappa' = \pi(\kappa)$, the following set belongs to M_α:

$$C = \bigcap_{\gamma \in a} (\kappa' \setminus Z_\gamma).$$

Moreover, letting $U'_\gamma = \pi(U_\gamma)$, for $\gamma < \alpha$, it follows that $C \in U'_\gamma$, and in general, whenever $\bar{\kappa} < \kappa'$ is measurable in M_α, then $C \cap \bar{\kappa}$ belongs to any normal ultrafilter on $\bar{\kappa}$.

Let $\beta = \sup a$. First, assume that $\beta = \alpha$ (so α is a limit ordinal of cofinality ω). Then $\sup_{\gamma<\beta} d'(\gamma) = \kappa_\alpha$. Consider the sequence $\vec{X} = \langle X_\gamma \mid \gamma < \alpha\rangle$ defined by $X_\gamma = C$. Then $\vec{X} \in \prod_{\gamma<\alpha} U'_\gamma$, so for sufficiently large ξ, it should be the case that $d'(\xi) \in X_\xi = C$, by 1. of Theorem 4.4, since d' is $\mathbb{M}'$-generic over M_α. But this fails for all $\xi \in a$, which is unbounded in α (since $d'(\xi) \in Z_\xi \subseteq \kappa_\alpha \setminus C$).

Now assume that $\beta < \alpha$. Note that since d' is a normal function, it follows that $d'(\beta) = \sup_{\gamma<\beta} d'(\gamma)$. Define a constant sequence $\vec{Y} = \langle Y_\gamma \mid \gamma < \beta\rangle$ by setting $Y_\gamma = C \cap d_\beta$. Then $\vec{Y} \in \prod_{\gamma<\beta} f'^{\beta}_{\gamma}(d_\beta)$, where $\vec{f'} = \pi(\vec{f})$. So by condition 2. of Theorem 4.4, it should again be true that for all large enough $\xi < \beta$, $d'(\xi) \in Y_\xi$, but this fails, as before, for all $\gamma \in a$. But a is unbounded in β, a contradiction. This concludes case 1.

Main case 2: $g \neq \emptyset$.

In this case, the iteration in case 1 doesn't work; the generic filter corresponding to the critical sequence won't contain the image of p. In order for this to be true, we would have to have that $\vec{\kappa}\restriction\mathrm{dom}(\pi_{0,\alpha}(g)) = \pi_{0,\alpha}(g)$. But this is clearly impossible, as the critical points never belong to the image of $\pi_{0,\alpha}$. So I will construct an iteration "along g", so that a sequence $\vec{\lambda}$ which is almost always (i.e., with at most finitely many exceptions) equal to the sequence $\vec{\kappa}$ of the critical points of the iteration, which is generic, and whose filter will contain the image of p under the iteration map.

Let $\mathrm{dom}(g) = a$. The "iteration along g" starts out with $M_0 = \mathrm{V}$, and at limit stages, direct limits will be formed, as before. The difference will lie in the successor case. So suppose $\langle\langle M_\gamma \mid \gamma \leq \beta\rangle, \langle \pi_{i,j} \mid i \leq j \leq \beta\rangle\rangle$ has been defined already. If $\beta \in a$, then let $M_{\beta+1} = M_\beta$, $\pi_{\beta,\beta+1} = \mathrm{id}\restriction M_\beta$. If $\beta \notin a$, then let

$$\pi_{\beta,\beta+1} : M_\beta \longrightarrow_{W_\beta} M_{\beta+1},$$

where in the case that $r_a(\beta) < \alpha$, we set

$$W_\beta = \pi_{0,\beta}(f_\beta^{r_a(\beta)}(g(r_a(\beta)))),$$

and in case $r_a(\beta) = \alpha$, we set

$$W_\beta = U^\beta_\beta = \pi_{0,\beta}(U_\beta).$$

As before, let κ_γ be the critical point of $\pi_{\gamma,\gamma+1}$, if $\gamma \notin a$, and let $\kappa_\alpha = \pi_{0,\alpha}(\kappa)$.

The generic sequence will be the sequence $\vec{\lambda} = \langle \lambda_\xi \mid \xi < \alpha\rangle$ defined by:

$$\lambda_\xi = \begin{cases} \kappa_\xi & \text{if } \xi \notin a, \\ \pi_{0,\xi}(g(\xi)) & \text{if } \xi \in a. \end{cases}$$

Let $\pi = \pi_{0,\alpha}$. It follows that

(1) $\mathrm{dom}(\pi(g)) = a$, *and for* $\xi \in a$, $\pi(g)(\xi) = \pi_{0,\xi}(g(\xi)) = \lambda_\xi$.

Proof of (1). $\pi(g)(\xi) = \pi(g(\xi)) = \pi_{\xi,\alpha}(\pi_{0,\xi}(g(\xi))$. Since $\pi_{\xi,\xi+1}$ is the identity, this is equal to $\pi_{\xi+1,\alpha}(\pi_{0,\xi}(g(\xi))$. Let ζ be the least ordinal less than α that is greater than ξ and that is not in a. In case α is a successor ordinal, it could be that ζ doesn't exist, but then $\pi_{\xi,\alpha}$ is the identity, so there is nothing to be shown. So suppose this is not the case. Then $\pi_{\xi,\zeta}$ is the identity, and we get that $\pi_{\xi,\alpha}(\pi_{0,\xi}(g(\xi)) = \pi_{\zeta,\alpha}(\pi_{0,\xi}(g(\xi))$. The critical point of $\pi_{\zeta,\alpha}$ is $\pi_{0,\zeta}(g(r_a(\zeta)))$, if $r_a(\zeta) < \alpha$, and it is $\pi_{0,\zeta}(\kappa)$ otherwise. So we have that

$$\lambda_\xi = \pi_{0,\xi}(g(\xi)) = \pi_{0,\zeta}(g(\xi)) < \pi_{0,\zeta}(g(r_a(\zeta)))$$

in case $r_a(\zeta) < \alpha$ (since then $\xi < \zeta$, so $\xi < r_a(\zeta)$, so $g(\xi) < g(r_a(\xi))$). But $\pi_{0,\zeta}(g(r_a(\zeta)))$ is the critical point of $\pi_{\zeta,\alpha}$, and so, it follows that

$$\pi_{\zeta,\alpha}(\lambda_\xi) = \lambda_\xi = \pi_{0,\xi}(g(\xi)).$$

In case $r_a(\zeta) = \alpha$, the critical point of $\pi_{\zeta,\alpha}$ is $\pi_{0,\zeta}(\kappa)$, which is obviously greater than $\lambda_\xi = \pi_{0,\zeta}(g(\xi))$, and the same argument as above proves the claim. $\square_{(1)}$

(2) *For every* $\xi < \alpha$, $\lambda_\xi \in \pi(B_\xi)$.

Proof of (2). If $\xi \in a$, then $\lambda_\xi = \pi(g(\xi))$, by (1), and $g(\xi) \in B_\xi$ since $\langle g, G\rangle$ is a condition. This implies the claim.

If $\xi \notin a$, then there are two cases. First, suppose that $r_a(\xi) < \alpha$. Then $B_\xi \cap g(r_a(\xi)) \in f_\xi^{r_a(\xi)}(g(r_a(\xi)))$, since $g(r_a(\xi)) \in B_{r_a(\xi)}$ (see the discussion in the beginning of Section 4). It follows that

$$\pi_{0,\xi}(B_\xi) \cap \lambda_\xi \in W_\xi = \pi_{0,\xi}(f_\xi^{r_a(\xi)}(g(r_a(\xi)))),$$

and since $\pi_{\xi,\xi+1} : M_\xi \longrightarrow_{W_\xi} M_{\xi+1}$, this readily implies that

$$\lambda_\xi \in \pi_{0,\xi+1}(B_\xi).$$

Applying $\pi_{\xi+1,\alpha}$ on both sides gives

$$\lambda_\xi \in \pi(B_\xi),$$

once again using (1).

In the second case, $r_a(\xi) = \alpha$, and the argument is a little easier: $B_\xi \in U_\xi$, so $\pi_{0,\xi}(B_\xi) \in W_\xi = \pi_{0,\xi}(U_\xi)$. It follows that $\lambda_\xi \in \pi_{0,\xi+1}(B_\xi)$, and finally that $\lambda_\xi \in \pi(B_\xi)$. $\square_{(2)}$

So $\vec{\lambda} \in \prod_{\gamma<\alpha} \pi(\vec{B})_\gamma$, and in order to see that $\vec{\lambda}$ is $\pi(\mathbb{M})$-generic over M_α, conditions 1. and 2. of Theorem 4.4 need to be verified.

For 1., let $\vec{X} \in M_\alpha \cap \prod_{\gamma<\alpha} \pi(\vec{U})_\gamma$. Assume α is a limit ordinal (otherwise, there is nothing to show). I claim that if $\xi < \alpha$ is large enough that $\mathrm{dom}(g) = a \subseteq \xi$ and $\vec{X} \in \mathrm{ran}(\pi_{\xi,\alpha})$, then $\lambda_\xi = \kappa_\xi \in X_\xi$. The argument is similar to the original argument establishing the genericity of the straightforward iteration: if $\pi_{\xi,\alpha}(\vec{Y}) = \vec{X}$, then $Y_\xi \in \pi_{0,\xi}(U_\xi) = W_\xi$ by definition of the iteration. So $\lambda_\xi = \kappa_\xi \in \pi_{\xi,\xi+1}(Y_\xi)$, and finally, using (1), $\lambda_\xi \in \pi_{\xi,\alpha}(Y_\xi) = X_\xi$.

For 2., let β be a limit ordinal less than α, and let $\vec{X} \in M_\alpha \cap \prod_{\gamma<\beta} \pi(\vec{f})^\beta_\gamma(\lambda_\beta)$. It follows that $\vec{X} \in M_\beta$, since $\mathrm{V}^{M_\beta}_{\lambda_\beta+1} = \mathrm{V}^{M_\alpha}_{\lambda_\beta+1}$. I claim that if $\xi < \beta$ is large enough that $a \cap \beta \subseteq \xi$ and $\vec{X} \in \mathrm{ran}(\pi_{\xi,\beta})$, then $\lambda_\xi \in X_\xi$. For such ξ, let $\pi_{\xi,\beta}(\vec{Y}) = \vec{X}$. The main claim to prove here is

(3) $\pi(f^\beta_\xi)(\lambda_\beta) = \pi_{\xi,\beta}(W_\xi)$.

Proof of (3). I will address several cases separately.
Case 1: $\beta \in a$.

Then $W_\xi = \pi_{0,\xi}(f^\beta_\xi(g(\beta)))$, so

$$\pi_{\xi,\beta}(W_\xi) = \pi_{0,\beta}(f^\beta_\xi(g(\beta))) = \pi_{0,\beta}(f^\beta_\xi)(\pi_{0,\beta}(g(\beta)) = \pi_{0,\beta}(f^\beta_\xi)(\lambda_\beta),$$

since in the present case, $\lambda_\beta = \pi_{0,\beta}(g(\beta))$. Moreover, in the present case, $\pi_{\beta,\beta+1}$ is the identity, and so, the critical point of $\pi_{\beta,\alpha}$, if there is any, is greater than λ_β. So $\pi_{\beta,\alpha}$ can be applied to both sides of this equation to get

$$\pi_{\xi,\beta}(W_\xi) = \pi_{0,\alpha}(f^\beta_\xi)(\lambda_\beta),$$

as claimed.
Case 2: $\beta \notin a$.

We have then that $\lambda_\beta = \kappa_\beta$. We know that $X_\xi \in \pi(f^\beta_\xi)(\lambda_\beta)$. To prove the claim, I will have to split into two subcases. Note that since $a \cap \beta \subseteq \xi$, $r_a(\xi) = r_a(\beta)$.
Case 2.1: $r_a(\beta) = \alpha$.

Then $W_\beta = \pi_{0,\beta}(U_\beta)$ and $\lambda_\beta = \pi_{0,\beta}(\kappa)$. Since $[f^\beta_\xi]_{U_\beta} = U_\xi$, it follows that $[\pi_{0,\beta}(f^\beta_\xi)]_{\pi_{0,\beta}(U_\beta)} = \pi_{0,\beta}(U_\xi)$, so $[\pi_{0,\beta}(f^\beta_\xi)]_{W_\beta} = \pi_{\xi,\beta}(W_\xi)$, and $[\pi_{0,\beta}(f^\beta_\xi)]_{W_\beta} = \pi_{\beta,\beta+1}(\pi_{0,\beta}(f^\beta_\xi))(\lambda_\beta)$. So $\pi_{0,\beta+1}(f^\beta_\xi)(\lambda_\beta) = \pi_{\xi,\beta}(W_\xi)$. So $\pi_{0,\alpha}(f^\beta_\xi)(\lambda_\beta) = \pi_{\xi,\beta}(W_\xi)$.
Case 2.2: $\theta := r_a(\beta) < \alpha$ (and again, $\theta = r_a(\xi)$ as well).

I first claim that $\lambda_\beta \in \pi_{0,\beta}(B_\theta)$. This is because by (2), $\lambda_\theta \in \pi_{0,\alpha}(B_\theta)$, so $\lambda_\theta \in \pi_{0,\theta+1}(B_\theta) = \pi_{0,\theta}(B_\theta)$ (since $\pi_{\theta,\theta+1}$ is the identity). But since $r_a(\beta) = \theta$, it follows that

$$\pi_{\beta,\theta}(\lambda_\beta) = \lambda_\theta \in \pi_{\beta,\theta}(\pi_{0,\beta}(B_\theta)),$$

which implies, by applying $\pi^{-1}_{\beta,\theta}$ to both sides, that $\lambda_\beta \in \pi_{0,\beta}(B_\theta)$, as claimed. It follows that

$$[\pi_{0,\beta}(f^\beta_\xi)\restriction\lambda_\beta]_{\pi_{0,\beta}(f^\theta_\beta)(\lambda_\beta)} = \pi_{0,\beta}(f^\theta_\xi)(\lambda_\beta).$$

This means that

$$\pi_{0,\beta+1}(f^\beta_\xi)(\lambda_\beta) = \pi_{0,\beta}(f^\theta_\xi)(\lambda_\beta),$$

since $W_\beta = \pi_{0,\beta}(f^\theta_\beta)(\lambda_\beta)$ and $\pi_{\beta,\beta+1} : M_\beta \longrightarrow_{W_\beta} M_{\beta+1}$. Note that the right hand side of this equation is $\pi_{\xi,\beta}(W_\xi)$. Applying $\pi_{\beta+1,\alpha}$ to both sides yields

$$\pi_{0,\alpha}(f^\beta_\xi)(\lambda_\beta) = \pi_{\xi,\beta}(\pi_{0,\xi}(f^\theta_\xi)(\lambda_\xi)) = \pi_{\xi,\beta}(W_\xi).$$

$\square_{(3)}$

Now, the argument proceeds as before. Since $X_\xi \in \pi(f^\beta_\xi)(\lambda_\beta) = \pi_{\xi,\beta}(W_\xi)$, it follows that $Y_\xi \in W_\xi$, and it follows from this that $\lambda_\xi \in \pi_{\xi,\xi+1}(Y_\xi)$, since $\pi_{\xi,\xi+1} : M_\xi \longrightarrow_{W_\xi} M_{\xi+1}$. Finally, applying $\pi_{\xi+1,\beta}$ yields that $\lambda_\xi \in X_\xi$.

This shows that $\vec{\lambda}$ is $\pi_{0,\alpha}(\mathbb{M})$-generic over M_α. Moreover, a similar argument shows that for all $\xi \in \alpha \setminus a$,

(4) $\lambda_\xi = \kappa_\xi \in \pi_{0,\alpha}(G)(\xi)$.

Proof of (4). To see this, fix $\xi \in \alpha \setminus a$, and let $r_a(\xi) = \theta$. It follows as before that $\pi(G)(\xi) \in \pi_{\xi,\theta}(W_\xi)$: if $\theta < \alpha$, then $G(\xi) \in f^\theta_\xi(g(\theta))$, so that, applying π, and noting that $\pi(g(\theta)) = \lambda_\theta$, $\pi(G)(\xi) \in \pi(f^\theta_\xi)(\lambda_\theta)$. Note that $a \cup \theta \subseteq \xi$, so that by (3), $\pi(G)(\xi) \in \pi_{\xi,\theta}(W_\xi)$. On the other hand, if $\theta = \alpha$, then $G(\xi) \in U_\xi$, so $\pi(G)(\xi) \in \pi(U_\xi) = \pi_{\xi,\alpha}(\pi_{0,\xi}(U_\xi)) = \pi_{\xi,\alpha}(W_\xi)$. In any case, the argument above shows that $\lambda_\xi \in \pi(G)(\xi)$ (for this, it would suffices to know that $\pi(G)(\xi) \in \mathrm{ran}(\pi_{\xi,\alpha})$, which is clearly the case). $\square_{(4)}$

By design, we will have

(5) *For* $\beta \in a$, $\pi_{0,\alpha}(g(\beta)) = \pi_{0,\beta}(g(\beta)) = \lambda_\beta$.

So, $\pi_{0,\alpha}(g) = \vec{\lambda}\restriction\mathrm{dom}(g)$. Taking these last points, (4) and (5), together, this shows that $\pi(\langle g, G\rangle)$ belongs to the generic filter associated to $\vec{\lambda}$. The argument can now be completed as in Case 1, because, with finitely many exceptions, the generic sequence we're dealing with is the critical sequence. $\square$

The argument of the proof of the Maximality Theorem generalizes to many other situations. The following corollary illustrates this. There will be more examples in the following section as well.

Corollary 6.2. *Let* $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$*, and let* c *be* $\mathbb{M}$*-generic over* V*. Then any* $d \in \mathrm{V}[c]$ *that is* $\bar{\mathbb{M}}$*-generic over* V *for any Magidor forcing* $\bar{\mathbb{M}}$ *is almost contained in* c*, i.e.,* $\mathrm{ran}(d) \setminus \mathrm{ran}(c)$ *is finite.*

Proof. Let $\mathbb{M}$ be based at κ and have length α. The proof of Theorem 6.1 goes through, after straightforward modifications.

If the claim failed, then there would be a condition p in the generic filter associated to c and an $\mathbb{M}$-name $\dot{d}$ with $\dot{d}^c = d$, so that p forces that $\dot{d}$ is $\check{\mathbb{M}}'$-generic over $\check{\mathrm{V}}$ but $\mathrm{ran}(\dot{d}) \setminus \mathrm{ran}(\Gamma)$ is infinite, where Γ is the canonical $\mathbb{M}$-name for the generic sequence. Let $\pi : \mathrm{V} \longrightarrow M_\alpha$ be the iteration corresponding to $\mathbb{M}$ along p, and let $\vec{\lambda}$ be the generic sequence associated to the iteration. Then $\pi(p)$ belongs to the $\pi(\mathbb{M})$-generic filter (over M_α) associated to $\vec{\lambda}$, and so, letting $d' := \pi(\dot{d})^{\vec{\lambda}}$, it follows that $d' \setminus \{\vec{\lambda}\}$ is infinite. Let e be the initial segment of $d' \setminus \{\vec{\lambda}\}$ of order type ω. Since d' is a normal function, generic for $\bar{\mathbb{M}}' := \pi(\bar{\mathbb{M}})$, it follows that $\bar{\kappa} := \sup e$ is measurable in M_α. The argument of the proof of Theorem 6.1 shows that there is a set $C \in M_\alpha$ such that for all $\epsilon \in e$, $\epsilon \notin C$, yet C belongs to any normal ultrafilter in M_α on $\bar{\kappa}$. Thus, the sequence with constant value C with the appropriate length can be used to show that d' does not satisfy the Characterization Theorem 4.4 over M_α. □

7 Uniqueness

In the previous section, I showed that if $d \in \mathrm{V}[c]$, where c, d are Magidor-sequences over V, then $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$. This phenomenon is known from Příkrý forcing, but there are generalizations of Příkrý forcing where a stronger conclusion, which I call uniqueness, can be made. In the above situation, if c and d are generic for the same Magidor forcing $\mathbb{M}$ of length α, then I want to conclude not only that the range of d is almost contained in the range of c, but that for almost all γ, $c(\gamma) = d(\gamma)$. In particular, it follows that $\mathrm{V}[c] = \mathrm{V}[d]$. This is a considerable strengthening that is true of diagonal Příkrý forcing (the terminology is due to Magidor, and the forcing is treated in [Git10, Section 1.3]. It is a special case of the "simple" forcings introduced earlier in [Fuc05, Section 7], which add one point below each measurable cardinal in a discrete set of measurables. Uniqueness of these simple sequences, and in particular of diagonal Příkrý sequences, is a consequence of the more general Maximality Theorem of [Fuc05, Section 6]). It turns out to be true of Magidor sequences as well, as I will show in the present section.

The following simple tool will be useful in many situations to follow.

Observation 7.1. *Let $\langle W_\gamma \mid \gamma < \alpha\rangle$ be a sequence of distinct $<\kappa$-complete ultrafilters on κ, where $\alpha < \kappa$. Then there is a sequence $\langle X_\gamma \mid \gamma < \alpha\rangle$ of pairwise disjoint sets such that for each $\gamma < \alpha$, $X_\gamma \in U_\gamma$.*

Proof. For $\gamma, \delta < \alpha$ with $\gamma \neq \delta$, let $X_{\gamma,\delta} \in W_\gamma \setminus W_\delta$. Let $W_\gamma = \bigcap_{\delta \in \alpha\setminus\{\gamma\}} X_{\gamma,\delta}$. By the closure of W_γ, $Y_\gamma \in W_\gamma$. And for $\delta < \alpha$ with $\delta \neq \gamma$, $Y_\gamma \subseteq X_{\gamma,\delta} \notin W_\delta$. So $Y_\gamma \notin W_\delta$. This means that $(\kappa \setminus Y_\gamma) \in W_\delta$. So let $X_\gamma = Y_\gamma \cap \bigcap_{\delta\in\alpha\setminus\{\gamma\}}(\kappa \setminus Y_\delta)$. Then $X_\gamma \in W_\gamma$, and for $\gamma \neq \delta$, $\gamma, \delta < \alpha$, $X_\gamma \cap X_\delta = \emptyset$: Suppose $\xi \in X_\gamma$. Then $\xi \in \kappa \setminus Y_\delta$, so $\xi \notin Y_\delta$. But $X_\delta \subseteq Y_\delta$. So $\xi \notin X_\delta$, i.e., X_γ and X_δ are disjoint. □

In fact, in the case of normal ultrafilters, this can be strengthened as follows (this may be relevant for a version of Magidor forcing where a κ-sequence is being added to κ).

Observation 7.2. *Let $\langle W_\gamma \mid \gamma < \kappa\rangle$ be a sequence of distinct normal ultrafilters on κ. Then there is a sequence $\langle X_\gamma \mid \gamma < \alpha\rangle$ of pairwise disjoint sets such that for each $\gamma < \alpha$, $X_\gamma \in U_\gamma$.*

Proof. For $\gamma, \delta < \kappa$ with $\gamma \neq \delta$, let $X_{\gamma,\delta} \in W_\gamma \setminus W_\delta$, and let $X_{\gamma,\gamma} = \kappa$. Let $Y_\gamma = \triangle_{\delta<\kappa} X_{\gamma,\delta}$. By normality of W_γ, $Y_\gamma \in W_\gamma$. For $\gamma, \delta < \kappa$ with $\gamma \neq \delta$, $Y_\gamma \setminus (\delta + 1) \subseteq X_{\gamma,\delta} \notin W_\delta$ (because if $\xi \in Y_\delta$ with $\xi > \delta$, then $\xi \in X_{\gamma,\delta}$). So $Y_\gamma \setminus (\delta + 1) \notin W_\delta$, since $X_{\gamma,\delta} \notin W_\delta$. So $Y_\gamma \notin W_\delta$.

Define, for $\gamma < \kappa$, $Z_\gamma = Y_\gamma \cap \bigcap_{\mu<\gamma}(\kappa \setminus Y_\mu)$. Clearly, $Z_\gamma \in W_\gamma$. Moreover, if $\gamma, \delta < \kappa$, $\gamma \neq \delta$ then obviously $Z_\gamma \cap Z_\delta = \emptyset$ (for wlog, let $\gamma < \delta$ – then $Z_\delta \subseteq (\kappa \setminus Y_\gamma)$ while $Z_\gamma \subseteq Y_\gamma$). □

The following two lemmas illustrate applications of Observations 7.1 or 7.2. Taken together, they will make it possible to show that Magidor sequences are unique in their forcing extension, up to finite changes.

Lemma 7.3. *Suppose $d \in \mathrm{V}[c]$, where c and d are $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$-generic over V. Let α, the length of $\mathbb{M}$, be a limit ordinal. Then for sufficiently large $\xi < \alpha$, $d(\xi) = c(\xi)$.*

Proof. By Theorem 6.1, there is a $\zeta < \alpha$ such that for all $\xi \in (\zeta, \alpha)$, there is a $\xi' = l(\xi)$ such that $d(\xi) = c(\xi')$. Using Observation 7.1, choose $\vec{X} \in \prod_{\gamma<\alpha} U_\gamma$ pairwise disjoint. By Theorem 4.4, there is a ζ' such that for all $\xi \in (\zeta', \alpha)$, $c(\xi), d(\xi) \in X(\xi)$ and $d(\xi) = c(l(\xi))$. Now if there were unboundedly many $\xi < \alpha$ with $d(\xi) \neq c(\xi)$, we could pick such a $\xi \in (\zeta', \alpha)$. So $l(\xi) \neq \xi$, $d(\xi) \in X(\xi)$, yet $d(\xi) = c(l(\xi)) \in X(l(\xi))$. So $X(\xi) \cap X(l(\xi)) \neq \emptyset$, even though $\xi \neq l(\xi)$, contradicting the choice of $\vec{X}$. □

Lemma 7.4. *If c is $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$-generic over V, where α, the length of $\vec{U}$, is an infinite successor ordinal, and if $d \in \mathrm{V}[c]$ is $\mathbb{M}$-generic over V as well, then $c(\bar{\alpha}) = d(\bar{\alpha})$, where $\bar{\alpha}$ is the largest limit ordinal less than α.*

Proof. By Corollary 6.2, $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$. It follows that for every limit ordinal $\lambda \leq \bar{\alpha}$, $d(\lambda) \geq c(\lambda)$. For if not, let λ be the smallest counterexample. If λ is a limit of limit ordinals, it follows that $d(\lambda) = \sup\{d(\bar{\lambda}) \mid \bar{\lambda} < \lambda \text{ is a limit}\} \geq \sup\{c(\bar{\lambda}) \mid \bar{\lambda} < \lambda \text{ is a limit}\} = c(\lambda)$, a contradiction. Otherwise, let $\bar{\lambda}$ be the largest limit ordinal below λ, so $\lambda = \bar{\lambda} + \omega$, or $\lambda = \omega$ and $\bar{\lambda} = 0$. Then $d(\bar{\lambda}) \geq c(\bar{\lambda})$. Let f, the finite set of all $\xi \in (\bar{\lambda}, \lambda)$ with $d(\xi) \notin \mathrm{ran}(c)$, have m elements. It follows for $n < \omega$ with $f \subseteq d(\bar{\lambda}+n)$ that $d(\bar{\lambda}+n+m) \geq c(\bar{\lambda}+n)$. So $d(\lambda) = \sup\{d(\bar{\lambda}+n+m) \mid n < \omega\} \geq \sup\{c(\bar{\lambda}+n) \mid n < \omega\} = c(\lambda)$, which is also a contradiction.

In particular, $d(\bar{\alpha}) \geq c(\bar{\alpha})$. Now assume towards a contradiction that $d(\bar{\alpha})$ was greater than $c(\bar{\alpha})$. Then there would be a $\zeta < \bar{\alpha}$ with $d(\zeta) > c(\bar{\alpha})$. Then for all $\xi \in (\zeta, \bar{\alpha})$, $d(\xi) > c(\bar{\alpha})$, but there are only finitely many ordinals greater than $c(\bar{\alpha})$ in $\mathrm{ran}(c)$. This is a contradiction. □

Putting these last two lemmas together yields the following Uniqueness Theorem.

Theorem 7.5 (Uniqueness)**.** *Let $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$ of length α, and let c be $\mathbb{M}$-generic over V. If $d \in \mathrm{V}[c]$ is also $\mathbb{M}$-generic, then for all but finitely many $\gamma < \alpha$, $c(\gamma) = d(\gamma)$. It follows in particular that for all limit ordinals $\gamma < \alpha$, $c(\gamma) = d(\gamma)$.*

Proof. Assume the contrary. Let α be least such that there is a counterexample $\mathbb{M}$ of length α. It cannot be that α is a limit ordinal, for if it were, then by Lemma 7.3, it would follow that there is an $\bar{\alpha} < \alpha$ such that for all $\gamma \in [\bar{\alpha}, \alpha)$, $c(\gamma) = d(\gamma)$. But then, $\bar{c} = c{\restriction}\bar{\alpha}$ and $\bar{d} = d{\restriction}\bar{\alpha}$ would be $\mathbb{M}^{-}_{\langle\bar{\alpha},\theta\rangle}$-generic, where

$\theta = c(\bar{\alpha}) = d(\bar{\alpha})$. By [Mag78, Lemma 5.3], it follows that $\bar{d} \in \mathrm{V}[\bar{c}]$. But since d and c differ infinitely many times, though never beyond $\bar{\alpha}$, it follows that $\bar{d}$ and $\bar{c}$ differ infinitely many times, contradicting the minimality of α. So α has to be a successor ordinal. Of course, α has to be infinite, so let $\alpha = \bar{\alpha} + n$, where $\bar{\alpha}$ is a limit ordinal. By Lemma 7.4, it follows that $\theta := c(\bar{\alpha}) = d(\bar{\alpha})$. Letting $\bar{c} = c{\restriction}\bar{\alpha}$ and $\bar{d} = d{\restriction}\bar{\alpha}$, it is now clear that $\bar{c}$ and $\bar{d}$ are $\mathbb{M}^{-}_{\langle\bar{\alpha},\theta\rangle}$-generic. As before, it follows that $\bar{d} \in \mathrm{V}[\bar{c}]$. Since there are infinitely many γ with $c(\gamma) \neq d(\gamma)$, yet there are only finitely many such γ that are greater than $\bar{\alpha}$, it follows that there are infinitely many γ with $\bar{c}(\gamma) \neq \bar{d}(\gamma)$. This again contradicts the minimality of α.

It remains to show that for all limit ordinals $\lambda < \alpha$, $c(\lambda) = d(\lambda)$. Assume the contrary, and let λ be a counterexample. Note that it is clear at this point that $\mathrm{V}[c] = \mathrm{V}[d]$, so by symmetry, it may be assumed that $c(\lambda) < d(\lambda)$. Since d is a normal function, $d(\lambda) = \sup_{\xi<\lambda} d(\xi)$. Let $\xi_0 < \lambda$ be such that $c(\lambda) < d(\xi_0)$. Then for all $\xi \in [\xi_0, \lambda)$, $c(\xi) < c(\lambda) < d(\xi)$. There are infinitely many such ξ, a contradiction. □

Note that a similar result does not hold in the case of Příkrý forcing. If c is a Příkrý sequence and d enumerates $\mathrm{ran}(c) \setminus \{\min(\mathrm{ran}(c))\}$, then $d \in \mathrm{V}[c]$ is also Příkrý-generic, but $c(n) \neq d(n)$ for all $n < \omega$. Diagonal Příkrý forcing and other forcings adding one point below every measurable cardinal in a discrete set, as introduced in [Fuc05], do have a similar uniqueness property. It seems that ultimately, the reason for the difference in behavior of classical Příkrý sequences on the one hand and sequences that were added by a variant of diagonal Příkrý forcing, or, as it will turn out, by Magidor forcing, may be that in Příkrý forcing, the whole infinite sequence comes from one normal measure, while in the other variants, every point in the sequence comes from its own measure. In terms of the iterations the critical points of which form a generic sequence, this can be formulated more precisely: In Příkrý forcing, the same normal measure (and its images) is used infinitely many times, while in diagonal Příkrý forcing (and its variants) each measure is applied only once (or finitely many times), and similarly for Magidor forcing.

Corollary 7.6. *Let $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$. Let c be* V*-generic for $\mathbb{M}$. Then the following are equivalent, for a function $d \in \mathrm{V}[c] \cap \prod_{\gamma<\alpha}(B_\gamma \setminus (\alpha+1))$:*

1. *d is $\mathbb{M}$-generic over* V.
2. *The set of $\xi < \alpha$ with $c(\xi) \neq d(\xi)$ is finite and contains no limit ordinals.*
3. $\mathrm{V}[c] = \mathrm{V}[d]$.

Proof. This is obvious, given Theorem 7.5 and Corollary 4.6. □

I want to explore now how these uniqueness results can be extended to the situation where $d \in \mathrm{V}[c]$, where c is Magidor generic and d is generic for a different Magidor forcing or even another kind of Příkrý type forcing. First, I show that it is impossible for d to be Příkrý generic, and in the following lemma, I show that d cannot be generic for any of the Příkrý type forcings introduced in [Fuc05].

Lemma 7.7. *Let $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$, and let c be $\mathbb{M}$-generic over* V*. Then* V$[c]$ *contains no sequence that is Příkrý-generic over* V*.*

Proof. Suppose $d \in \mathrm{V}[c]$ was generic over V for Příkrý-forcing with respect to a normal ultrafilter $W \in \mathrm{V}$ on $\bar{\kappa}$. The argument of the proof of the Maximality Theorem 6.1, or Corollary 6.2, shows that the range of d is almost contained in the range of c. It follows that $\bar{\kappa}$ is a limit point of $\mathrm{ran}(c)$. Let's assume that $\bar{\kappa} = c(\beta)$, where β is a limit ordinal less than α, the length of $\vec{U}$. Since W can be equal to at most one of the ultrafilters $f^{\beta}_{\gamma}(c(\beta))$, for $\gamma < \beta$, let β_0 be such that for all $\gamma \in (\beta_0, \beta)$, $W \neq f^{\beta}_{\gamma}(c(\beta))$. Apply the previous observation to pick, in V, a set $X \in W$ and a sequence $\langle X_\gamma \mid \beta_0 < \gamma < \beta \rangle$ so that $X_\gamma \in f^{\beta}_{\gamma}(c(\beta))$, and so that all of these measure one sets are pairwise disjoint. Now by Mathias' criterion for Příkrý genericity, a tail of d is contained in X. But by the Characterization Theorem 4.4, for sufficiently large $\gamma < \beta$, $c(\gamma) \in X_\gamma$. Since a tail of d is contained in the range of c, it follows that there are arbitrarily large $\gamma < \beta$ such that $c(\gamma) \in \mathrm{ran}(d)$, $c(\gamma) \in X$ and $c(\gamma) \in X_\gamma$. This is a contradiction. One can argue similarly if $\sup(\mathrm{ran}(d)) = \sup(\mathrm{ran}(c))$, by using U_γ in place of $f^{\beta}_{\gamma}(c(\beta))$. □

Lemma 7.8. *Let $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$, and let c be $\mathbb{M}$-generic over* V*. Then* V$[c]$ *contains no sequence that is generic over* V *for any of the generalized Příkrý forcings introduced in [Fuc05].*

Proof. I will use the notation from [Fuc05] freely here. Suppose $d \in \mathrm{V}[c]$ was generic for the generalized Příkrý forcing $\mathbb{P}(\vec{\kappa}, \vec{U}, \vec{\eta})$. Since $\mathrm{V}[c]$ contains no Příkrý sequence, by Lemma 7.7, it may be assumed that $\eta_i < \omega$, for all i. It suffices to show that $d{\restriction}\omega \notin \mathrm{V}[c]$. Since $d{\restriction}\omega$ is generic for $\mathbb{P}(\vec{\kappa}{\restriction}\omega, \vec{U}{\restriction}\omega, \vec{\eta}{\restriction}\omega)$, we may assume that the domain of $\vec{\kappa}$, $\vec{U}$ and $\vec{\eta}$ is already ω. The argument of Theorem 6.1 or Corollary 6.2 shows that $\bigcup \mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$. But then, it follows that $\sup_{i<\omega} \kappa_i = \sup(\bigcup \mathrm{ran}(d))$ is either an element of $\mathrm{ran}(c)$ or equal to κ, where κ is the measurable cardinal on which the ultrafilters $\vec{U}$ with respect to which $\mathbb{M}$ is constructed. In both cases, it would follow that $\sup_{i<\omega} \kappa_i$ is measurable in the ground model, which is absurd, since this sequence exists there and witnesses that it has cofinality ω. □

These last two lemmas show that Příkrý sequences and generalized Příkrý sequences are very different from Magidor sequences, despite all the similarities I pointed out before. There is another way in which they differ substantially, which I want to emphasize here. If $\mathbb{P}$ is a (generalized) Příkrý forcing and $\varphi(x)$ is a formula, then for any a in the ground model, if there is a condition $p \in \mathbb{P}$ which forces $\varphi(\check{a})$, then $\mathbb{1}_{\mathbb{P}}$ forces $\varphi(\check{a})$ (i.e., any condition $q \in \mathbb{P}$ forces $\varphi(\check{a})$. Let's describe this property by saying that $\mathbb{1}_{\mathbb{P}}$ *decides all statements about check names.* So $\mathbb{P}$ behaves similarly to almost homogeneous forcing. The following lemma shows that Magidor forcing does not have that property iff the length of the forcing is strictly greater than ω.

Lemma 7.9. *Let $\mathbb{M}$ be a Magidor forcing of length α.*

1. *If $\alpha \leq \omega$, then $\mathbb{1}_{\mathbb{M}}$ decides all statements about check names.*

2. *If $\alpha > \omega$, then $\mathbb{1}_{\mathbb{M}}$ doesn't decide all statements about check names.*

Proof. I will first show point 2. Let $\mathbb{M}(\vec{U}, \vec{f})$ have length α, where α is greater than ω. For $\gamma < \alpha$, let $B_\gamma = B^{\mathbb{M}}_\gamma$. Let $\delta \in B_\omega$. Let $p = p^\delta$ be the condition $\langle s, T\rangle$, where $s = \{\langle \omega, \delta\rangle\}$, $T(n) = \delta$ for $n < \omega$ and $T(\gamma) = \kappa \setminus (\delta + 1)$, for $\omega < \gamma < \alpha$, where κ is the measurable cardinal of Mitchell order α on which $\mathbb{M}$ is based. Then p forces that the cofinality of δ is ω. But if I pick $\delta' \in B_0$, $\delta' > \delta$, and let $q = \langle s', T'\rangle \in \mathbb{M}$ be defined by $s' = \{\langle 0, \delta'\rangle\}$ and $T'(\gamma) = \kappa \setminus (\delta' + 1)$, for $0 < \gamma < \alpha$, then q forces that δ is regular (in fact, measurable), by Fact 2.11.

To show point 1, let $\mathbb{M}$ have length ω (if the length is finite, then the forcing is trivial, and there is nothing to show). Suppose p and q were conditions in $\mathbb{M}$ such that p forces $\varphi(\check{a})$ and $q = \langle s, T\rangle$ forces $\neg\varphi(\check{a})$, for some formula φ and some $a \in \mathrm{V}$. Let $G \ni p$ be $\mathbb{M}$-generic over V, and let c be the Magidor sequence associated to G. So $\varphi(a)$ holds in $\mathrm{V}[c]$. There is a sequence $c' \in \mathrm{V}[c]$ which results from modifying c at finitely many coordinates, such that $s \subseteq c'$. It follows from Corollary 4.6 that c' is $\mathbb{M}$-generic over V. Clearly, q belongs to the $\mathbb{M}$-generic filter associated to c', so $\varphi(a)$ fails in $\mathrm{V}[c']$. But equally clearly, $\mathrm{V}[c] = \mathrm{V}[c']$, so $\varphi(a)$ holds in $\mathrm{V}[c']$. This is a contradiction. □

I now want to turn to the case that $d \in \mathrm{V}[c]$, where both c and d are generic for Magidor forcings, but possibly different Magidor forcings, and analyze the coordinate-wise relationship between d and c. Let's first see what the situation is like in the case of Příkrý forcing.

Lemma 7.10. *If c is a Příkrý sequence generic for the Příkrý forcing with respect to the normal ultrafilter on U, then in $\mathrm{V}[c]$, there is no Příkrý forcing generic with respect to Příkrý forcing with respect to a different normal ultrafilter U'.*

Proof. Suppose there was $d \in \mathrm{V}[c]$ generic for the Příkrý forcing with respect to U', where $U' \neq U$. Clearly, U' has to be on κ (U' can't be on a $\bar{\kappa} < \kappa$, since adding c adds no bounded subset of κ, and it can't be on a measurable cardinal greater than κ, because adding c would preserve the measurability of such a cardinal, while adding d makes it ω-cofinal). The argument of Corollary 6.2 shows that $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$. Let $X \in U \setminus U'$. Then $\mathrm{ran}(c)$ is almost contained in X. But since $\kappa \setminus X \in U'$, $\mathrm{ran}(d)$ is almost contained in $\kappa \setminus X$. So there are at most finitely many members of $\mathrm{ran}(d)$ that belong to $\mathrm{ran}(c)$, contradicting that $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$. □

The situation in the case of Magidor forcing is a little more complicated.

Lemma 7.11. *Let $d \in \mathrm{V}[c]$, where c and d are generic for the Magidor forcings $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$ and $\bar{\mathbb{M}} = \mathbb{M}(\vec{\bar{U}}, \vec{\bar{f}})$, respectively. Let α be the length of $\mathbb{M}$ and $\bar{\alpha}$ be the length of $\bar{\mathbb{M}}$. Then there is a weakly increasing, continuous function $t : \bar{\alpha} \longrightarrow \alpha$ in V such that*

1. *For all but finitely many $\gamma < \bar{\alpha}$, and for all limit $\gamma < \bar{\alpha}$, $d(\gamma) = c(t(\gamma))$.*

2. For all limit ordinals $\lambda < \bar{\alpha}$ and for all sufficiently large $\gamma < \lambda$,

$$\bar{f}^{\lambda}_{\gamma}(d(\lambda)) = f^{t(\lambda)}_{t(\gamma)}(c(t(\lambda)))$$

3. If $\bar{\alpha}$ is a limit ordinal, then let $\theta = \sup \operatorname{ran}(t)$. If $\theta = \alpha$, then set $f^{\theta}_{\gamma}(\kappa) = U_\gamma$, for $\gamma < \alpha$. Then, for all sufficiently large $\gamma < \bar{\alpha}$,

$$\bar{U}_\gamma = f^{\theta}_{t(\gamma)}(\sup c``\theta)$$

Note: It follows from 1. that there are at most finitely many $\delta < \alpha$ such that the preimage $t^{-1}``\{\delta\}$ has more than one member, and each of these preimages is finite. This is because if $t(\gamma_0) = t(\gamma_1)$ yet $\gamma_0 < \gamma_1$, then $d(\gamma_0) < d(\gamma_1)$, yet $c(t(\gamma_0)) = c(t(\gamma_1))$. So $d(\gamma_0) \neq c(t(\gamma_0))$ or $d(\gamma_1) \neq c(t(\gamma_1))$. But there are only finitely many γ for which $d(\gamma) \neq c(t(\gamma))$. So t has to be "almost strictly increasing".

Proof. By Corollary 6.2, $\operatorname{ran}(d)$ is almost contained in $\operatorname{ran}(c)$. We may assume that $\sup(\operatorname{ran}(d)) \leq \sup(\operatorname{ran}(c))$, and that $\bar{\alpha}$ is a limit ordinal. It follows that $\bar{\alpha} \leq \alpha$. Define in $\mathrm{V}[c]$: $t(\xi) = c^{-1}(\min(\operatorname{ran}(c) \setminus d(\xi)))$ (in particular, if $d(\xi) \in \operatorname{ran}(c)$, then $t(\xi) = c^{-1}(d(\xi))$). Note that $t \subseteq \bar{\alpha} \times \alpha$, and so, $t \in \mathrm{V}$, since $\alpha < c(0)$.

Since for almost all $\xi < \bar{\alpha}$, $d(\xi) \in \operatorname{ran}(c)$, t is continuous (since both c and d are), and point 1 is immediate.

For point 2, assume the contrary. Fix a limit ordinal $\lambda < \bar{\alpha}$ such that the set E of $\gamma < \lambda$ with

$$\bar{f}^{\lambda}_{\gamma}(\theta) \neq f^{t(\lambda)}_{t(\gamma)}(\theta)$$

is unbounded in λ, where $\theta = d(\lambda) = c(t(\lambda))$. Define, in V, a sequence $\langle X_\gamma \mid \gamma < \lambda\rangle$ such that for $\gamma \in E$, $X_\gamma \in \bar{f}^{\lambda}_{\gamma}(\theta)$ but $X_\gamma \notin f^{t(\lambda)}_{t(\gamma)}(\theta)$. For $\gamma < \lambda$ with $\gamma \notin E$, let $X_\gamma = \theta$. Now let $\langle Y_\gamma \mid \gamma < t(\lambda)\rangle$ be defined as follows. If $\gamma \in E$, then let $Y_{t(\gamma)} = \theta \setminus X_\gamma$, and $Y_\xi = \theta$ whenever $\xi < \lambda$ is not of the form $t(\gamma)$, for some $\gamma < \lambda$ with $\gamma \in E$. Then by part 2 of Theorem 4.4, there is a $\bar{\zeta} < \lambda$ and a $\zeta < t(\lambda)$ such that for all $\xi < \lambda$ with $\xi > \bar{\zeta}$, $d(\xi) \in X_\xi$, and for all $\xi < t(\lambda)$ with $\xi > \zeta$, $c(\xi) \in Y_\xi$. Now let $\gamma \in E$ be large enough that $\gamma > \bar{\zeta}$, $t(\gamma) > \zeta$ and $d(\gamma) = c(t(\gamma))$. Then $d(\gamma) \in X_\gamma$, $c(t(\gamma)) \in Y_{t(\gamma)}$, $Y_{t(\gamma)} = \theta \setminus X_\gamma$, and $d(\gamma) = c(t(\gamma))$, a contradiction.

The proof of point 3 is almost identical. □

Definition 7.12. Let $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$ be a Magidor forcing of length α, based at κ. Let $C \subseteq \alpha$ be closed in its supremum, let $t : \bar{\alpha} \longrightarrow C$ be the monotone enumeration of C, and let $\bar{\kappa} \leq \kappa$. Then the *collapse of* $\mathbb{M}{\restriction}C$ *of height* $\bar{\kappa}$ is the forcing $\bar{\mathbb{M}} = \mathbb{M}(\vec{\bar{U}}, \vec{\bar{f}})$ defined as follows.

If C is unbounded in α, then $\bar{\mathbb{M}}$ is only defined if $\bar{\kappa} = \kappa$, and in that case, it is defined by $\bar{U}_\gamma = U_{t(\gamma)}$, for $\gamma < \bar{\alpha}$, and $\bar{f}^{\nu}_{\mu}(\delta) = f^{t(\nu)}_{t(\mu)}$, for $\mu < \nu < \bar{\alpha}$.

If C is bounded in α, then let $\theta = \sup C$. $\bar{\mathbb{M}}$ is then defined if $\bar{\kappa} \in B_\theta$, and in that case, it is defined by $\bar{U}_\gamma = f^{\theta}_{t(\gamma)}(\bar{\kappa})$, for $\gamma < \bar{\alpha}$, and $\bar{f}^{\nu}_{\mu} = f^{t(\nu)}_{t(\mu)}{\restriction}\bar{\kappa}$.

Lemma 7.13. *If $\bar{\mathbb{M}}$ is the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$, then $\bar{\mathbb{M}}$ is a Magidor forcing as well. Moreover, letting $\bar{\alpha}$ be the length of $\bar{\mathbb{M}}$ (i.e., the order type of C) and $t:\bar{\alpha}\longrightarrow C$ the monotone enumeration, it follows that $B^{\mathbb{M}}_{t(\gamma)}\cap\bar{\kappa}\subseteq B^{\bar{\mathbb{M}}}_{\gamma}$, for $\gamma<\bar{\alpha}$.*

Proof. It is obvious that $\bar{\mathbb{M}}$ is a Magidor forcing in the case that C is unbounded in α and hence, $\bar{\kappa}=\kappa$. If C is bounded in α, then let $\theta=\sup C$. Letting $\mu<\nu<\bar{\alpha}$, it needs to be verified that $[\bar{f}\nu_\mu]_{\bar{U}_\nu}=\bar{U}_\mu$. By definition of $\vec{\bar{U}}$ and $\vec{\bar{f}}$, this means that $[f^{t(\nu)}_{t(\mu)}{\restriction}\bar{\kappa}]_{f^{\theta}_{t(\nu)}(\bar{\kappa})}=f^{\theta}_{t(\mu)}(\bar{\kappa})$, and this is true because $\bar{\kappa}\in B_\theta$.

For the "moreover" part of the lemma, this is again clear if C is unbounded in α and $\bar{\kappa}=\kappa$, so let's focus on the other case. Fix $\gamma<\bar{\alpha}$, and assume that $\delta\in B^{\mathbb{M}}_{t(\gamma)}$, $\delta<\bar{\kappa}$. To see that $\delta\in B^{\bar{\mathbb{M}}}_{\gamma}$, fix $\mu<\nu<\bar{\alpha}$. It needs to be checked that $[\bar{f}^{\nu}_{\mu}{\restriction}\delta]_{\bar{f}^{\gamma}_{\nu}(\delta)}=\bar{f}^{\gamma}_{\mu}(\delta)$. Unraveling the definitions, this is equivalent to $[f^{t(\nu)}_{t(\mu)}{\restriction}\delta]_{f^{t(\gamma)}_{t(\nu)}(\delta)}=f^{t(\gamma)}_{t(\mu)}(\delta)$. But this holds, because $\delta\in B^{\mathbb{M}}_{t(\gamma)}$. □

Definition 7.14. Let $\mathbb{M}$ be a Magidor forcing of length α at κ, and let c be V-generic for $\mathbb{M}$. Let $C\subseteq\alpha$ be closed in its supremum. Let $\theta=\sup C$. If $\theta=\alpha$, then let $\bar{\kappa}=\kappa$, and if $\theta<\alpha$, then let $\bar{\kappa}=c(\theta)$. Then the canonical collapse of $\mathbb{M}$ determined by C and c is the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$.

Observation 7.15. *The canonical collapse of a Magidor forcing $\mathbb{M}$, determined by C and c as in the previous definition, is always defined, and is a Magidor forcing.*

Proof. Using the terminology of the definition, either $\bar{\kappa}=\kappa$, or $\bar{\kappa}\in B_\theta$. So the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$ is defined, and by Lemma 7.13, it is a Magidor forcing. □

Theorem 7.16. *Let c be* V*-generic for the Magidor forcing $\mathbb{M}=\mathbb{M}(\vec{U},\vec{f})$ of length α. Then*

1. *Let $C\subseteq\alpha$ be closed in its supremum, and let t be the monotone enumeration of C. Let $d:\bar{\alpha}\longrightarrow\bar{\kappa}$ be defined by*

$$d(\gamma)=c(t(\gamma)).$$

   Then d belongs to V$[c]$ *and is* V*-generic for the canonical collapse of $\mathbb{M}$ determined by C and c.*

2. *If $d\in$ V$[c]$ is a Magidor sequence, then let $C=\{\gamma\mid d(\gamma)\in\mathrm{ran}(c)\}$. Then C is closed in its supremum,* $\mathrm{dom}(d)\setminus C$ *is at most finite, and letting $t:\bar{\alpha}\longrightarrow C$ be the monotone enumeration, the function $\bar{d}(\gamma)=c(t(\gamma))$, for $\gamma<\bar{\alpha}$, is generic for $\bar{\mathbb{M}}$, the canonical collapse of $\mathbb{M}$ determined by C and c. Note that $\bar{d}$ is the monotone enumeration of* $\mathrm{ran}(d)\cap\mathrm{ran}(c)$*, which is almost equal to* $\mathrm{ran}(d)$*, with at most finitely many points missing.*

Proof. For 1., I will first deal with the case that $\theta=\alpha$ (and hence, $\bar{\kappa}=\kappa$). In this case, it is clear that the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$ is defined. To see that d is V-generic for $\bar{\mathbb{M}}$, I have to verify the conditions of the Characterization

Theorem 4.4. Before verifying 1. and 2. of that theorem, recall that by Lemma 7.13, for $\gamma < \bar{\alpha}$, $d(\gamma) = c(t(\gamma)) \in B^{\mathbb{M}}_{t(\gamma)} \cap \bar{\kappa} \subseteq B^{\bar{\mathbb{M}}}_{\gamma}$. To check condition 1. now, let $X \in \mathrm{V} \cap \prod_{\gamma<\bar{\alpha}} \bar{U}_\gamma$. So for $\gamma < \bar{\alpha}$, $X(\gamma) \in U_{t(\gamma)}$. Let $Y \in \mathrm{V} \cap \prod_{\gamma<\alpha} U_\gamma$ be defined by $Y(\gamma) = X(t^{-1}(\gamma))$ if $\gamma \in \mathrm{ran}(t)$, and let $Y(\gamma) = \kappa$ otherwise. Then since c is generic, for sufficiently large $\gamma < \alpha$, $c(\gamma) \in Y(\gamma)$. But this implies that for sufficiently large $\gamma < \bar{\alpha}$, $d(\gamma) \in X(\gamma)$. The second condition follows in a similar fashion.

I will now turn to the case $\theta < \alpha$, $\bar{\kappa} < \kappa$. Since $\bar{\kappa} = c(\theta) \in B^{\mathbb{M}}_{\theta}$, the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$ is defined, and by Lemma 7.13, it is a Magidor forcing. So it remains to check that d is $\bar{\mathbb{M}}$-generic over V, by checking that the conditions of the Characterization Theorem 4.4 hold, as above. As before, for $\gamma < \bar{\alpha}$, $d(\gamma) = c(t(\gamma)) \in B^{\mathbb{M}}_{t(\gamma)} \cap \bar{\kappa} \subseteq B^{\bar{\mathbb{M}}}_{\gamma}$, again by Lemma 7.13. For the first condition, let $\vec{X} \in \prod_{\gamma<\bar{\alpha}} \bar{U}_\gamma$. Define $\vec{Y} \in \prod_{\delta<\alpha} f^{\theta}_{\delta}(\bar{\kappa})$ by setting $Y(\delta) = X(t^{-1}(\gamma))$ if $\gamma \in \mathrm{ran}(t)$, and $Y(\delta) = \bar{\kappa}$ otherwise. Keeping in mind that $\bar{\kappa} = c(\theta)$, and that c is $\mathbb{M}$-generic over V, it follows from the fact that c satisfies the second condition of Theorem 4.4 that for sufficiently large $\delta < \theta$, $c(\delta) \in Y(\delta)$. Pulling back via t, this shows that for sufficiently large $\gamma < \bar{\alpha}$, $d(\gamma) \in X(\gamma)$. The argument establishing that d satisfies the second condition of Theorem 4.4 is similar. This concludes the proof of 1.

For 2., let d be V-generic for some Magidor forcing $\tilde{\mathbb{M}}$. Then $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$, by Theorem 6.2. Let $\bar{d}$ enumerate $\mathrm{ran}(c) \cap \mathrm{ran}(d)$. It is easy to see that, letting $C := \{\gamma \mid d(\gamma) \in \mathrm{ran}(c)\}$, $\bar{d}$ is generic for the canonical collapse $\mathbb{M}' = \mathbb{M}(\vec{U}', \vec{f}')$ of $\tilde{\mathbb{M}}$ determined by C and d. So $\bar{d} \in \mathrm{V}[c]$ is also generic for a Magidor forcing, and $\mathrm{ran}(\bar{d}) \subseteq \mathrm{ran}(c)$. Let $\theta = \sup C$, and in case C is unbounded α, let $\bar{\kappa} = \kappa$, and $\bar{\kappa} = c(\theta)$ otherwise. By (the proof of) Lemma 7.11, there is a normal function $t : \bar{\alpha} \longrightarrow \alpha$, where $\bar{\alpha}$ is the length of $\mathbb{M}'$, such that

1. For all $\gamma < \bar{\alpha}$, $\bar{d}(\gamma) = c(t(\gamma))$.
2. For all limit ordinals $\lambda < \bar{\alpha}$ and for all sufficiently large $\gamma < \lambda$, ${f'}^{\lambda}_{\gamma}(d(\lambda)) = f^{t(\lambda)}_{t(\gamma)}(c(t(\lambda)))$.
3. Suppose $\bar{\alpha}$ is a limit ordinal. If $\theta = \alpha$, then for all sufficiently large $\gamma < \bar{\alpha}$, $U'_\gamma = U_{t(\gamma)}$, and if $\theta < \alpha$, then for all sufficiently large $\gamma < \bar{\alpha}$, $U'_\gamma = f^{\theta}_{t(\gamma)}(\bar{\kappa})$.

I will use this in order to verify conditions 1. and 2. of the Characterization Theorem 4.4 for $\bar{d}$ with respect to $\bar{\mathbb{M}} = \mathbb{M}(\vec{\bar{U}}, \vec{\bar{f}})$, the collapse of $\mathbb{M}{\restriction}C$ at $\bar{\kappa}$. First, note that for $\gamma < \bar{\alpha}$, $\bar{d}(\gamma) = c(t(\gamma)) \in B^{\mathbb{M}}_{t(\gamma)} c \cap \bar{\kappa} \subseteq B^{\bar{\mathbb{M}}}_{\gamma}$. Now, for the first characterization condition, if $\bar{\alpha}$ is a limit ordinal, let $X \in \mathrm{V} \cap \prod_{\gamma<\bar{\alpha}} \bar{U}_\gamma$. Note that by 3. above, for sufficiently large $\gamma < \bar{\alpha}$, $U'_\gamma = \bar{U}_\gamma$. So for sufficiently large $\gamma < \bar{\alpha}$, $X_\gamma \in U'_\gamma$. Since $\bar{d}$ is generic for $\mathbb{M}'$, it follows that for sufficiently large $\gamma < \bar{\alpha}$, $\bar{d}(\gamma) \in U'_\gamma = \bar{U}_\gamma$. The argument for the second characterizing condition is similar. Let $\lambda < \bar{\alpha}$ be a limit ordinal, and let $X \in \mathrm{V} \cap \prod_{\gamma<\lambda} \bar{f}^{\lambda}_{\gamma}(\bar{d}(\lambda))$. By 1. and

2. above, for sufficiently large $\gamma < \lambda$, ${f'}^{\lambda}_{\gamma}(\bar{d}(\lambda)) = f^{t(\lambda)}_{t(\gamma)}(c(t(\lambda))) = \bar{f}^{\lambda}_{\gamma}(\bar{d}(\lambda))$. Since $\bar{d}$ is $\mathbb{M}'$-generic, it follows that for sufficiently large $\gamma < \lambda$, $\bar{d}(\gamma) \in {f'}^{\lambda}_{\gamma}(\bar{d}(\lambda))$. So, for sufficiently large $\gamma < \lambda$, $\bar{d}(\lambda) \in \bar{f}^{\lambda}_{\gamma}(\bar{d}(\lambda))$. This completes the proof. $\square$

Lemma 7.17. *Let c be $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$-generic and d be $\mathbb{M}' = \mathbb{M}(\vec{U}', \vec{f}')$-generic over* V*, and assume that* $\mathrm{V}[c] = \mathrm{V}[d]$. *Let α be the length of $\mathbb{M}$ and α' be the length of $\mathbb{M}'$. Let $\bar{\alpha}$ be the largest limit ordinal less than or equal to α. Then the following hold.*

1. *The symmetric difference of* $\mathrm{ran}(c)$ *and* $\mathrm{ran}(d)$ *is finite.*
2. *$\bar{\alpha}$ is also the largest limit ordinal less than or equal to α'.*
3. *For all limit $\gamma < \bar{\alpha}$, $c(\gamma) = d(\gamma)$, and if $\bar{\alpha} < \alpha$ and $\bar{\alpha} < \alpha'$, then $c(\bar{\alpha}) = d(\bar{\alpha})$.*
4. *If $\lambda < \bar{\alpha}$ is a limit of limits, then for sufficiently large $\xi < \lambda$, $c(\xi) = d(\xi)$, and moreover, $f^{\lambda}_{\xi}(c(\lambda)) = {f'}^{\lambda}_{\xi}(d(\lambda))$. The same is true for $\lambda = \bar{\alpha}$, if $\bar{\alpha}$ is a limit of limits and $\bar{\alpha} < \alpha, \alpha'$.*

Proof. Since $c \in \mathrm{V}[d]$, $\mathrm{ran}(c)$ is almost contained in $\mathrm{ran}(d)$ by Corollary 6.2, and vice versa, since $d \in \mathrm{V}[c]$, the same corollary shows that $\mathrm{ran}(d)$ is almost contained in $\mathrm{ran}(c)$, establishing 1. Clearly, 1 implies 2.

To avoid trivialities, assume c and d are infinite sequences, and let θ be the largest limit point of $\mathrm{ran}(c)$. By 1, θ is also the largest limit point of $\mathrm{ran}(d)$. Let $\bar{c}$ and $\bar{d}$ be the maximal initial segments of c and d, respectively, so that $\theta = \sup \mathrm{ran}(\bar{c}) = \sup \mathrm{ran}(\bar{d})$. Then $\bar{\alpha} = \mathrm{dom}(\bar{c}) = \mathrm{dom}(\bar{d})$ is the largest limit ordinal less than or equal to α as well as to α'.

Since $d \in \mathrm{V}[\bar{c}]$ and $c \in \mathrm{V}[\bar{d}]$, there are weakly monotonic functions $s : \alpha \longrightarrow \alpha'$ and $t : \alpha' \longrightarrow \alpha$ such that the conclusions of Lemma 7.11 hold in both directions. So

(a) For all but finitely many $\gamma < \alpha$, and for all limit $\gamma < \alpha$, $c(\gamma) = d(s(\gamma))$, and for all but finitely many $\gamma < \alpha'$, $d(\gamma) = c(t(\gamma))$.

(b) For all limit ordinals $\lambda < \alpha$ and for all sufficiently large $\gamma < \lambda$, $f^{\lambda}_{\gamma}(c(\lambda)) = {f'}^{s(\lambda)}_{s(\gamma)}(d(s(\lambda)))$, and for all limit ordinals $\lambda < \alpha'$ and for all sufficiently large $\gamma < \lambda$, ${f'}^{\lambda}_{\gamma}(d(\lambda)) = f^{t(\lambda)}_{t(\gamma)}(c(t(\lambda)))$.

(c) If α is a limit ordinal, then let $\theta' = \sup \mathrm{ran}(s)$. If $\theta = \alpha'$, then set ${f'}^{\theta}_{\gamma}(\kappa') = U'_{\gamma}$, for $\gamma < \alpha'$. Then, for all sufficiently large $\gamma < \alpha$,

$$U_{\gamma} = {f'}^{\theta}_{s(\gamma)}(\sup d``\theta)$$

Similarly, for all sufficiently large $\gamma < \alpha'$,

$$U'_{\gamma} = f^{\sup \mathrm{ran}(t)}_{t(\gamma)}(\sup c``(\sup \mathrm{ran}(t)))$$

It follows from the note after the statement of Lemma 7.11 that for limit ordinals γ in the domain of s, $s(\gamma)$ is a limit ordinal greater than or equal to γ, and similarly, $t(\gamma)$ is a limit ordinal greater than or equal to γ if γ belongs to the domain of t. The existence of the functions s and t immediately implies 3, because if $\gamma < \bar{\alpha}$ is a limit, then by (a), $c(\gamma) = d(s(\gamma))$, and, since $s(\gamma)$ is a limit, $d(s(\gamma)) = c(t(s(\gamma)))$. So $c(\gamma) = c(t(s(\gamma)))$, which implies that $\gamma \leq s(\gamma) \leq t(s(\gamma)) = \gamma$, and so, $c(\gamma) = d(\gamma)$. Note that this shows that if $\gamma < \bar{\alpha}$ is a limit ordinal, then $\gamma = s(\gamma) = t(\gamma)$. It follows similarly that $s(\bar{\alpha}) = t(\bar{\alpha})$ if $\bar{\alpha} < \alpha, \alpha'$.

For 4, let $\lambda < \bar{\alpha}$ be a limit of limits. Let $\bar{\lambda} < \lambda$ be a limit ordinal such that for all $\gamma < \lambda$ with $\gamma \geq \bar{\lambda}$, $\gamma \leq s(\gamma)$ and $\gamma \leq t(\gamma)$, and (a) above holds at γ. So, for such γ, $c(\gamma) = d(s(\gamma))$, $d(s(\gamma)) = c(t(s(\gamma)))$, which implies that $t(s(\gamma)) = \gamma$. But then, $\gamma \leq s(\gamma) \leq t(s(\gamma)) = \gamma$, so $\gamma = s(\gamma) = t(\gamma)$. This means that for $\gamma < \lambda$ with $\gamma \geq \bar{\lambda}$, $c(\gamma) = d(\gamma)$. The rest of statement 4 follows similarly. □

One cannot conclude, in the setting of the previous lemma, that for all but finitely many ξ, $c(\xi) = d(\xi)$. For example, $\mathbb{M}$ could be the collapse of $\mathbb{M}' \restriction \kappa \setminus \{0\}$ of height κ and $c(n) = d(n+1)$, for all $n < \omega$. This also shows that the requirement in 4. of the previous lemma that λ be a limit of limits cannot be dropped.

8 Minimal Magidor forcing

Observation 7.1 suggests that for fixed $\mu < \alpha$, the functions $\langle f_\mu^\nu \mid \mu < \nu < \alpha \rangle$ can be merged into one function g_μ as follows. Let $\vec{X} \in \prod_{\gamma<\alpha} U_\gamma$ be pairwise disjoint, and let $\bigcup_{\gamma<\alpha} X_\gamma = \kappa$. Define, for $\mu < \alpha$, a function $g_\mu : \kappa \longrightarrow \mathrm{V}$ by

$$g_\mu(\delta) = \begin{cases} f_\mu^\nu(\delta) & \text{if } \delta \in X_\nu, \text{ where } \nu > \mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that for $\mu < \nu < \alpha$,

$$U_\mu = [g_\mu]_{U_\nu},$$

because $U_\mu = [f_\mu^\nu]_{U_\nu}$, and $[f_\mu^\nu]_{U_\nu} = [g_\mu]_{U_\nu}$, since for all $\delta \in X_\nu$, $g_\mu(\delta) = f_\mu^\nu(\delta)$, and $X_\nu \in U_\nu$. One arrives at a "minimal" version of Magidor forcing by replacing the sequence $\vec{f}$ by the sequence $\vec{g}$. Let's say that $\mathbb{M}(\vec{U}, \vec{f})$ is minimal if for all $\mu < \nu < \nu' < \alpha$, $f_\mu^\nu = f_\mu^{\nu'}$. So in this case, it is not necessary to keep track of the superscripts, and one can just write f_μ in place of f_μ^ν. The above construction shows the following.

Observation 8.1. *Let $\vec{U}$ be a sequence of normal ultrafilters on κ of length $\alpha < \kappa$, increasing in the Mitchell order. Then there is a sequence $\vec{f}$ such that $\mathbb{M}(\vec{U}, \vec{f})$ is minimal.*

It seems that using minimal Magidor forcing reduces the combinatorial complexity considerably. The question arises whether restricting to minimal Magidor forcing is really a restriction - i.e., can there be Magidor sequences that are not generic for any minimal Magidor forcing? The following theorem shows that this is not the case.

Theorem 8.2. *Every Magidor sequence is generic for a minimal Magidor forcing.*

Proof. Suppose not. Let $p = \langle g, G\rangle \in \mathbb{M} := \mathbb{M}(\vec{U}, \vec{f})$. Let $\mathbb{M}$ be at κ and have length α. Let a be the domain of g. Let $\langle X_\mu \mid \mu < \alpha\rangle$ be a partition of κ such that

1. For all $\mu < \alpha$, $X_\mu \in U_\mu$.
2. For all $\gamma \in a$ and for all $\nu < \gamma$, $X_\nu \cap g(\gamma) \in f^\gamma_\nu(g(\gamma))$.
3. For all $\gamma \in a$, $g(\gamma) \in X_\gamma$.

This is easily achieved: For each $\gamma \in a$, apply Observation 7.2 to the sequence $\langle f^\gamma_\mu(g(\gamma)) \mid \mu < \gamma\rangle$ of normal ultrafilters on $g(\gamma)$. This gives a partition $\langle Y^\gamma_\mu \mid \mu < \gamma\rangle$ of $g(\gamma)$ such that for each $\mu < \gamma$, $Y_\mu \in f^\gamma_\mu(g(\gamma))$. In addition, by the same observation, let $\langle Y^\alpha_\mu \mid \mu < \alpha\rangle$ be a partition of κ such that for all $\mu < \alpha$, $Y^\alpha_\mu \in U_\mu$. Now define a partition $\langle Y_\mu \mid \mu < \alpha\rangle$ of κ by setting:

$$\xi \in Y_\mu \iff \xi \in Y^\gamma_\mu, \qquad \begin{array}{l}\text{where } \gamma < \alpha \text{ is the least } \mu \text{ such that } \mu \in a \text{ and } \xi < g(\mu),\\ \text{or } \gamma = \alpha \text{ if } g(\mu) < \xi \text{ for all } \mu \in a.\end{array}$$

It is clear that $\langle Y_\mu \mid \mu < \alpha\rangle$ satisfies conditions 1. and 2. above. By making finitely many modifications to this sequence, it is easy to arrange condition 3. in addition.

Let $\vec{g}$ be the resulting minimal sequence, i.e., for $\mu < \alpha$, $g_\mu : \kappa \longrightarrow \mathrm{V}$ is defined by

$$g_\mu(\delta) = \begin{cases} f^\nu_\mu(\delta) & \text{if } \delta \in X_\nu, \text{ where } \nu > \mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

so that for $\mu < \nu < \alpha$, $g_\mu{\restriction}X_\nu = f^\nu_\mu{\restriction}X_\nu$. Let $G'(\gamma) = G(\gamma) \cap X_\gamma$, for $\gamma \in \mathrm{dom}(G)$. So by 1. and 2. above, $\langle g, G'\rangle \in \mathbb{M}(\vec{U}, \vec{f})$, and clearly, $\langle g, G'\rangle$ is a strengthening of $\langle g, G\rangle$.

Claim: If $\langle f, F\rangle \leq \langle g, G'\rangle$ in $\mathbb{M}(\vec{U}, \vec{f})$, then $\langle f, F\rangle \in \mathbb{M}(\vec{U}, \vec{g})$.

For $\gamma \in \mathrm{dom}(F)$, if there is no $\gamma' > \gamma$ with $\gamma' \in \mathrm{dom}(f)$, then $F(\gamma) \in U_\gamma$, by 2., and if there is such a γ', then, letting β be the least, it follows that $f(\beta) \in X_\beta$ (either because $f(\beta) = g(\beta)$, or because $f(\beta) \in G'(\beta) = G(\beta) \cap X_\beta$). So $F(\gamma) \in f^\beta_\gamma(f(\beta)) = g_\gamma(f(\beta))$, by definition of g_γ. The only non-obvious condition that needs to be verified is that $f(\gamma) \in B^{\mathbb{M}(\vec{U}, \vec{g})}_\gamma$, for $\gamma \in \mathrm{dom}(f)$. First, note that, again, $f(\gamma) \in X_\gamma$. Fix $\mu < \nu < \gamma$. It needs to be shown that $[g_\mu{\restriction}f(\gamma)]_{g_\nu(f(\gamma))} = g_\mu(f(\gamma))$. Since $\langle f, F\rangle \in \mathbb{M}(\vec{U}, \vec{f})$, by definition, $f(\gamma) \in B^{\mathbb{M}(\vec{U}, \vec{f})}_\gamma$, which implies that $[f^\nu_\mu{\restriction}f(\gamma)]_{f^\gamma_\nu(f(\gamma))} = f^\gamma_\mu(f(\gamma))$. Since $f(\gamma) \in X_\gamma$, it follows that $W := g_\nu(f(\gamma)) = f^\gamma_\nu(f(\gamma))$ and $g_\mu(f(\gamma)) = f^\gamma_\mu(f(\gamma))$. So it remains to show that $[g_\mu{\restriction}f(\gamma)]_W = [f^\nu_\mu{\restriction}f(\gamma)]_W$. By definition of g_μ, for all $\xi \in X_\nu$,

$g_\mu(\xi) = f_\mu^\nu(\xi)$. But by condition 2. above, and because $\langle f, F\rangle \leq \langle g, G'\rangle$, it follows that $X_\nu \cap f(\gamma) \in f_\nu^\gamma(f(\gamma))$, which proves that $g(\gamma) \in B_\gamma^{\mathbb{M}(\vec{U},\vec{g})}$.

In particular, $\langle g, G'\rangle \in \mathbb{M}(\vec{U}, \vec{g})$. It is now easy to see that the converse of the previous claim holds as well. So whenever $\langle f, F\rangle \leq \langle g, G'\rangle$ in $\mathbb{M}(\vec{U}, \vec{g})$, then $\langle f, F\rangle \in \mathbb{M}(\vec{U}, \vec{f})$. It follows then easily that the restrictions of $\mathbb{M}(\vec{U}, \vec{g})$ and $\mathbb{M}(\vec{U}, \vec{f})$ to conditions below $\langle g, G'\rangle$ are equal.

Recall that $\langle g, G\rangle \in \mathbb{M}(\vec{U}, \vec{f})$ was arbitrary, and we found $\langle g, G'\rangle$ below it. This shows that the set of conditions $\langle h, H\rangle$ in $\mathbb{M}(\vec{U}, \vec{f})$ such that the restriction of $\mathbb{M}(\vec{U}, \vec{f})$ to conditions below $\langle h, H\rangle$ is the same as the restriction of some minimal Magidor forcing $\mathbb{M}(\vec{U}, \vec{g})$ to conditions below $\langle h, H\rangle$ is dense in $\mathbb{M}(\vec{U}, \vec{f})$. This proves the theorem. $\square$

I want to conclude this paper with a few general questions indicating possible further research ideas in the areas of characterization and uniqueness of Příkrý-type generic sequences. Firstly, there is the obvious question whether there are similar characterizations of genericity for other Příkrý-type forcings. Do they lead to the genericity of the critical sequences of certain accompanying iterations, and further, to similar uniqueness properties? Secondly, similar questions can be asked about Namba forcing. It was shown by Jensen in [Jen08] that any ω-sequence cofinal in ω_2^{V} which exists in $\mathrm{V}[G]$, where G is generic for Namba forcing, is also Namba-generic. So this is a simple characterization of the Namba-generic sequences that are present in a fixed Namba-generic extension. At the same time, it shows that uniqueness completely fails for Namba forcing. There are other versions of Namba forcing, and the question arises whether there is a version of Namba forcing that adds sequences having some kind of uniqueness property, and how these generic sequences can be characterized.

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