Closed Maximality Principles and Large Cardinals

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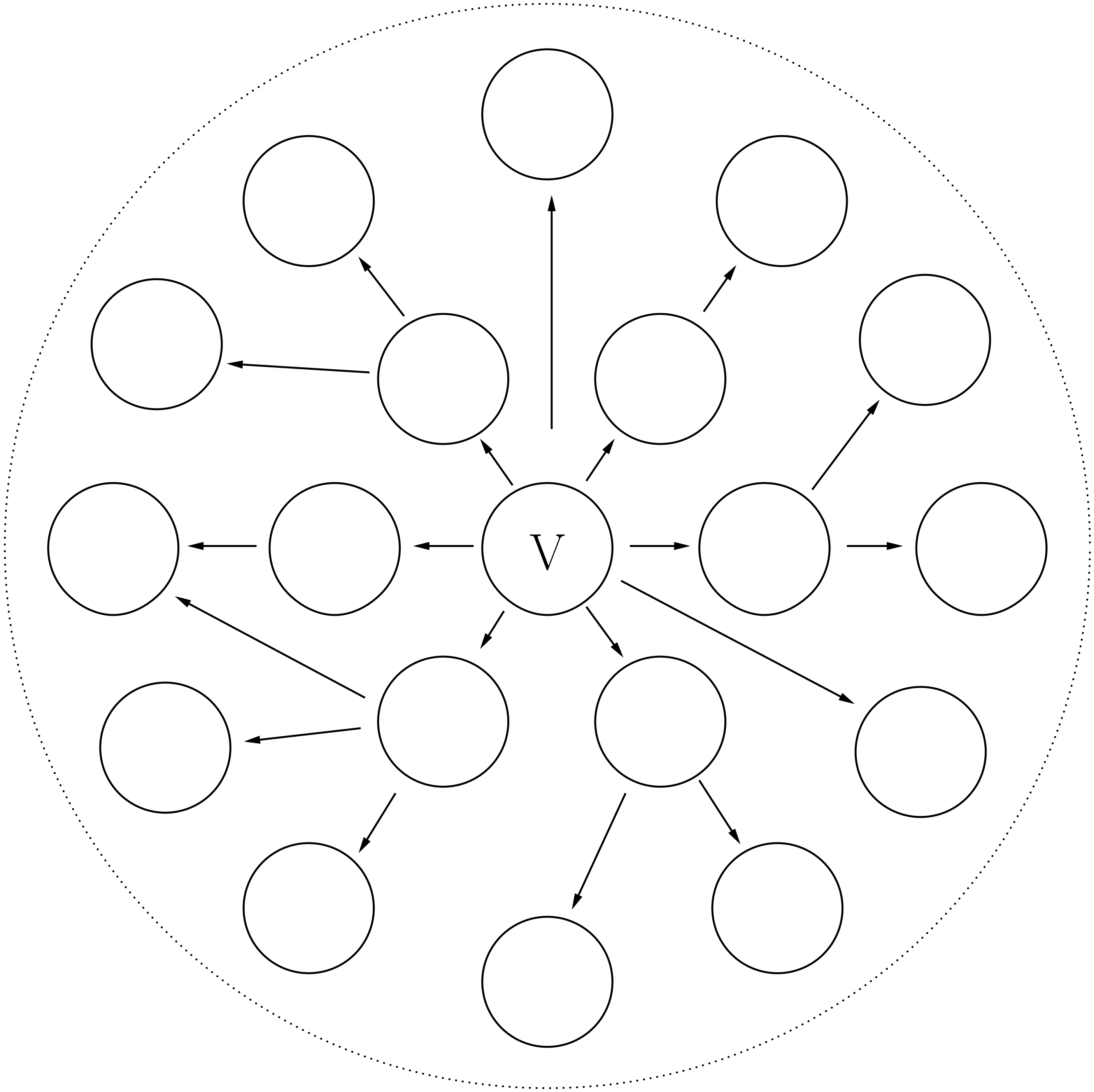
First European Set Theory Meeting, Bedlewo, 2007

July 12, 2007

Let's view the universe
and its possible generic extensions
as a Kripke model
for modal logic.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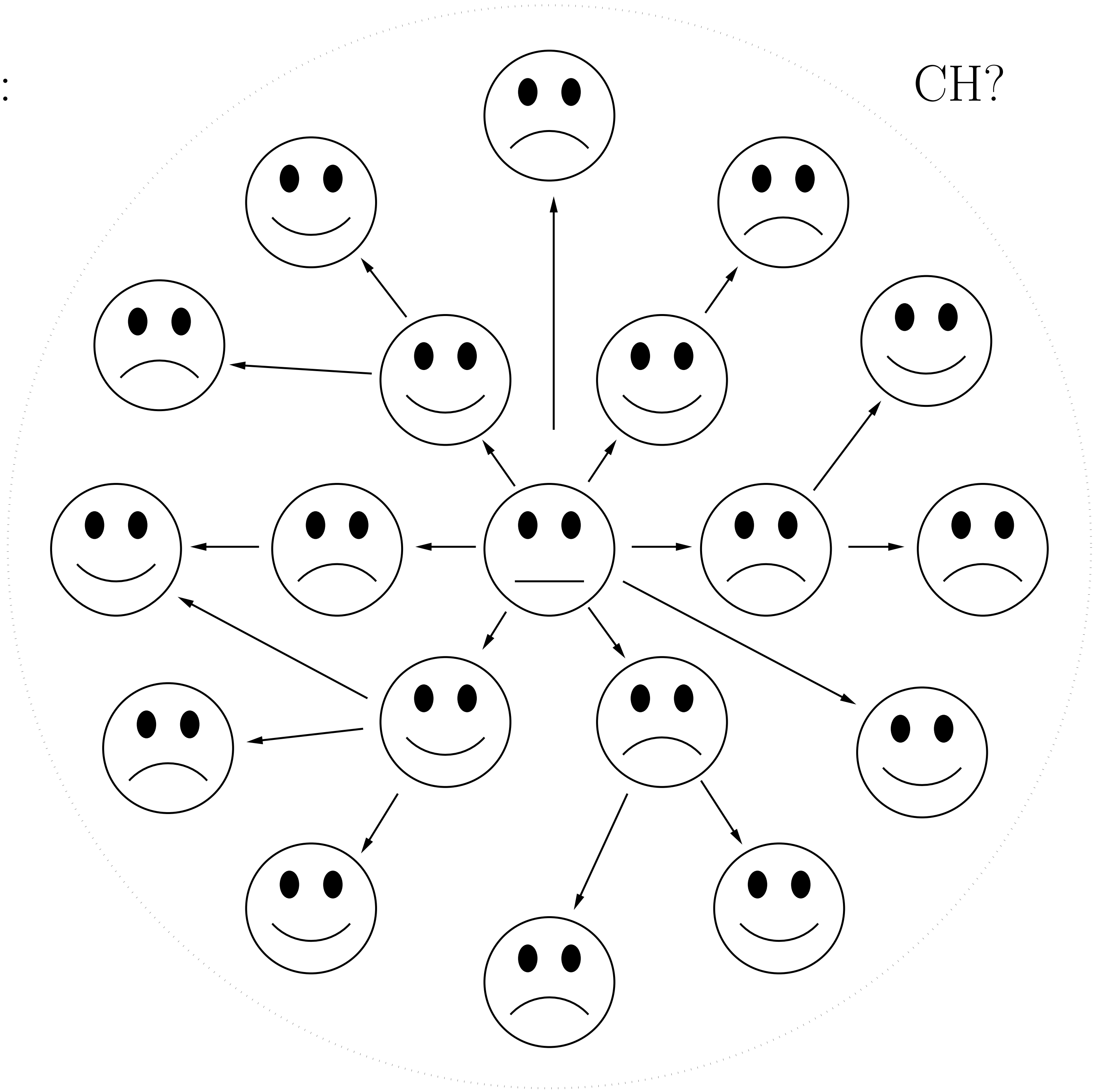
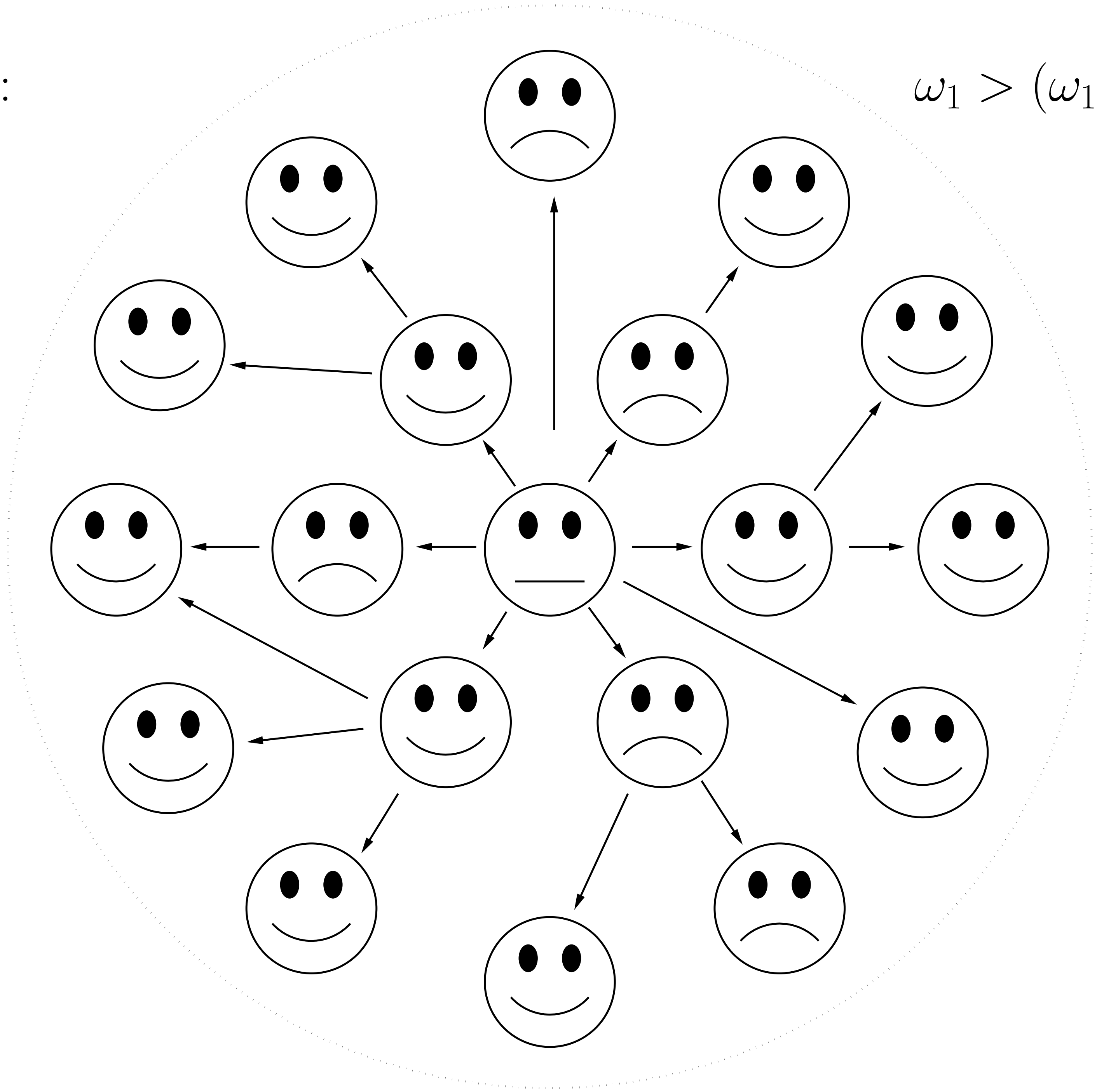
Question:

CH?



Question: $\omega_1 > (\omega_1)^L$?



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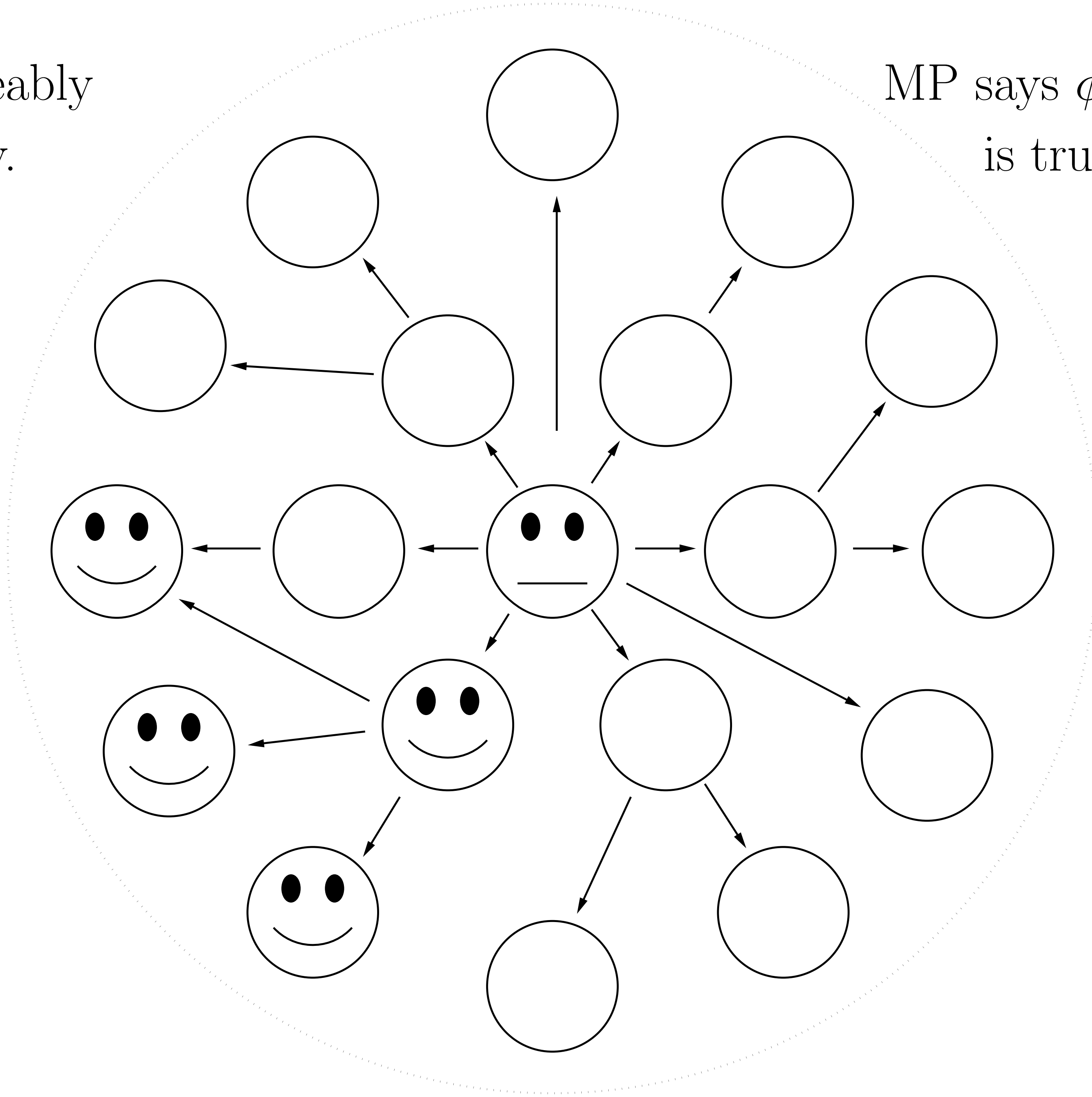

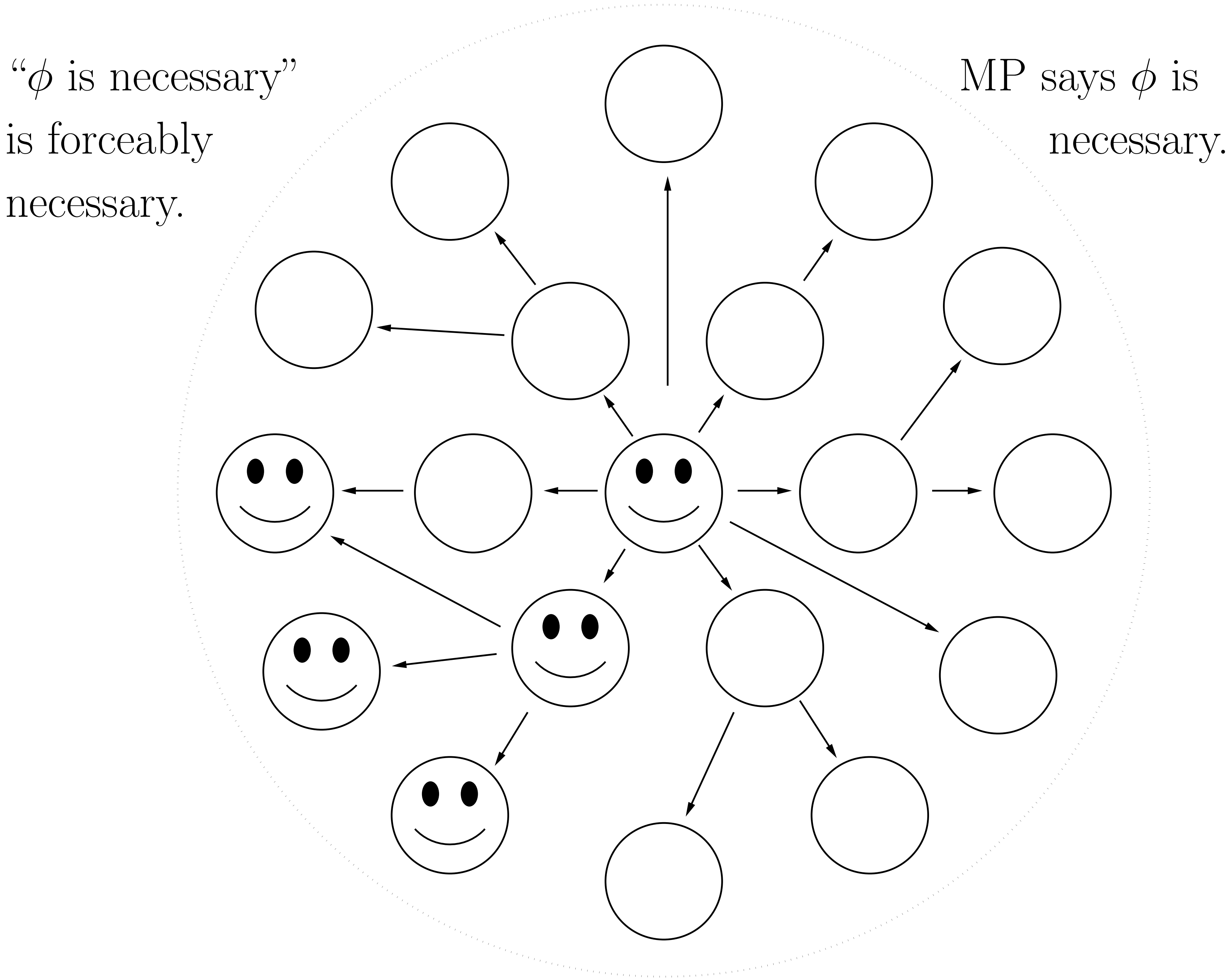
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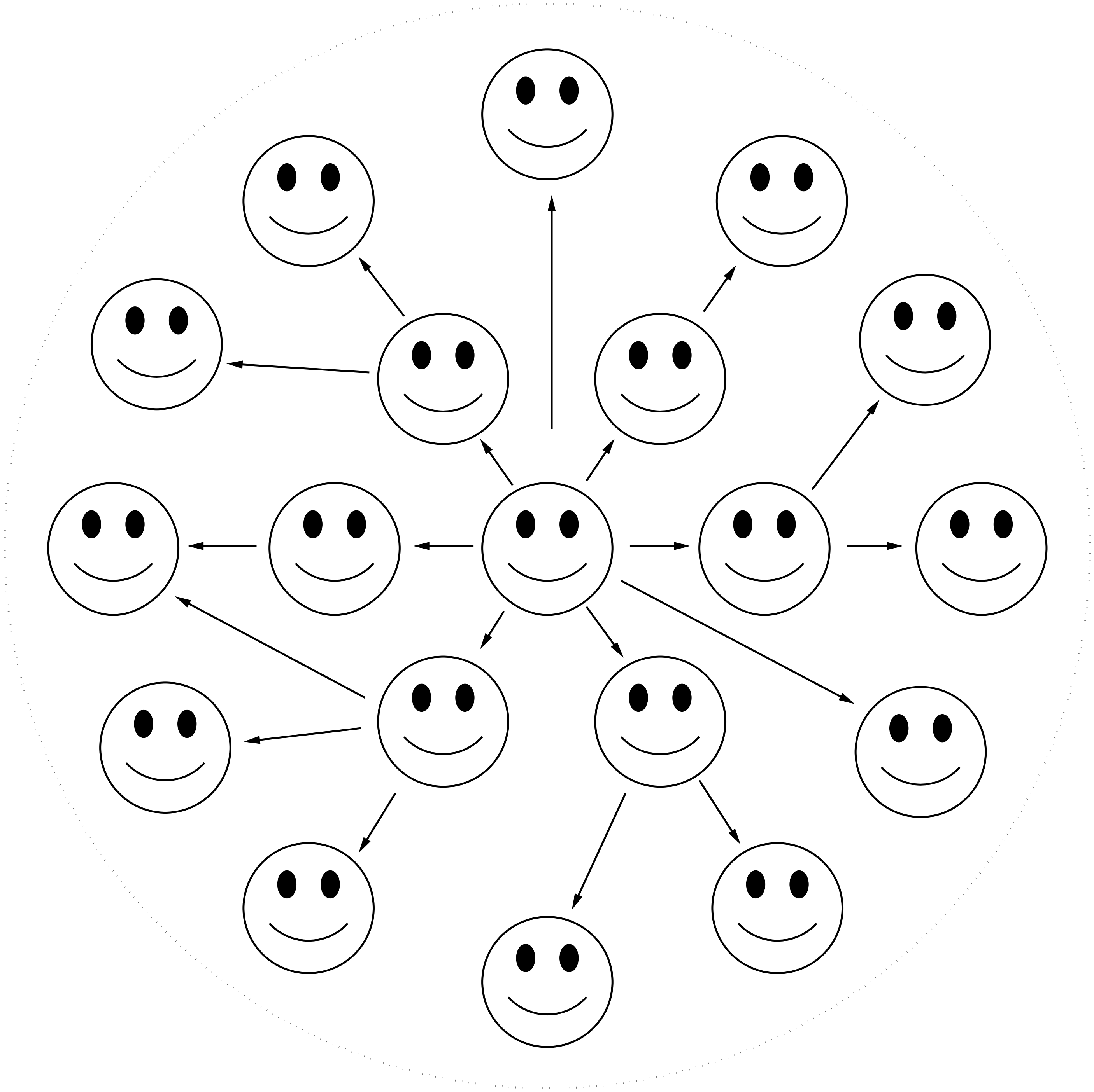

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The Maximality Principle MP is the scheme consisting of the formulae

$$(\Diamond\Box\varphi) \implies \varphi,$$

for every sentence φ. It was introduced by Joel Hamkins, and a close relative was introduced earlier and independently by Stavi and Väänänen.

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General form of the principle:

$$\mathsf{MP}_\Gamma(X),$$

where Γ is a class of partial orders and X is the parameter set.

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The implications between the principles are as follows:

Relationships

$$\begin{array}{ccc}
\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+}) \\
\Downarrow & & \Downarrow \\
\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+}) \\
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In general, none of the implications can be reversed.

Consistency Results

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- Moreover, in that model, $\kappa_1 < \delta_1$ are regular, and δ_1 is still fully reflecting.
- So since $\kappa_1 \geq \kappa_0^+$, further forcing with $\mathrm{Col}(\kappa_1, <\delta_1)$ preserves $\mathsf{MP}_{<\kappa_0-\mathrm{dir.\ cl.}}(H_{\kappa_0^+})$ and makes $\mathsf{MP}_{<\kappa_1-\mathrm{dir.\ cl.}}(H_{\kappa_1^+})$ true, in addition.

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- For every $\delta \in A$, $\langle \mathrm{V}_\delta, \in \rangle \prec \langle \mathrm{V}, \in \rangle$.

For $\delta \in A$, let $\bar{\delta}$ be the least regular cardinal which is greater than or equal to $\sup(A \cap \delta)$. The forcing which produces the desired model is then a reverse Easton iteration of collapses of the form $\mathrm{Col}(\bar{\delta}, {<}\delta)$, for $\delta \in A$. Call this forcing iteration $\mathbb{P}_A$.

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Definition 2. *A forcing has a strong closure point at a cardinal* δ *if it factors as* $\mathbb{P} * \dot{\mathbb{Q}}$, *where* $\mathbb{P}$ *has size at most* δ *and* $\mathbb{P}$ *forces that* $\dot{\mathbb{Q}}$ *is* $<\delta^{++}$*-strategically closed.*

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The crucial point is the following:

Lemma 3. *There is a formula $\psi(\cdot,\cdot)$ with the following property:*

If $\mathrm{V} = M[G]$, where G is generic over M for a forcing which has a strong closure point at δ then

$$M = \{x \mid \mathrm{V} \models \psi(x, z)\},$$

where $z = \mathcal{P}(\delta^+)^M$.

This uses Hamkins' approximation and cover properties and ideas of Reitz.

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So $\kappa^+ > (\kappa^+)^M$, which is impossible, since $\mathbb{P}$ has size less than κ.

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This leads to a study of techniques which lift embeddings of a model to generic extensions.

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6. $\pi : M[G] \longrightarrow_{\mathcal{F}} M'$ witnesses that κ is large in $\mathrm{V}[G]$.

Below a weakly compact cardinal

Lemma 4. *Let κ be weakly compact and $\mathrm{V}_\kappa \prec \mathrm{V}$. Then there is a forcing $\mathbb{P}$ such that in any $\mathbb{P}$-generic extension, $\mathrm{V}[G]$, κ is still weakly compact, and the boldface maximality principle for directed closed forcings holds at every regular cardinal $\bar{\kappa} < \kappa$.*

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Note: This is an equiconsistency; we get the reflecting weakly compact back in L.

On an unbounded measure 0 set below a measurable

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1. *$\mathsf{MP}_{<\bar{\kappa}-\mathrm{dir.\ cl.}}(H_{\bar{\kappa}^+})$ is true, for every $\bar{\kappa} \in A$, and*
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In this case, let U be a normal ultrafilter on κ, let $j : \mathrm{V} \longrightarrow N$ be the ultrapower by U, and let $\mathbb{P}$ force MP at all $\bar{\kappa} \in A$. Let G be generic for $\mathbb{P}$. $N[G]$ is closed under κ-sequences and thinks that the tail forcing $\dot{\mathbb{Q}}_G$, where $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, is $<\kappa^+$-closed, so that it is $<\kappa^+$-closed in $\mathrm{V}[G]$. Since moreover, $\mathcal{P}(j(\mathbb{P})) \cap N[G]$ has size κ^+ in $\mathrm{V}[G]$, it is possible to construct a generic G' for $\dot{\mathbb{Q}}_G$ over $N[G]$ in $\mathrm{V}[G]$. Then j lifts to $j' : \mathrm{V}[G] \longrightarrow N[G][G']$. $\square$

Remark 6.

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3. *The strength of an indestructible weakly compact is at least that of a non-domestic mouse, by methods of Jensen, Schindler and Steel (cf. "Stacking Mice"), as was observed by Schindler and myself.*

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Definition 7. *Let κ be a cardinal, γ an ordinal, and A a set. Then κ is supercompact up to γ wrt. A if for every $\bar{\gamma} < \gamma$, there is a $\bar{\gamma}$-supercompact embedding j of the universe, with critical point κ, such that $j(A) \cap \mathrm{V}_{\bar{\gamma}} = A \cap \mathrm{V}_{\bar{\gamma}}$ and $j(\kappa) > \bar{\gamma}$.*

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The notion "almost huge to γ wrt. A" is defined analogously. It is all just like in the case of strong cardinals.

On a measure one set up to and including a measurable

Lemma 8. *Assume that* $\kappa < \rho$, $\mathrm{V}_\kappa \prec \mathrm{V}_\rho \prec \mathrm{V}$ *and* κ *is supercompact up to* $\rho + 1$ *wrt.* A, *where* $A = \{\bar{\rho} \mid \bar{\rho} \leq \rho \wedge \mathrm{V}_{\bar{\rho}} \prec \mathrm{V} \wedge \bar{\rho}$ *is regular.*$\}$. *Then there is a forcing* $\mathbb{P}$ *such that if* G *is* $\mathbb{P}$*-generic over* V, *then in* $\mathrm{V}[G]$, κ *is measurable,* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$ *holds, and the set of* $\lambda < \kappa$ *which are regular and at which* $\mathsf{MP}_{<\lambda-\mathrm{dir.\ cl.}}(H_{\lambda^+})$ *holds has measure 1 wrt. any normal ultrafilter on* κ *in* $\mathrm{V}[G]$.

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Proof. The Silver argument works. Supercompactness wrt. A is used in order to guarantee that $\mathbb{P}$ is an initial segment of $j(\mathbb{P})$, where $\mathbb{P}$ is the forcing iteration of length $\kappa + 1$ which forces the desired maximality principles.

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Up to (and including) a weakly compact

Lemma 9. *Let* $\kappa < \rho$*,* ρ *regular,* $\mathrm{V}_\rho \prec \mathrm{V}$*,* $A = \{\bar\rho \mid \bar\rho \leq \rho \wedge \mathrm{V}_{\bar\rho} \prec \mathrm{V}\}$*, and let* κ *be* A*-supercompact to* $\rho + 1$*. Then there is a forcing* $\mathbb{P}$ *which yields an extension* $\mathrm{V}[G]$ *such that* κ *is weakly compact in* $\mathrm{V}[G]$ *and* $\mathsf{MP}_{<\bar\kappa-\mathrm{dir.\ cl.}}(H_{\bar\kappa^+})$ *holds at every regular* $\bar\kappa \leq \kappa$*.*

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Lemma 9. *Let $\kappa < \rho$, ρ regular, $\mathrm{V}_\rho \prec \mathrm{V}$, $A = \{\bar{\rho} \mid \bar{\rho} \leq \rho \wedge \mathrm{V}_{\bar{\rho}} \prec \mathrm{V}\}$, and let κ be A-supercompact to $\rho + 1$. Then there is a forcing $\mathbb{P}$ which yields an extension $\mathrm{V}[G]$ such that κ is weakly compact in $\mathrm{V}[G]$ and* $\mathsf{MP}_{<\bar{\kappa}-\mathrm{dir.\ cl.}}(H_{\bar{\kappa}^+})$ *holds at every regular $\bar{\kappa} \leq \kappa$.*

Proof. Let $\mathbb{P}$ be the length $\kappa + 1$ iteration forcing the desired Maximality Principles, and let G be generic. Given a size κ transitive model M in the extension, it shows up in an extension of the form $\mathrm{V}[G \restriction \kappa][\bar{G}]$, where $\bar{G}$ is the restriction of the last coordinate of G to $\mathrm{Col}(\kappa, {<}\bar{\rho})$, for some $\bar{\rho} < \rho$. Now a supercompact embedding $j : \mathrm{V} \longrightarrow P$ lifts to an embedding $j' : \mathrm{V}[G \restriction \kappa][\bar{G}] \longrightarrow P[G][H]$ (H is generic over $\mathrm{V}[G]$); the Silver master condition argument goes through because the size of the forcing on the left is less than the closure of the tail forcing. The ultrafilter derived from the restriction of j' to M is in $\mathrm{V}[G]$, because it has size $\kappa < \rho$ and the tail forcing is ${<}\rho$-closed.

Up to (and including) a measurable

Lemma 10. *Let κ be almost huge up to $\rho+2$ wrt. A, where*

1. $\kappa < \rho \in A = \{\bar\rho \leq \rho \mid \bar\rho$ *is regular and* $\mathrm{V}_{\bar\rho} \prec \mathrm{V}\}$,

2. $\rho = \min(A \setminus (\kappa+1))$.

Then there is a forcing extension of V *in which the following statements hold:*

1. κ *is measurable and*

2. $\mathsf{MP}_{<\bar\kappa-\mathrm{dir.\ cl.}}(H_{\bar\kappa^+})$ *holds at every regular* $\bar\kappa \leq \kappa$.

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Up to (and including) a (partial) supercompact

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Proof. Force with $j(\mathbb{P}_{\kappa+1})$, extend j to an embedding from $\mathrm{V}[G \restriction \lambda]$ to $N[G][H]$, for arbitrarily large $\lambda < j(\kappa)$. H is generic for a tail of $j(j(\mathbb{P})_\lambda)$. Derive a supercompactness measure and argue that it can be found in $\mathrm{V}[G]$ and is a supercompactness measure there.

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I don't know yet how to get a fully supercompact cardinal κ such that the boldface closed maximality principles hold up to and including κ.

Large Cardinals, Woodinized

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I am aiming at producing a model in which the boldface closed maximality principle holds up to a Woodin cardinal. I seem to need strong assumptions, and get the result for a Woodinized supercompact cardinal.

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Proof. Forcing with $\mathbb{P}_A$ does the trick, where A is the set of fully reflecting cardinals below κ.

Up to (and including) a Woodinized supercompact cardinal

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