Euler's Polyhedral Formula

Abhijit Champanerkar

College of Staten Island, CUNY

Talk at William Paterson University

Euler

Leonhard Euler (1707-1783)

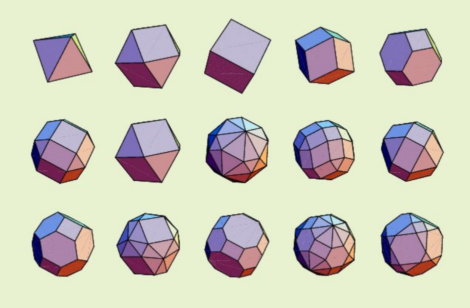
Leonhard Euler was a Swiss mathematician who made enormous contibutions to a wide range of fields in mathematics.

Euler: Some contributions

- Euler introduced and popularized several notational conventions through his numerous textbooks, in particular the concept and notation for a function.
- In analysis, Euler developed the idea of power series, in particular for the exponential function e^x. The notation e made its first appearance in a letter Euler wrote to Goldbach.
- For complex numbers he discovered the formula $e^{i\theta} = \cos\theta + i\sin\theta$ and the famous identity $e^{i\pi} + 1 = 0$.
- In 1736, Euler solved the problem known as the Seven Bridges of Königsberg and proved the first theorem in Graph Theory.
- Euler proved numerous theorems in Number theory, in particular he proved that the sum of the reciprocals of the primes diverges.

Convex Polyhedron

A polyhedron is a solid in $\mathbb{R}^3$ whose faces are polygons.



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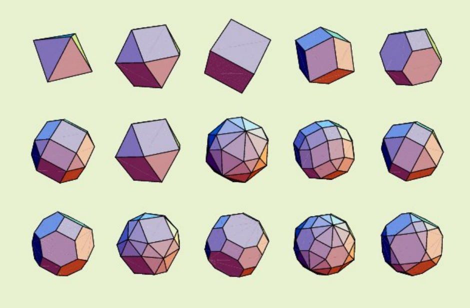
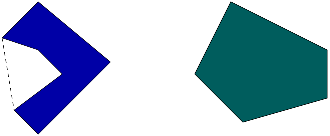


A polyhedron P is convex if the line segment joining any two points in P is entirely contained in P.



Euler's Polyhedral Formula

Euler's Formula

Let P be a convex polyhedron. Let v be the number of vertices, e be the number of edges and f be the number of faces of P. Then $v - e + f = 2$.

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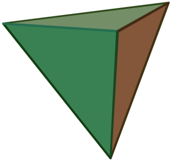
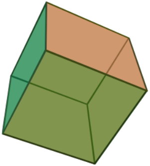
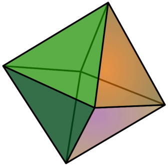


Tetrahedron



Cube



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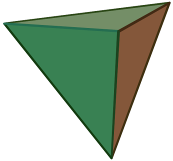
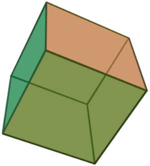


Tetrahedron

$v = 4, \ e = 6, \ f = 4$



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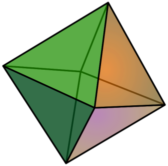

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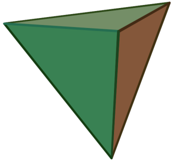

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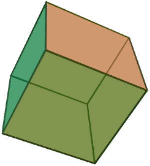

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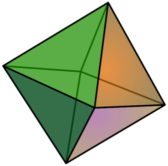
$v = 8, \ e = 12, \ f = 6$



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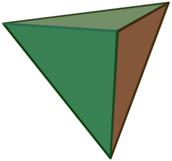

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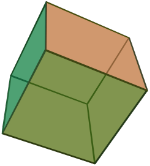

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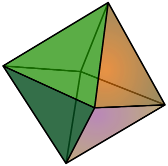

Octahedron
$v = 6, \; e = 12, \; f = 8$

Euler's Polyhedral Formula

Euler mentioned his result in a letter to Goldbach (of Goldbach's Conjecture fame) in 1750. However Euler did not give the first correct proof of his formula.

It appears to have been the French mathematician Adrian Marie Legendre (1752-1833) who gave the first proof using Spherical Geometry.

Adrien-Marie Legendre (1752-1833)

Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.

Parallel Postulate

Parallel Postulate: If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

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Failure of Parallel Postulate gives rise to non-Euclidean geometries.

No lines $\longrightarrow$ Spherical geometry (positively curved)
Infinitely many lines $\longrightarrow$ Hyperbolic geometry (negatively curved)

Spherical geometry

Let S^2 denote the unit sphere in $\mathbb{R}^3$ i.e. the set of all unit vectors i.e. the set $\{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = 1 \ \}$.

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A great circle in S^2 is a circle which divides the sphere in half. In other words, a great circle is the interesection of S^2 with a plane passing through the origin.

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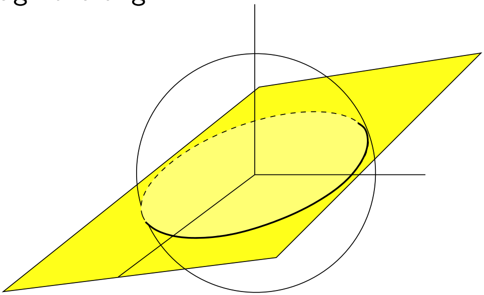

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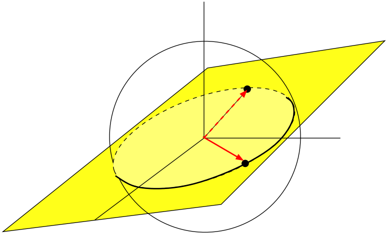

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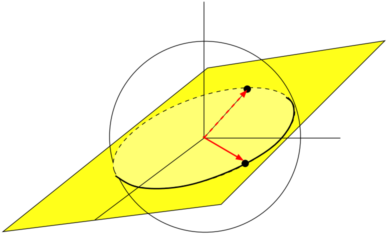
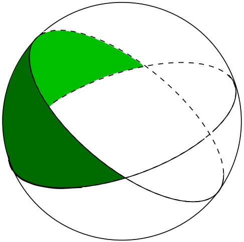


You can similarly verify the other three Euclid's posulates for geometry.

Diangles

Any two distinct great circles intersect in two points which are negatives of each other.



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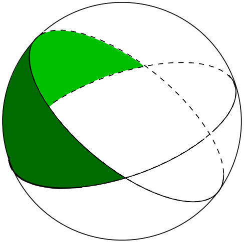

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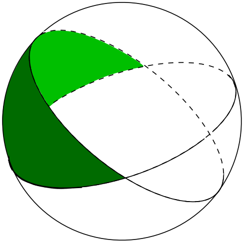

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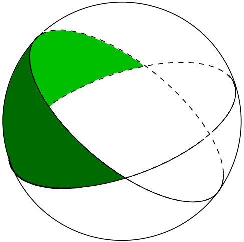

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The angle at both the vertices are equal. Both diangles bounded by two great circles are congruent to each other.

Area of a diangle

Proposition

Let θ be the angle of a diangle. Then the area of the diangle is 2θ.

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Alternatively, one can compute this area directly as the area of a surface of revolution of the curve $z = \sqrt{1 - y^2}$ by an angle θ. This area is given by the integral $\int_{-1}^{1} \theta z \sqrt{1 + (z')^2}\, dy$. ∎

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If the radius of the sphere is r then the area of the diangle is $2\theta r^2$.

This is very similar to the formula for the length of an arc of the unit circle which subtends an angle θ is θ.

Spherical polygons

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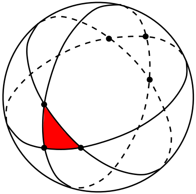
More Examples. Take ballon, ball and draw on it.



Spherical Triangle

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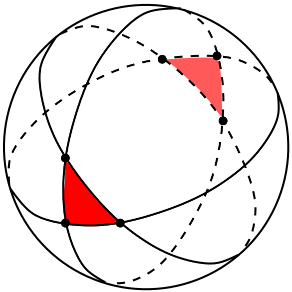

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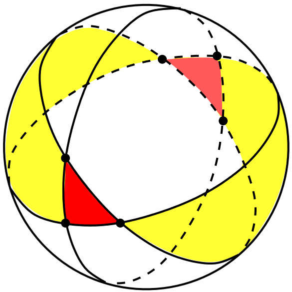

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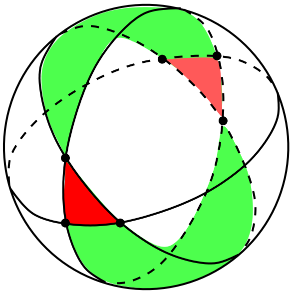

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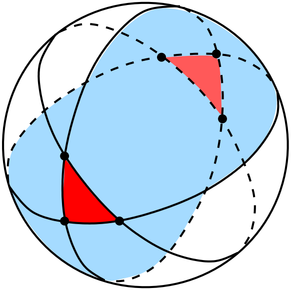

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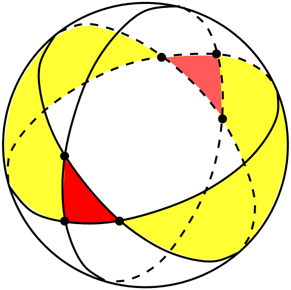

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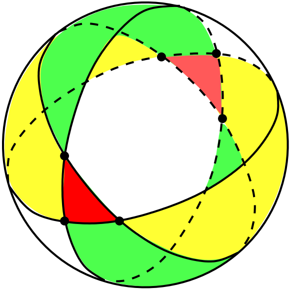

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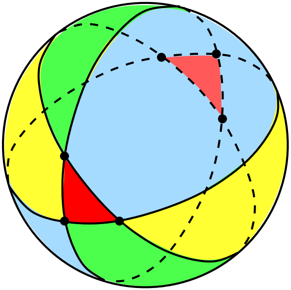

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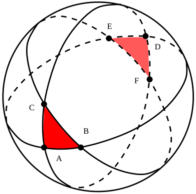
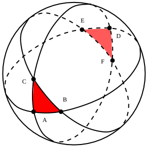


$\triangle ABC$ as shown above is formed by the intersection of three great circles.

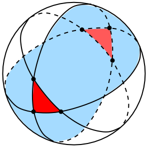
Vertices A and D are antipodal to each other and hence have the same angle. Similarly for vertices B, E and C, F. Hence the triangles $\triangle ABC$ and $\triangle DEF$ are antipodal (opposite) triangles and have the same area.

Assume angles at vertices A, B and C to be α, β and γ respectively.

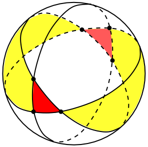
Area of a spherical triangle



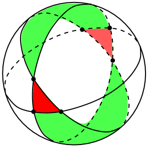
$\triangle ABC$



R_{AD}



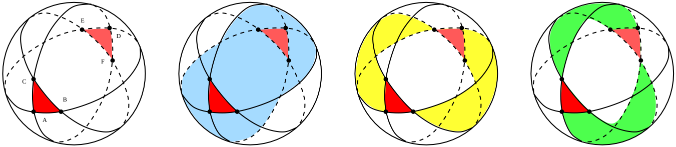
R_{BE}



R_{CF}

Let R_{AD}, R_{BE} and R_{CF} denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.

Area of a spherical triangle

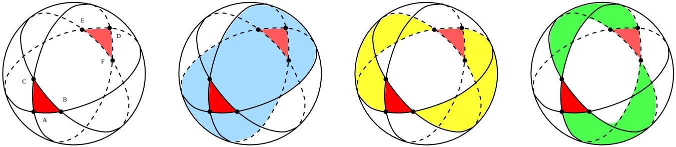


$\triangle ABC$ $\qquad$ R_{AD} $\qquad$ R_{BE} $\qquad$ R_{CF}

Let R_{AD}, R_{BE} and R_{CF} denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.

$R_{AD} \cup R_{BE} \cup R_{CF} = S^2$, $\mathrm{Area}(\triangle ABC) = \mathrm{Area}(\triangle DEF) = X$.

Area of a spherical triangle



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$R_{AD} \cup R_{BE} \cup R_{CF} = S^2$, $\mathrm{Area}(\triangle ABC) = \mathrm{Area}(\triangle DEF) = X$.

$$\begin{aligned}
\mathrm{Area}(S^2) &= \mathrm{Area}(R_{AD}) + \mathrm{Area}(R_{BE}) + \mathrm{Area}(R_{CF}) - 4X \\
4\pi &= 4\alpha + 4\beta + 4\gamma - 4X \\
X &= \alpha + \beta + \gamma - \pi
\end{aligned}$$

Area of a spherical polygon

Corollary

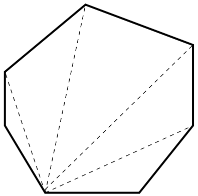
Let R be a spherical polygon with n vertices and n sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\mathrm{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

Area of a spherical polygon

Corollary

Let R be a spherical polygon with n vertices and n sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\mathrm{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

Proof: Any polygon with n sides for $n \geq 4$ can be divided into $n-2$ triangles.



The result follows as the angles of these triangles add up to the interior angles of the polygon. ■

Proof of Euler's Polyhedral Formula

Let P be a convex polyhedron in $\mathbb{R}^3$. We can "blow air" to make (boundary of) P spherical.

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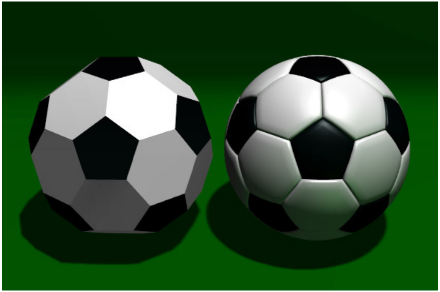

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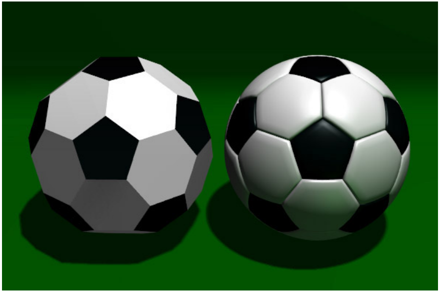

This can be done rigourously by arranging P so that the origin lies in the interior of P and projecting the boundary of P on S^2 using the function $f(x, y, z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$.

It is easy to check that vertices of P go to points on S^2, edges go to parts of great circles and faces go to spherical polygons.

Proof of Euler's Polyhedral Formula

Let v, e and f denote the number of vertices, edges and faces of P respectively. Let $R_1, \ldots, R_f$ be the spherical polygons on S^2.

Since their union is S^2, $\mathrm{Area}(R_1) + \ldots + \mathrm{Area}(R_f) = \mathrm{Area}(S^2)$.

Let n_i be the number of edges of R_i and α_{ij} for $j = 1, \ldots, n_i$ be its interior angles.

$$\begin{aligned}\sum_{i=1}^{f}(\sum_{j=1}^{n_i}\alpha_{ij} - n_i\pi + 2\pi) &= 4\pi \\ \sum_{i=1}^{f}\sum_{j=1}^{n_i}\alpha_{ij} - \sum_{i=1}^{f} n_i\pi + \sum_{i=1}^{f} 2\pi &= 4\pi\end{aligned}$$

Proof of Euler's Polyhedral Formula

Since every edge is shared by two polygons

$$\sum_{i=1}^{f} n_i \pi = 2\pi e.$$

Since the sum of angles at every vertex is 2π

$$\sum_{i=1}^{f} \sum_{j=1}^{n_i} \alpha_{ij} = 2\pi v.$$

Hence $2\pi v - 2\pi e + 2\pi f = 4\pi$ that is $v - e + f = 2$ $\blacksquare$

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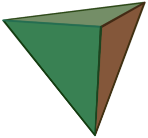	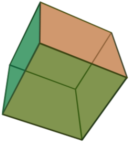	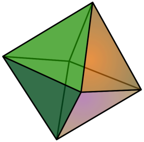	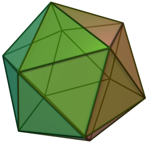	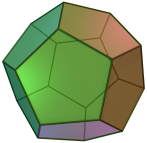
Tetrahedron	Cube	Octahedron	Icosahedron	Dodecahedron

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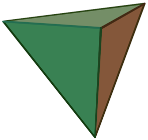	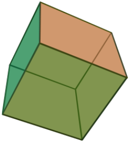	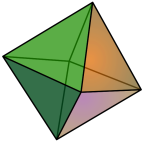	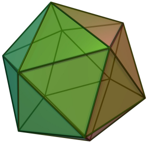	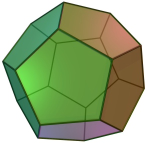
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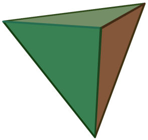	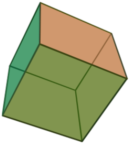	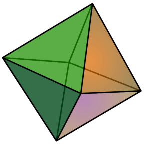	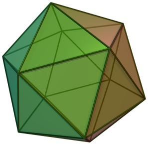	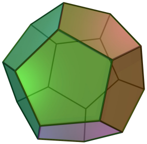
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$$\begin{aligned} \frac{2e}{a} - e + \frac{2e}{b} &= 2 \\ \frac{1}{a} + \frac{1}{b} &= \frac{1}{2} + \frac{1}{e} > \frac{1}{2} \end{aligned}$$

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If $a \geq 6$ or $b \geq 6$ then $\frac{1}{a} + \frac{1}{b} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Hence $a < 6$ and $b < 6$ which gives us finitely many cases to check.

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Plane graphs

Note that we actually proved the Theorem for any (geodesic) graph on the sphere.

Any plane graph can be made into a graph on a sphere by tying up the unbounded face (like a balloon). However one may need to make some modifications (which do not change the count $v - e + f$) to make the graph geodesic on the sphere (keywords: k-connected for $k = 1, 2, 3$).

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Theorem

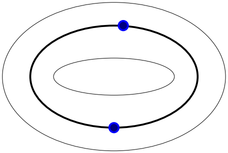
If G is a connected plane graph with v vertices, e edges and f faces (including the unbounded face), then $v - e + f = 2$.

This theorem from graph theory can be proved directly by induction on the number of edges and gives another proof of Euler's Theorem !

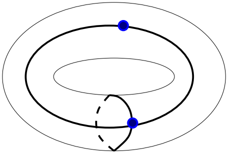
Surfaces

What about graphs on other surfaces ?

Other surfaces



2−2+1=1



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We need the restriction that every face of the graph on the surface is a disk.

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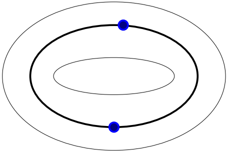

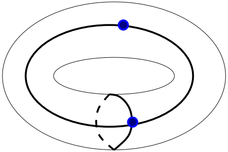

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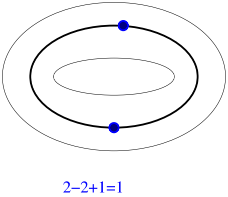

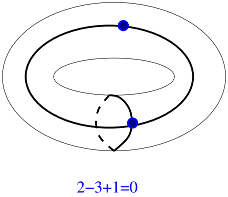

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The number $\chi = v - e + f$ is called the Euler characteristic of the surface. $\chi = 2 - 2g$ where g is the genus of the surface i.e. the number of holes in the surface.

Thank You

Slides available from:
`http://www.math.csi.cuny.edu/abhijit/talks.html`