Scissors Congruence & Hilbert's 3rd Problem

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CSI and The Graduate Center, CUNY

CSI Math Club Talk

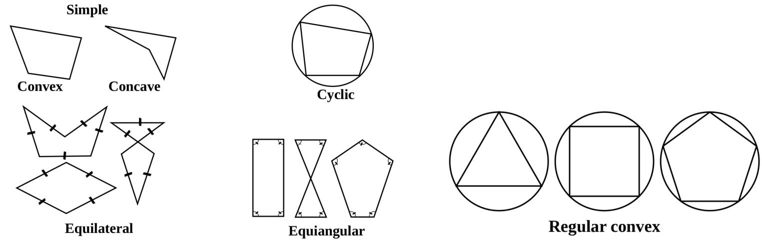
Feb 20, 2020

Euclidean Polygons

A polygon is a plane region bounded by finitely many straight lines, connected to form a polygonal chain.

polygon = polus (many) + gonia (corner)

Example:



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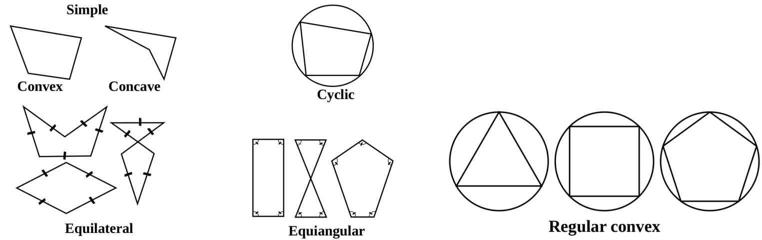

By polygon, we will mean a simple polygon i.e. a polygon that does not intersect itself and has no holes, equivalently, whose boundary is a single closed polygonal path (simple closed curve).

Scissors Congruence

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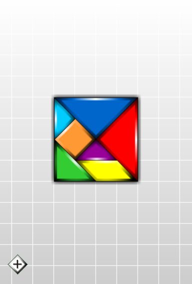
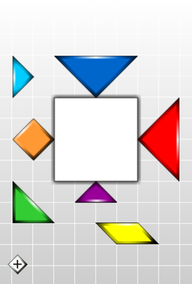
Example: Tangrams





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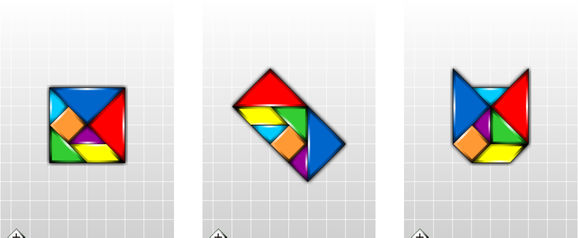
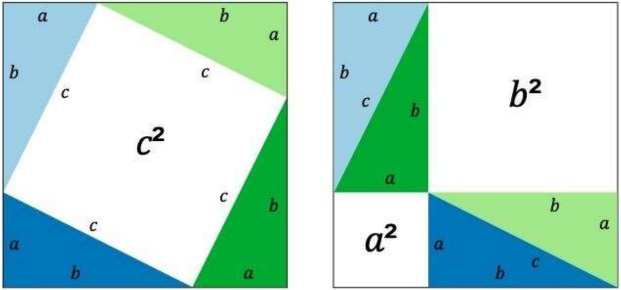
Example: All the polygons below are scissors congruent.



Scissors Congruence

The idea of scissors congruence goes back to Euclid. By "equal area" Euclid really meant scissors congruent (though not using this term and without proof !).

Scissors congruence proofs of the Pythagorean Theorem



$$c^2 = a^2 + b^2$$

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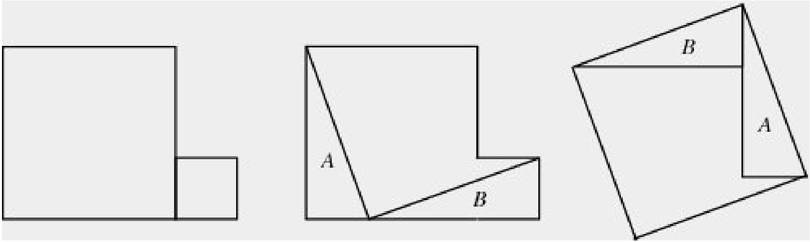

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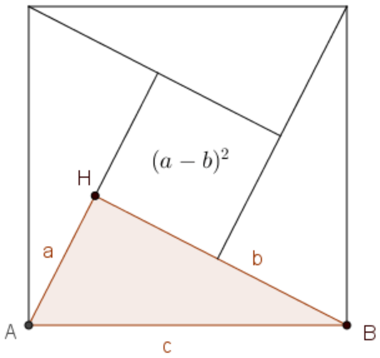

Scissors Congruence is an equivalence relation

$\sim_{sc}$ is an equivalence relation on the set of all polygons in the Euclidean plane.

- (Reflexive) $P \sim_{sc} P$.
- (Symmetric) $P \sim_{sc} Q$ then $Q \sim_{sc} P$.
- (Transitive) $P \sim_{sc} Q$ and $Q \sim_{sc} R$ then $P \sim_{sc} R$.

Transitivity follows by juxtaposing the two decompositions of Q and using the resulting common sub-decomposition of Q to reassemble into P and R, thus showing that $P \sim_{sc} R$.

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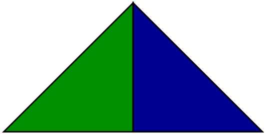
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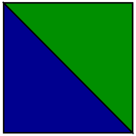
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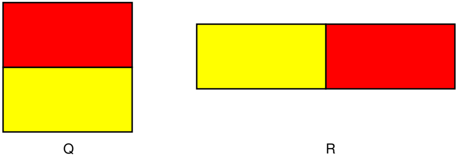

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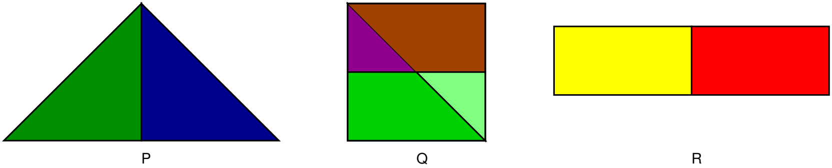

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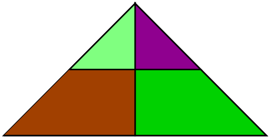

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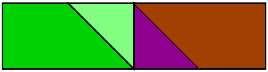

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Theorem (Wallace-Bolyai-Gerwien)

Any two simple polygons of equal area are scissors congruent, i.e. they can be dissected into a finite number of congruent polygonal pieces.

Some history

Wallace-Bolyai-Gerwien Theorem

Any two simple polygons of equal area are equidecomposable.

This apparently simple statement is of relatively recent origins. It's been associated with the names of Lowry, W. Wallace, Farkas Bolyai, and P. Gerwien and usually goes by the name of *Wallace-Bolyai-Gerwien Theorem*. The accounts, however, differ. According to *Greg Frederickson*, Lowry (1814) provided a simple explanation in answer to a problem posed by Wallace around 1808. (Wallace presumably had a solution at that time, which he gave in expanded form in 1831.) Frederickson acknowledges the methods of Bolyai (1832) and Gerwien (1833).

According to *Ian Stewart*, the statement is usually called Bolyai-Gerwien Theorem, because Wolfgang Bolyai raised the question, and P. Gerwien answered it in 1833. However, Stewart adds, William Wallace got there earlier: he gave a proof in 1807.

According to *Andreescu and Gelca*, the property was proved independently by F. Bolyai (1833) and Gerwien (1835). A Russian math encyclopedia concurs.

(It should be mentioned that the Bolyai in question was a noted Hungarian mathematician Wolfgang Farkas Bolyai (1775-1856) and, sometimes *Farkas Wolfgang Bolyai*, father of Janos Bolyai, the co-inventor of the *non-Euclidean geometry*, and a dear friend of Johann Carl Friedrich Gauss. Like the *birthplace of I. Kant*, the birthplace of F. Bolyai has also changed hands. It is now a part of Romania.)

Source: `https://www.cut-the-knot.org/do_you_know/Bolyai.shtml`

Sketch of Proof

Step 1: Triangulate the polygon.

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Step 4: Finish proof of Theorem.

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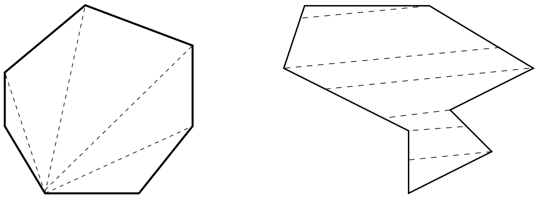

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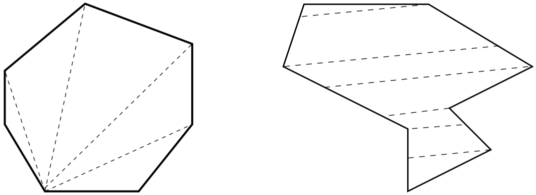


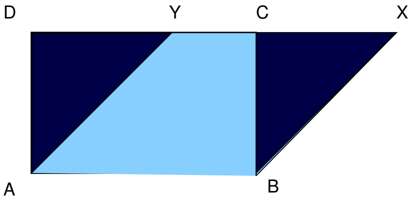
For a polygon, choose a line of slope m which is distint from the slopes of all its sides. Lines of slope m through the vertices of the polygon decompose it into triangles and trapezoids, which again can be decomposed into (acute) angled triangles. ∎

Proof

Step 2: Scissors congruence for parallelograms and triangles of same base and equal height.

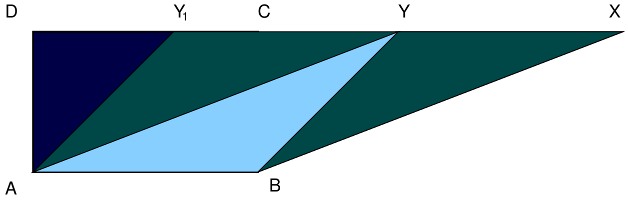
Proof: Let $ABCD$ be a rectangle with base AB and height AD. Let $ABXY$ be a parallelogram with height AD. Assume $|DY| \leq |DC|$. Then

$$ABCD \sim_{sc} AYD + ABCY \sim_{sc} ABCY + BXC \sim_{sc} ABXY.$$



Proof

If $|DY| > |DC|$, then cutting along the diagonal BY and regluing the triangle BXY, we obtain the scissors congruent parallelogram $ABYY_1$ such that $|DY_1| = |DY| - |DC|$. Continuing this process k times, for $k = [|DY|/|DC|]$, we obtain the parallelogram $ABY_{k-1}Y_k$ such that $|DY_k| < |DC|$, which is scissors congruent to $ABCD$ as above.



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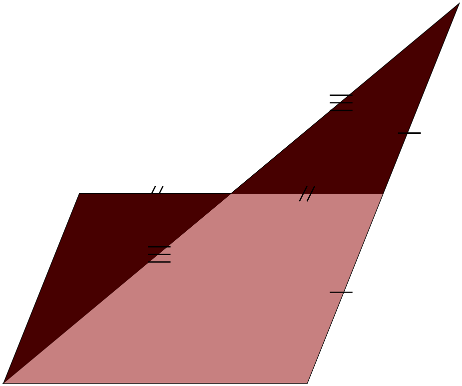
Since any triangle is scissors congruent to a parallelogram with the same base and half height, this implies that any two triangles with same base and height are scissors congruent. ■.



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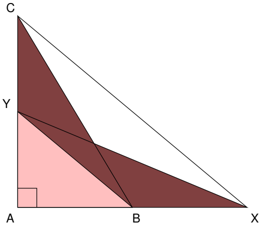
$$\text{Let Area}(ABC) = \text{Area}(AXY)$$
$$\implies \frac{|AB||AC|}{2} = \frac{|AY||AX|}{2}$$
$$\implies \frac{|AY|}{|AC|} = \frac{|AB|}{|AX|}$$
$$\implies ABY \sim AXC \text{ SAS test}$$



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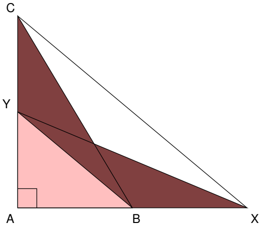

This implies BY is parallel to XC. Hence triangles BYC and BYX have same base and same height which implies by Step 2 that they are scissors congruent i.e.

$ABC \sim_{sc} ABY + BYC \sim_{sc} ABY + BYX \sim_{sc} AXY$. ■

Proof

Step 4: Putting it all together.

Any triangle T is scissors congruent to a right triangle with height 2 and base equal to the area of T, which is scissors congruent to a rectangle with **unit height** and base equal to area of T. Lets denote such a rectangle by R_x where x is its area ($=$ base).

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Thus for any polygon P ,

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P &\sim_{sc} T_1 + \ldots + T_n \text{ by Step 1} \\
&\sim_{sc} R_{\text{Area}(T_1)} + \ldots + R_{\text{Area}(T_n)} \text{ by Step 3} \\
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Hence, if polygons P, Q have equal area then
$P \sim_{sc} R_{\text{Area}(P)} = R_{\text{Area}(Q)} \sim_{sc} Q$. ■

Visualizing scissors congruence

Visualization application by Satyan L. Devadoss, Ziv Epstein, and Dmitriy Smirnov is implemented in HTML5 and JavaScript.

The interface allows the user to input her own intial and terminal polygons. It then rescales the polygons so that they are of the same area, by calculating the optimal scaling factor for each polygon such that the following two constraints are satisfied: both polygons are of equal area, and the wider of the two is not too wide that is goes off the screen.

`http://dmsm.github.io/scissors-congruence/.`

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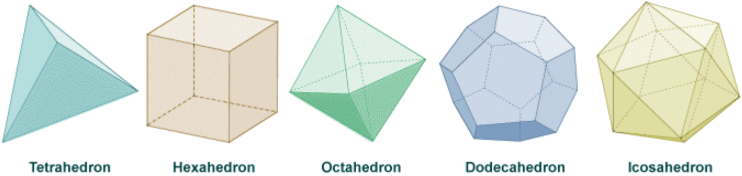

The 5 regular polyhedra called the Platonic solids

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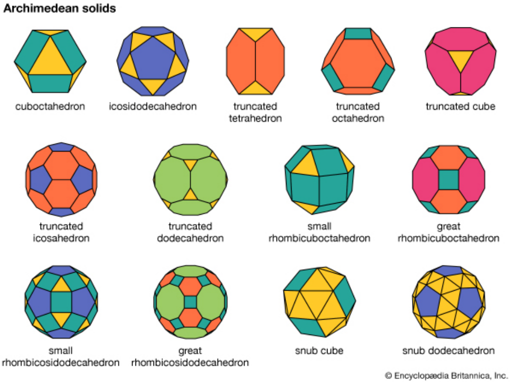

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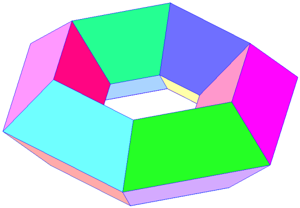

We will not allow solids whose boundary in not a sphere (i.e. $\mathbb{S}^2$)

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As before, transitivity follows by juxtaposing the two decompositions of Q and using the resulting common sub-decomposition of Q to reassemble into P and R, thus showing that $P \sim_{sc} R$. This is harder to visualize or draw.

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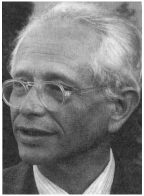
Hilbert made clear that he expected a negative answer.

Solution to Hilbert's Third Problem



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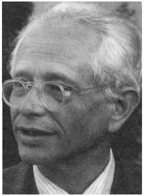

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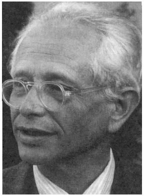

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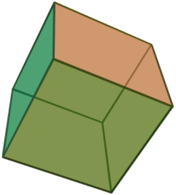
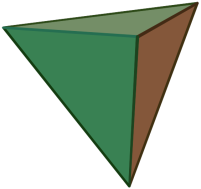
Dehn showed that the regular tetrahedron and the cube of the same volume were not scissors congruent.



$\not\sim_{sc}$



Dehn's solution

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The $\otimes$ symbol is called **tensor product** and implies that $\delta(P)$ does not change when you cut along an edge or cut along an angle i.e. $\delta(P)$ in an invariant of scissors congruence.

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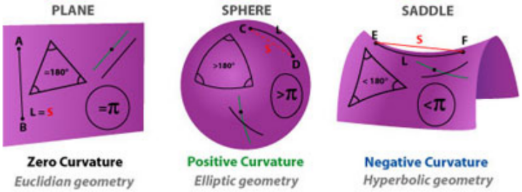
- $\delta(\text{unit cube}) = 0 \neq 6 \times a \otimes \arccos(\frac{1}{3}) = \delta(\text{tetrahedra})$
- Thus the unit cube and the unit tetrahedra are not scissors congruent !

Scissors congruence in other geometries and higher dimensions

The possible 2-dimensional geometries are Euclidean, spherical and hyperbolic.



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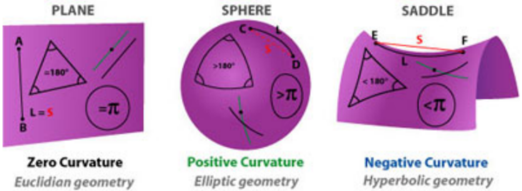

It is known that area determines scissors congruence in 2-dimensional spherical geometry $\mathbb{S}^2$ and hyperbolic geometry $\mathbb{H}^2$.

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- Walter Neumann and Jun Yang (1999) used a "complexified" Dehn invariant in $\mathbb{H}^3$ to define invariants of hyperbolic 3-manifolds.

Video

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`https://www.youtube.com/watch?v=ysV6iF3Rmjo`