Derivatives of self-intersection local times

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Abstract

We show that the renormalized self-intersection local time $\gamma_t(x)$ for the Brownian motion and symmetric stable process in R^1 is differentiable in the spatial variable and that $\gamma_t'(0)$ can be characterized as the continuous process of zero quadratic variation in the decomposition of a natural Dirichlet process. This Dirichlet process is the potential of a random Schwartz distribution. Analogous results for fractional derivatives of self-intersection local times in R^1 and R^2 are also discussed.

1   Introduction

In their study of the intrinsic Brownian local time sheet and stochastic area integrals for Brownian motion, [13, 14, 15], Rogers and Walsh were led to analyze the functional

$$(1.1) \qquad A(t, B_t) = \int_0^t 1_{[0,\infty)}(B_t - B_s)\, ds$$

where B_t is a 1-dimensional Brownian motion. They showed that $A(t, B_t)$ is not a semimartingale, and in fact show that

$$(1.2) \qquad A(t, B_t) - \int_0^t L_s^{B_s}\, dB_s$$

has finite non-zero 4/3-variation. Here L_s^x is the local time at x, which is formally $L_s^x = \int_0^s \delta(B_r - x)\, dr$, where $\delta(x)$ is Dirac's 'δ-function'. A formal

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application of Ito's lemma, using $\frac{d}{dx}1_{[0,\infty)}(x) = \delta(x)$ and $\frac{d^2}{dx^2}1_{[0,\infty)}(x) = \delta'(x)$, yields

$$(1.3) \qquad A(t, B_t) - \int_0^t L_s^{B_s} \, dB_s = t + \frac{1}{2} \int_0^t \int_0^s \delta'(B_s - B_r) \, dr \, ds$$

which motivates the subject matter of this paper. We study the process which is formally defined as

$$(1.4) \qquad \gamma_t' = -\int_0^t \int_0^s \delta'(X_s - X_r) \, dr \, ds$$

where δ' is the derivative of the delta-function, and X_t is Brownian motion or, more generally, a symmetric stable process in R^1. The process γ_t' is related to the self-intersection local time process which is formally defined as

$$(1.5) \qquad \alpha_t = \int_0^t \int_0^s \delta(X_s - X_r) \, dr \, ds.$$

If we set

$$(1.6) \qquad \alpha_t(y) = \int_0^t \int_0^s \delta(X_s - X_r - y) \, dr \, ds$$

we obtain a 'near intersection' local time, and formally differentiating in x suggests that (1.4) is the derivative $\frac{d}{dy}\alpha_t(y)|_{y=0}$.

α_t has not been studied much in one dimension since it can be expressed in terms of the local time L_t^y of the process X_t: $\alpha_t = \frac{1}{2}\int (L_t^y)^2 \, dy$ and $\alpha_t(x) = \frac{1}{2}\int L_t^{y+x}L_t^y \, dy$. The fact that L_t^y is not differentiable in the spatial variable y indicates that the existence of (1.4) and its identification as a derivative requires some care.

In two dimensions, even for Brownian motion, α_t does not exist and must be 'renormalized' by subtracting off a counterterm. This was first done by Varadhan [21], and has been the subject of a large literature, see Dynkin [4], Le Gall [11], Bass and Khoshnevisan [1], Rosen [19, 20]. The resulting renormalized self-intersection local time turns out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener and stable sausage in the plane and fluctuations of the range of stable random walks. (See Le Gall [10, 9], Le Gall-Rosen [12] and Rosen [18]).

γ_t' in R^1, in a certain sense, is even more singular than self-intersection local time in R^2, but as we shall see, due to the symmetry properties of δ', there is no need for a counterterm. We begin with a precise definition of γ_t', show

that it exists, is the spatial derivative of the renormalized self-intersection local time $\gamma_t(x) = \alpha_t(x) - E\left(\alpha_t(x)\right)$, and has zero quadratic variation. We then show how it can be characterized as the continuous process of zero quadratic variation in the decomposition of a natural Dirichlet process. This Dirichlet process is the potential of a random Schwartz distribution.

Let

$$(1.7) \qquad \alpha_{t,\epsilon}(y) \stackrel{def}{=} \int_0^t \int_0^s f_\epsilon(X_s - X_r - y)\, dr\, ds$$

and

$$(1.8) \qquad \alpha'_{t,\epsilon}(y) \stackrel{def}{=} -\int_0^t \int_0^s f'_\epsilon(X_s - X_r - y)\, dr\, ds$$

where f_ϵ is an approximate $\delta-$function at zero, i.e. $f_\epsilon(x) = f(x/\epsilon)/\epsilon$ with f a positive, C^1, even function of x supported in the unit interval with $\int f\, dx = 1$. We then define

$$(1.9) \qquad \alpha_t(y) = \lim_{\epsilon \to 0} \alpha_{t,\epsilon}(y), \qquad \alpha'_t(y) = \lim_{\epsilon \to 0} \alpha'_{t,\epsilon}(y)$$

$$(1.10) \qquad \gamma_t(y) = \lim_{\epsilon \to 0} \alpha_{t,\epsilon}(y) - E\left(\alpha_{t,\epsilon}(y)\right)$$

and

$$(1.11) \qquad \gamma'_t(y) = \lim_{\epsilon \to 0} \alpha'_{t,\epsilon}(y) - E\left(\alpha'_{t,\epsilon}(y)\right)$$

whenever the limit exists. We set $\gamma'_t = \gamma'_t(0)$. Let

$$(1.12) \qquad h_\beta(x) = \begin{cases} c(\beta)\mathrm{sgn}(x)x^{\beta-2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where $c(\beta) = \pi^{-1}\Gamma(2-\beta)\cos((1-\beta)\pi/2)$ if $1 < \beta < 2$ and $c(2) = 1$. Note that $h_\beta(x)$ is not continuous at $x = 0$.

Theorem 1 *Let X_t denote the symmetric stable process of order $\beta > 3/2$ in R^1. Then $\alpha_{t,\epsilon}(y)$ and $\alpha'_{t,\epsilon}(y)$ converge a.s. and in all L^p spaces as $\epsilon \to 0$ for any $(t, y) \in R_+ \times R^1$.*

The following hold almost surely:

1. For any continuous function $g(x)$ we have

$$(1.13) \qquad \int_0^t \int_0^s g(X_s - X_r)\, dr\, ds = \int g(y)\alpha_t(y)\, dy.$$

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and if $g \in C^1$

$$(1.14) \qquad \int_0^t \int_0^s g'(X_s - X_r) \, dr \, ds = - \int g(y) \alpha_t'(y) \, dy.$$

2. *$\alpha_t'(y) - h_\beta(y)t$ is continuous in y.*

3. *$\{\gamma_t(y), (t,y) \in R_+ \times R^1\}$ and $\{\gamma_t'(y), (t,y) \in R_+ \times R^1\}$ are continuous and $\gamma_t(y)$ is differentiable in y with $\gamma_t'(y) = \frac{d}{dy}\gamma_t(y)$.*

4. *$\alpha_t'(0) = \gamma_t'(0)$.*

We see from (1.13) and (1.14) that $\alpha_t'(y)$ is the distributional derivative of $\alpha_t(y)$. However, we see from 2. that $\alpha_t'(y)$ is not continuous at $y = 0$ so that these equations do not allow us to characterize $\alpha_t'(0)$.

For any function g_t and any sequence $\tau = \{\tau_n\}$ of partitions $\tau_n = \{0 = t_0 < t_{n,1} < \cdots < t_{n,n} = T\}$ of $[0, T]$, with mesh size $|\tau_n| = \max_i |t_{n,i} - t_{n,i-1}|$ going to 0, we set

$$(1.15) \qquad V_p(g; \tau) = \lim_{n \to \infty} \sum_{i=1}^n |g_{t_{n,i}} - g_{t_{n,i-1}}|^p.$$

In [15], Rogers and Walsh show that for Brownian motion $V_{4/3}(\gamma'; \tau)$ is a finite non-zero constant, independent of τ. Let β' denote the usual conjugate exponent to β, i.e. $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.

Theorem 2 *Let X be a symmetric stable process of order $\beta > 3/2$ in R^1. Then $V_p(\gamma'; \tau) = 0$ for any τ and any $p > \frac{2\beta'}{3}$.*

Note that for Brownian motion this shows that $V_p(\gamma'; \tau) = 0$ for any $p > 4/3$. We conjecture that for the symmetric stable process of order $\beta > 3/2$ in R^1 we have that $V_{\frac{2\beta'}{3}}(\gamma'; \tau)$ is a finite non-zero constant, independent of τ.

We now obtain an intrinsic characterization for γ_t', which doesn't involve limits. The key idea is that γ_t' has zero quadratic variation. In the case of renormalized intersection local time γ_t for Brownian motion in the plane this was observed by Bertoin, [2] and extended by us in [20].

Recall that a continuous adapted process Z_t is said to have zero quadratic variation, if for each $T > 0$ and any sequence of partitions $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$, with mesh size $|\tau_n| = \max_i |t_i - t_{i-1}|$ going to 0

$$(1.16) \qquad \lim_{n \to \infty} E\left(\sum_{t_i \in \tau_n} (Z_{t_i} - Z_{t_{i-1}})^2\right) = 0.$$

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Föllmer, [7] has coined the term "Dirichlet process" to refer to any process which can be written as the sum of a martingale and a process of zero quadratic variation. It is important to note that such a decomposition is unique. The class of Dirichlet processes is much wider than the class of semimartingales.

We use Y_t to denote our stable process X_t killed at an independent exponential time λ. In the following theorem γ_t' will be defined for the process Y_t in place of X_t.

Let us begin with a special case of the Doob-Meyer decomposition for semimartingales. Let L_t^μ denotes the continuous additive functional of X_t with Revuz measure μ. Using the additivity of L_t^μ and the Markov property we have

$$(1.17) \qquad E^x(L_\lambda^\mu \,|\, \mathcal{F}_t) = L_{t\wedge\lambda}^\mu + U^1\mu(Y_t)$$

where $\mathcal{F}_t = \sigma(Y_s\,,\ s \leq t)$. Equivalently, $U^1\mu(Y_t) = M_t - L_{t\wedge\lambda}^\mu$ where $M_t = E^x(L_\lambda^\mu \,|\, \mathcal{F}_t)$ is a martingale. This is the Doob-Meyer decomposition for the potential $U^1\mu(Y_t)$. We will show that γ_t' arises in a similar decomposition for the potential of a random Schwartz distribution. This new potential will no longer be a semimartingale but a Dirichlet process, and γ_t' will correspond to the process of zero quadratic variation in the decomposition of this Dirichlet process.

Let F_t be the random additive distribution-valued process defined by

$$(1.18) \qquad F_t(g) = -\lim_{\epsilon \to 0} \int_0^t \int g(y + Y_r) f_\epsilon'(y)\, dy\, dr$$

whenever the limit exists. It is easy to check that for all $g \in C_0^\infty(R^1)$

$$(1.19) \qquad F_t(g) = \int_0^t g'(Y_s)\, ds.$$

With this definition it is natural to set

$$(1.20) \qquad U^1 F_t(x) = -\lim_{\epsilon \to 0} \int_0^t \int u^1(x - (y + Y_r)) f_\epsilon'(y)\, dy\, dr$$

whenever the limit exists.

Note that formally

$$(1.21) \qquad U^1 F_t(x) = -\int_0^t \frac{du^1}{dx}(x - Y_s)\, ds.$$

We have the following analogue of the Doob-Meyer decomposition.

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Theorem 3 *Let Y be a symmetric stable process of order $\beta > 3/2$ in R^1, killed at an independent exponential time λ. Then γ_t' is continuous a.s. with zero quadratic variation and*

$$(1.22) \qquad\qquad U^1 F_t(Y_t) = M_t - \gamma_t'$$

where M_t is the martingale $E^x(\gamma_\infty' \,|\, \mathcal{F}_t)$.

In view of Theorem 3 we can characterize the renormalized intersection local time γ_t' as the continuous process of zero quadratic variation in the decomposition of the random potential $U^1 F_t(Y_t)$ which is formally $-\int_0^t \frac{du^1}{dx}(Y_t - Y_s)\,ds$.

1.1   fractional derivatives

All our results can be extended to fractional derivatives. There are in fact several natural candidates for the fractional derivative of order $0 < \rho < 1$ in R^d: $g^{(\rho)}(x) = (2\pi)^{-d} \int e^{ipx} w(p)\hat{g}(p)\,dp$ with $w(p)$ positively homogeneous of index ρ. Our results can be extended for any such $w(p)$, but for simplicity we work with symmetric fractional derivatives: $w(-p) = -w(p)$ which allows us to avoid introducing counterterms. In one dimension this determines $w(p)$ up to a constant factor: $w(p) = \mathrm{sgn}(p)|p|^\rho$. Then we study

$$(1.23) \qquad\qquad \gamma_t^{(\rho)} = -\int_0^t \int_0^s \delta^{(\rho)}(X_s - X_r)\,dr\,ds.$$

More precisely, let

$$(1.24) \qquad\qquad \alpha_{t,\epsilon}^{(\rho)}(y) \overset{def}{=} -\int_0^t \int_0^s f_\epsilon^{(\rho)}(X_s - X_r - y)\,dr\,ds,$$

$$(1.25) \qquad\qquad \alpha_t^{(\rho)}(y) = \lim_{\epsilon \to 0} \alpha_{t,\epsilon}^{(\rho)}(y)$$

and
$$(1.26) \qquad\qquad \gamma_t^{(\rho)}(y) = \lim_{\epsilon \to 0} \alpha_{t,\epsilon}^{(\rho)}(y) - E\left(\alpha_{t,\epsilon}^{(\rho)}(y)\right)$$

whenever the limit exists. We set $\gamma_t^{(\rho)} = \gamma_t^{(\rho)}(0)$.

Theorem 4 *Let X_t denote the symmetric stable process of order $\beta > 1 \vee (\rho + 1/2)$ in R^1. Then $\alpha_{t,\epsilon}^{(\rho)}(y)$ converges a.s. and in all L^p spaces as $\epsilon \to 0$ for any $(t, y) \in R_+ \times R^1$.*
    The following hold almost surely:

1. For any C^ρ function $g(x)$ we have

$$(1.27) \qquad \int_0^t \int_0^s g^{(\rho)}(X_s - X_r)\, dr\, ds = -\int g(y)\alpha_t^{(\rho)}(y)\, dy.$$

2. $\alpha_t^{(\rho)}(y) - h_{\beta+1-\rho}(y)t$ is continuous in y.

3. $\{\gamma_t^{(\rho)}(y),\ (t,y) \in R_+ \times R^1\}$ is a.s. continuous and $\gamma_t^{(\rho)}(y)$ is the derivative of order ρ in y of $\gamma_t(y)$.

4. $\alpha_t^{(\rho)}(0) = \gamma_t^{(\rho)}(0)$.

5. $V_p(\gamma^{(\rho)}; \tau) = 0$ for any τ and any $p > \frac{2\beta}{3\beta - 1 - 2\rho}$.

Once again, we use Y_t to denote our R^1 valued Lévy process X_t killed at an independent exponential time λ and in the following theorem $\gamma_t^{(\rho)}$ will be defined for the process Y_t in place of X_t. Let now Φ_t be the random additive distribution-valued process defined by

$$(1.28) \qquad \Phi_t(g) = -\lim_{\epsilon \to 0} \int_0^t \int g(y + Y_r) f_\epsilon^{(\rho)}(y)\, dy\, dr$$

whenever the limit exists. It is easy to check that for all $g \in C_0^\infty(R^1)$

$$(1.29) \qquad \Phi_t(g) = \int_0^t g^{(\rho)}(Y_s)\, ds.$$

With this definition it is natural to set

$$(1.30) \qquad U^1\Phi_t(x) = -\lim_{\epsilon \to 0} \int_0^t \int u^1(x - (y + Y_r)) f_\epsilon^{(\rho)}(y)\, dy\, dr$$

whenever the limit exists.

Note that formally

$$(1.31) \qquad U^1\Phi_t(x) = -\int_0^t (u^1)^{(\rho)}(x - Y_s)\, ds.$$

We have the following analogue of the Doob-Meyer decomposition.

Theorem 5 *Let Y be a symmetric stable process of order $\beta > \rho + 1/2$ in R^1, killed at an independent exponential time λ. Then $\gamma_t^{(\rho)}$ is continuous a.s. with zero quadratic variation and*

$$(1.32) \qquad U^1\Phi_t(Y_t) = M_t - \gamma_t^{(\rho)}$$

where M_t is the martingale $E^x(\gamma_\lambda^{(\rho)} \,|\, \mathcal{F}_t)$.

We leave to the interested reader the task of formulating analogous results for fractional derivatives of renormalized self-intersection local times in the plane.

This paper is organized as follows. In section 2 we prove Theorem 1 and in section 3 prove Theorem 2. Section 4 contains the short proof of Theorem 3. The proofs of Theorems 4 and 5 are similar and left to the reader,

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2   Existence of $\alpha'_t(x)$

Proof of Theorem 1: For any $x \in R^1$ and Borel set $B \subseteq R^2_+$ let

$$(2.1) \qquad \alpha'_\epsilon(x, B) = - \int_B \int f'_\epsilon(X_s - X_r - x) \, dr \, ds.$$

$\alpha'_\epsilon(x, B)$ is clearly continuous in all parameters as long as $\epsilon > 0$. We use $|B|$ to denote the Lebesgue measure of $B \subseteq R^2_+$. For any random variable Y we set $\{Y\}_0 = Y - E(Y)$.

The following Lemma will be proven at the end of this section.

Lemma 1 *Let X be the symmetric stable process of index $\beta > 3/2$ in R^1. Then for some $\zeta > 0$*

$$(2.2) \qquad \left| E\left(\{\alpha'_\epsilon(x, B) - \alpha'_{\epsilon'}(x', B)\}_0^j \right) \right| \le c_0(\zeta, j) \, |(\epsilon, x) - (\epsilon', x')|^{j\zeta}$$

and
$$(2.3) \qquad \left| E\left(\{\alpha'_\epsilon(x, B)\}_0^j \right) \right| \le c_0(\zeta, j) \, |B|^{j\zeta}$$

for all $j \in Z_+$, $\epsilon, \epsilon' \in (0, 1]$, $x, x' \in R^1$ and all Borel sets $B \subseteq A_1^1 =: [0, 1/2] \times [1/2, 1]$.

Let

$$(2.4) \qquad A_k^n = [(2k-2)2^{-n}, (2k-1)2^{-n}] \times [(2k-1)2^{-n}, (2k)2^{-n}].$$

Using the scaling $X_{\lambda t} =_{\mathcal{L}} \lambda^{1/\beta} X_t$ and $f'_{\lambda \epsilon}(x) = \frac{1}{\lambda^2} f'_\epsilon(x/\lambda)$ we have

$$(2.5) \qquad \alpha'_\epsilon(x, B) = 2^{-n(2-2/\beta)} \alpha'_{2^{n/\beta}\epsilon}(2^{n/\beta} x, 2^n B)$$

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so that from (2.2) and (2.3) we have that for all Borel sets $B \subseteq A_k^{n+1}$

$$\left| E\left(\{\alpha_\epsilon'(x, B) - \alpha_{\epsilon'}'(x', B)\}_0^j \right) \right| \leq c_0(\zeta, j) 2^{-nj(2-2/\beta-\zeta/\beta)} |(\epsilon, x) - (\epsilon', x')|^{\zeta j}$$

(2.6)

and

(2.7)
$$\left| E\left(\{\alpha_\epsilon'(x, B)\}_0^j \right) \right| \leq c_0(\zeta, j) 2^{-nj(2-2/\beta-2\zeta)} |B|^{\zeta j}.$$

For any B let

(2.8)
$$\gamma_\epsilon'(x, B) = \{\alpha_\epsilon'(x, B)\}.$$

Following Le Gall [11] we write

(2.9)
$$\gamma_\epsilon'(x, B) = \sum_{n,k} \gamma_\epsilon'(x, B \cap A_k^n)$$

for any $B \subseteq \{0 \leq r \leq s \leq 1\}$. Using (2.6) and (2.7) together with independence we then have

(2.10)
$$\left| E\left(\left\{ \sum_{k=1}^{2^n} \gamma_\epsilon'(x, B \cap A_k^{n+1}) - \gamma_{\epsilon'}'(x', B \cap A_k^{n+1}) \right\}^j \right) \right|$$

$$\leq c_0(\zeta, j) 2^{nj/2} 2^{-nj(2-2/\beta-\zeta/\beta)} |(\epsilon, x) - (\epsilon', x')|^{\zeta j},$$

and

(2.11)
$$\left| E\left(\left\{ \sum_{k=1}^{2^n} \gamma_\epsilon'(x, B \cap A_k^{n+1}) \right\}^j \right) \right| \leq c_0(\zeta, j) 2^{nj/2} 2^{-nj(2-2/\beta-2\zeta)} |B|^{\zeta j}$$

so that

(2.12)
$$\|\gamma_\epsilon'(x, B) - \gamma_{\epsilon'}'(x', B)\|_j \leq c \, |(\epsilon, x) - (\epsilon', x')|^\zeta$$

and

(2.13)
$$\|\gamma_\epsilon'(x, B)\|_j \leq c \, |B|^\zeta$$

if we choose $\zeta > 0$ so that $1/2 - 2 + 2/\beta + \zeta(2 + 1/\beta) < 0$. This is possible for $\beta > 4/3$, (and we are assuming that $\beta > 3/2$).

Let $B_t = \{0 \leq r \leq s \leq t\}$ and set $\gamma_{\epsilon,t}'(x) =: \gamma_\epsilon'(x, B_t)$. If $t, t' \leq M < \infty$ then $|B_t - B_{t'}| \leq M|t - t'|$ so that by (2.13) for some $c < \infty$

(2.14)
$$\|\gamma_{\epsilon,t}'(x) - \gamma_{\epsilon,t'}'(x)\|_j \leq c \, |t - t'|^\zeta$$

and combined with (2.12) this shows that for some $\zeta > 0$

(2.15)
$$\|\gamma_{\epsilon,t}'(x) - \gamma_{\epsilon',t'}'(x')\|_j \leq c \, |(\epsilon, x, t) - (\epsilon', x', t')|^\zeta.$$

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Kolmogorov's lemma then shows that locally

$$(2.16) \quad |\gamma'_{\epsilon,t}(x) - \gamma'_{\epsilon',t'}(x')| \le c_\omega \, |(\epsilon, x, t) - (\epsilon', x', t')|^{\zeta'}, \qquad \epsilon, \epsilon' > 0$$

for some $\zeta' > 0$, which assures us of a locally uniform and hence continuous limit

$$(2.17) \qquad\qquad \gamma'_t(x) = \lim_{\epsilon \to 0} \gamma'_{\epsilon,t}(x).$$

The next Lemma is proven in section 5

Lemma 2 *For the symmetric stable process of index $\beta > 1$ we can find a continuous function $h(x)$ with $h(0) = 0$ such that for each x*

$$(2.18) \qquad \lim_{\epsilon \to 0} E\left(\alpha'_{\epsilon,t}(x)\right) - h(x) = \begin{cases} c(\beta)\,sgn(x)x^{\beta-2}t & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where $c(\beta) = 2\Gamma(2 - \beta)\cos((1 - \beta)\pi/2)$ if $1 < \beta < 2$ and $c(2) = 1$. Furthermore, (2.18) converges locally uniformly in x away from 0 and locally in L^1.

Using this Lemma and the locally uniform convergence (2.17) we see that

$$(2.19) \qquad\qquad \alpha'_t(x) = \lim_{\epsilon \to 0} \alpha'_{\epsilon,t}(x)$$

exists for all x, t, and $\alpha'_t(0) = \gamma'_t(0)$. Furthermore the convergence is locally uniform in x away from 0 and locally in L^1.

A similar and simpler analysis shows that locally

$$(2.20) \quad |\gamma_{\epsilon,t}(x) - \gamma_{\epsilon',t'}(x')| \le c_\omega \, |(\epsilon, x, t) - (\epsilon', x', t')|^{\zeta'}, \qquad \epsilon, \epsilon' > 0$$

for some $\zeta' > 0$, which assures us of a locally uniform and hence continuous limit

$$(2.21) \qquad\qquad \gamma_t(x) = \lim_{\epsilon \to 0} \gamma_{\epsilon,t}(x).$$

Using the bound $p_s(y) \le c/s^{1/\beta}$ we can check that $\int_0^t \int_0^s p_{s-r}(y)\,dr\,ds$ is bounded and continuous in y and that uniformly in x

$$
\begin{aligned}
(2.22) \quad \lim_{\epsilon \to 0} E\left(\alpha_{\epsilon,t}(x)\right) &= \lim_{\epsilon \to 0} \int_0^t \int_0^s \left(\int f_\epsilon(y - x) p_{s-r}(y)\,dy\right) dr\,ds \\
&= \lim_{\epsilon \to 0} \int f_\epsilon(y - x) \left(\int_0^t \int_0^s p_{s-r}(y)\,dr\,ds\right) dy \\
&= \int_0^t \int_0^s p_{s-r}(x)\,dr\,ds
\end{aligned}
$$

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Together with the locally uniform convergence of (2.21) we see that

$$(2.23) \qquad \alpha_t(x) = \lim_{\epsilon \to 0} \alpha_{\epsilon,t}(x).$$

locally uniformly.

Now let $g(x)$ be a C^1 function with compact support. The locally L^1 convergence in (2.19) and the fact that $\{X_s\,;\, 0 \le s \le t\}$ is bounded a.s. shows that

$$(2.24) \qquad \begin{aligned}
\int g(x)\alpha'_t(x)\, dx &= \lim_{\epsilon \to 0} \int g(x)\alpha'_{\epsilon,t}(x)\, dx \\
&= -\lim_{\epsilon \to 0} \int g(x) \left(\int_0^t \int_0^s f'_\epsilon(X_s - X_r - x)\, dr\, ds \right) dx \\
&= -\lim_{\epsilon \to 0} \int_0^t \int_0^s \left(\int g'(x) f_\epsilon(X_s - X_r - x)\, dx \right) dr\, ds \\
&= -\lim_{\epsilon \to 0} \int_0^t \int_0^s f_\epsilon * g'(X_s - X_r)\, dr\, ds \\
&= -\int_0^t \int_0^s g'(X_s - X_r)\, dr\, ds.
\end{aligned}$$

Since the path $\{X_s\,;\, 0 \le s \le t\}$ is bounded a.s. we have that $\alpha'_t(x)$ has compact support a.s. so that (2.24) holds for all C^1 functions $g(x)$.

Similarly, using the locally uniform convergence (2.23) we see that for any continuous function $h(x)$ we have

$$(2.25) \qquad \int h(x)\alpha_t(x)\, dx = \int_0^t \int_0^s h(X_s - X_r)\, dr\, ds.$$

Therefore

$$(2.26) \qquad \int g(x)\alpha'_t(x)\, dx = -\int g'(x)\alpha_t(x)\, dx$$

holds for all C^1 functions $g(x)$.

It is clear that $\frac{d}{dx}\gamma_{\epsilon,t}(x) = \gamma'_{\epsilon,t}(x)$ for any $\epsilon > 0$ and hence

$$(2.27) \qquad \gamma_{\epsilon,t}(x) = \gamma_{\epsilon,t}(y) + \int_y^x \gamma'_{\epsilon,t}(z)\, dz.$$

The locally uniform convergence shown above then implies that

$$(2.28) \qquad \gamma_t(x) = \gamma_t(y) + \int_y^x \gamma'_t(z)\, dz$$

and therefore $\frac{d}{dx}\gamma_t(x) = \gamma'_t(x)$. This completes the proof of Theorem 1.

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Proof of Lemma 1: We begin by showing how to find a bound

(2.29)
$$\left| E\left((\alpha'_\epsilon(x,B))^j \right) \right| \leq c^j$$

uniform in $\epsilon \in (0,1]$, $x \in R^1$ and $B \subseteq A_1^1$.

We use

(2.30)
$$f'_\epsilon(x) = \frac{i}{2\pi} \int e^{ipx} p \widehat{f}(\epsilon p)\, dp$$

and independence to write that

(2.31) $\quad E\left((\alpha'_\epsilon(x,B))^j \right)$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \exp(-ix \sum_{k=1}^{j} p_k) E\left(\exp(\sum_{k=1}^{j} ip_k(X_{s_k} - X_{r_k})) \right)$$

$$\prod_{k=1}^{j} p_k \widehat{f}(\epsilon p_k)\, ds_k\, dr_k\, dp_k$$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \exp(-ix \sum_{k=1}^{j} p_k) E\left(\exp(\sum_{k=1}^{j} ip_k(X_{1/2} - X_{r_k})) \right)$$

$$E\left(\exp(\sum_{k=1}^{j} ip_k(X_{s_k} - X_{1/2})) \right) \prod_{k=1}^{j} p_k \widehat{f}(\epsilon p_k)\, ds_k\, dr_k\, dp_k.$$

We write

(2.32)
$$\sum_{k=1}^{j} p_k(X_{1/2} - X_{r_k}) = \sum_{k=1}^{j} v_k(X_{t_{k+1}} - X_{t_k})$$

where the $t_1, \ldots, t_j$ are the r_i's relabeled so that $t_1 \leq t_2 \leq \ldots \leq t_j \leq t_{j+1} \overset{def}{=} 1/2$ and $v_i = \sum_{l:r_l \leq t_i} p_l$ so that the v_i's span R^j. Similarly we rewrite

(2.33)
$$\sum_{k=1}^{j} p_k(X_{s_k} - X_{1/2}) = \sum_{k=1}^{j} v'_k(X_{t'_k} - X_{t'_{k-1}})$$

with $t'_0 \overset{def}{=} 1/2$. Then using (2.31) and independence we have

(2.34) $\quad E\left((\alpha'_\epsilon(x,B))^j \right)$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \exp(-ix \sum_{k=1}^{j} p_k) \exp(-\sum_{k=1}^{j} |v_k|^\beta (t'_{k+1} - t'_k))$$

$$\exp(-\sum_{k=1}^{j} |v'_k|^\beta (t_k - t_{k-1})) \prod_{k=1}^{j} p_k \widehat{f}(\epsilon p_k)\, ds_k\, dr_k\, dp_k.$$

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Using this and the simple bound

$$(2.35) \qquad \int_0^1 e^{-t|v|^\beta}\, dt \le \frac{c}{1 + |v|^\beta}$$

we have the uniform bound

$$(2.36) \qquad \left| E\left((\alpha'_\epsilon(x, B))^j \right) \right| \le c^j \int \prod \frac{1}{1 + |v_k|^\beta} \prod \frac{1}{1 + |v'_k|^\beta} \prod_{k=1}^j |p_k|\, dp_k$$

$$\le c^j \|\prod \frac{|p_k|^{1/2}}{1 + |v_k|^\beta} \|_2 \, \| \prod \frac{|p_k|^{1/2}}{1 + |v'_k|^\beta} \|_2.$$

Since $v_i = \sum_{l : r_l \le t_i} p_l$, we see that each p_k can be represented as the difference $p_k = v_i - v_{i-1}$ for some i, and each v_i appears in the representation of at most two p_k's. Thus

$$(2.37) \qquad \| \prod \frac{|p_k|^{1/2}}{1 + |v_k|^\beta} \|_2^2 = \int \prod \frac{|p_k|}{1 + |v_k|^{2\beta}}\, dp_k$$

$$\le c^j \int \prod \frac{1 + |v_k| + |v_k|^2}{1 + |v_k|^{2\beta}}\, dv_k$$

which is bounded if $\beta > 3/2$.

We can now establish (2.2). To handle the variation in x we replace (2.34) by

$$(2.38) \quad E\left((\alpha'_\epsilon(x, B) - \alpha'_\epsilon(x', B))^j \right)$$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \left(\prod_{k=1}^j \{ \exp(-ip_k x) - \exp(-ip_k x') \} \right)$$

$$\exp(-\sum_{k=1}^j |v_k|^\beta (t'_{k+1} - t'_k)) \exp(-\sum_{k=1}^j |v'_k|^\beta (t_k - t_{k-1}))$$

$$\prod_{k=1}^j p_k \widehat{f}(\epsilon p_k)\, ds_k\, dr_k\, dp_k$$

and use the bound

$$(2.39) \qquad | \exp(-ip_k x) - \exp(-ip_k x')| \le C|p_k|^\zeta |x - x'|^\zeta$$

for any $0 \le \zeta \le 1$.

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Similarly to handle the variation in ϵ we replace (2.34) by

$$(2.40) \qquad E\left((\alpha'_\epsilon(x, B) - \alpha'_{\epsilon'}(x, B))^j\right)$$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \exp(-ix \sum_{k=1}^{j} p_k) \exp(-\sum_{k=1}^{j} |v_k|^\beta (t'_{k+1} - t'_k))$$

$$\exp(-\sum_{k=1}^{j} |v'_k|^\beta (t_k - t_{k-1})) \prod_{k=1}^{j} p_k \left(\widehat{f}(\epsilon p_k) - \widehat{f}(\epsilon' p_k)\right) ds_k \, dr_k \, dp_k$$

and use the bound

$$(2.41) \qquad\qquad |\widehat{f}(\epsilon p_k) - \widehat{f}(\epsilon' p_k)| \leq C|p_k|^\varsigma |\epsilon - \epsilon'|^\varsigma$$

for any $0 \leq \varsigma \leq 1$. Since (2.38)-(2.40) hold also when $j = 1$, we obtain (2.2). To prove (2.3) we first apply Holder's inequality to (2.34):

$$(2.42) \qquad E\left((\alpha'_\epsilon(x, B))^j\right)$$

$$= \frac{1}{(2\pi i)^j} \int \int_{B^j} \exp(-ix \sum_{k=1}^{j} p_k) \exp(-\sum_{k=1}^{j} |v_k|^\beta (t'_{k+1} - t'_k))$$

$$\exp(-\sum_{k=1}^{j} |v'_k|^\beta (t_k - t_{k-1})) \prod_{k=1}^{j} p_k \widehat{f}(\epsilon p_k) \, ds_k \, dr_k \, dp_k$$

$$\leq c^j |B|^{j/a} \int \left(\int_{B^j} \exp(-a' \sum_{k=1}^{j} |v_k|^\beta (t'_{k+1} - t'_k)) \right.$$

$$\left. \exp(-a' \sum_{k=1}^{j} |v'_k|^\beta (t_k - t_{k-1})) \prod_{k=1}^{j} ds_k \, dr_k \right)^{1/a'} \prod_{k=1}^{j} |p_k| \, dp_k$$

for any $1/a + 1/a' = 1$. The last integral can be bounded as before if a' is chosen close to 1. As in the proof of (2.2), this completes the proof of (2.3) and therefore of Lemma 1.

3   p-variation of γ'_t

Proof of Theorem 2: Since we know that $\alpha'_{t,\epsilon} \to \gamma'_t$ in L^2, we have

$$(3.1) \qquad E\left((\gamma'_t - \gamma'_{t'})^2\right)$$

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$$= \lim_{\epsilon \to 0} E\left(\left(\alpha'_{t,\epsilon} - \alpha'_{t',\epsilon}\right)^2\right)$$

$$= \lim_{\epsilon \to 0} E\left(\left(\int_{t'}^{t} \int_{0}^{s} f'_\epsilon(X_s - X_r)\, dr\, ds\right)^2\right)$$

$$= \lim_{\epsilon \to 0} \int \int_{\left\{\substack{0 \le s_1 \le s_2 \le s_3 \le s_4 \\ t' \le s_3 \le t\, ; \, t' \le s_4 \le t}\right\}} E\left(\exp(-ip(X_{s_2} - X_{s_1}) - i(p+q)(X_{s_3} - X_{s_2}))\right.$$

$$\exp(-iq(X_{s_4} - X_{s_3}))) \prod_{k=1}^{4} ds_k pq \widehat{f}(\epsilon p)\widehat{f}(\epsilon q)\, dp\, dq$$

$$+ \lim_{\epsilon \to 0} \int \int_{\left\{\substack{0 \le s_1 \le s_2 \le s_3 \le s_4 \\ t' \le s_3 \le t\, ; \, t' \le s_4 \le t}\right\}} E\left(\exp(-ip(X_{s_2} - X_{s_1}) - i(p+q)(X_{s_3} - X_{s_2}))\right.$$

$$\exp(-ip(X_{s_4} - X_{s_3}))) \prod_{k=1}^{4} ds_k pq \widehat{f}(\epsilon p)\widehat{f}(\epsilon q)\, dp\, dq$$

$$= \lim_{\epsilon \to 0} \int \int_{\left\{\substack{0 \le s_1 \le s_2 \le s_3 \le s_4 \\ t' \le s_3 \le t\, ; \, t' \le s_4 \le t}\right\}} e^{-(s_2-s_1)|p|^\beta - (s_3-s_2)|p+q|^\beta - (s_4-s_3)|q|^\beta}$$

$$\prod_{k=1}^{4} ds_k pq \widehat{f}(\epsilon p)\widehat{f}(\epsilon q)\, dp\, dq$$

$$+ \lim_{\epsilon \to 0} \int \int_{\left\{\substack{0 \le s_1 \le s_2 \le s_3 \le s_4 \\ t' \le s_3 \le t\, ; \, t' \le s_4 \le t}\right\}} e^{-(s_2-s_1)|p|^\beta - (s_3-s_2)|p+q|^\beta - (s_4-s_3)|p|^\beta}$$

$$\prod_{k=1}^{4} ds_k pq \widehat{f}(\epsilon p)\widehat{f}(\epsilon q)\, dp\, dq$$

Consider the first summand on the right hand side of (3.1). It is bounded by

$$(3.2) \qquad \int \int_{\left\{\substack{t' \le r_1 + r_2 + r_3 \le t \\ t' \le r_1 + r_2 + r_3 + r_4 \le t}\right\}} e^{-r_2|p|^\beta - r_3|p+q|^\beta - r_4|q|^\beta} \prod_{k=1}^{4} dr_k |p||q|\, dp\, dq$$

$$\le C(t-t')^{1+a} \int \frac{1}{1+|p|^\beta} \frac{1}{1+|p+q|^\beta} \frac{1}{1+|q|^{(1-a)\beta}} |p||q|\, dp\, dq$$

where we first integrated with respect to dr_4 using Hölder's inequality with $0 \le a \le 1$

$$(3.3) \qquad \int_{t'-(r_1+r_2+r_3)}^{t-(r_1+r_2+r_3)} e^{-r_4|q|^\beta}\, dr_4 \le (t-t')^a \left(\int_0^t e^{-r_4|q|^\beta/(1-a)}\, dr_4\right)^{(1-a)},$$

then with respect to dr_1 using $\int_{t'-(r_2+r_3)}^{t-(r_2+r_3)} dr_1 \leq t - t'$, and finally the dr_2, dr_3 integrals are bounded using (2.35). It is easily seen that (3.2) is bounded as long as $\beta - 1 + (1-a)\beta - 1 > 1$, i.e. $a < 2 - 3/\beta$.

Now consider the second summand on the right hand side of (3.1). An attempt to use a bound similar to (3.2) where we bound pq by $|pq|$ would be fatal. Rather, we first observe that $\int e^{-(s_3-s_2)|q|^\beta} q\, dq = 0$ and use this to rewrite the second summand on the right hand side of (3.1) as

$$(3.4)\quad \lim_{\epsilon \to 0} \int \int_{\left\{\substack{0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \\ t' \leq s_3 \leq t\,;\, t' \leq s_4 \leq t}\right\}} e^{-(s_2-s_1)|p|^\beta - (s_4-s_3)|p|^\beta}$$

$$\left\{ e^{-(s_3-s_2)|p+q|^\beta} - e^{-(s_3-s_2)|p|^\beta} e^{-(s_3-s_2)|q|^\beta} \right\} \prod_{k=1}^{4} ds_k\, pq\, \widehat{f}(\epsilon p)\widehat{f}(\epsilon q)\, dp\, dq.$$

Now we bound this as in (3.2) by

$$(3.5)\int \int_{\left\{\substack{t' \leq r_1+r_2+r_3 \leq t \\ t' \leq r_1+r_2+r_3+r_4 \leq t}\right\}} e^{-r_2|p|^\beta - r_4|p|^\beta}$$

$$\left| e^{-r_3|p+q|^\beta} - e^{-r_3|p|^\beta} e^{-r_3|q|^\beta} \right| \prod_{k=1}^{4} dr_k\, |p||q|\, dp\, dq$$

$$\leq C(t-t')^{1+a} \int \frac{1}{1+|p|^{(2-a)\beta}} \left| \frac{1}{1+|p+q|^\beta} - \frac{1}{1+|p|^\beta+|q|^\beta} \right| |p||q|\, dp\, dq.$$

Here we proceeded as in (3.2) except that for the dr_3 integral we used

$$(3.6)\int e^{-r} \left| e^{-r|p+q|^\beta} - e^{-r(|p|^\beta+|q|^\beta)} \right| dr \leq \left| \frac{1}{1+|p+q|^\beta} - \frac{1}{1+|p|^\beta+|q|^\beta} \right|$$

by arguing seperately depending on whether or not $|p|^\beta + |q|^\beta > |p+q|^\beta$. We claim that once again the integral on the right hand side of (3.5) is finite whenever $\beta - 1 + (1-a)\beta - 1 > 1$, i.e. $a < 2 - 3/\beta$. This is clear in the region where $|q| \leq 2|p|$ since we can use the bound $\frac{1}{1+|p|^{(2-a)\beta}} \leq \frac{1}{1+|p|^\beta} \frac{c}{1+|q|^{(1-a)\beta}}$. If, however, $|q| > 2|p|$, we can use the bound

$$(3.7)\qquad\qquad \left| \frac{1}{1+|p+q|^\beta} - \frac{1}{1+|p|^\beta+|q|^\beta} \right|$$

$$= \frac{\left| |p+q|^\beta - |q|^\beta - |p|^\beta \right|}{(1+|p+q|^\beta)(1+|p|^\beta+|q|^\beta)}$$

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$$\leq \frac{c|p||q|^{\beta-1} + |p|^{\beta}}{(1+|p+q|^{\beta})(1+|p|^{\beta}+|q|^{\beta})}$$

$$\leq \frac{c|p||q|^{\beta-1}}{(1+|q|^{\beta})^2}.$$

This allows us to bound the resulting integral from the right hand side of (3.5) by

$$(3.8) \qquad c \int \frac{|p|^2}{1+|p|^{(2-a)\beta}} \frac{1}{(1+|q|^{\beta})} \, dp \, dq$$

which leads to the same result as before.

Finally, for $h \leq 2$ we have

$$(3.9) \qquad E\left((\gamma'_t - \gamma'_{t'})^h\right) \leq \left\{ E\left((\gamma'_t - \gamma'_{t'})^2\right)\right\}^{h/2} \leq C(t-t')^{(1+a)h/2}.$$

We will have h-variation 0 when $(1+a)h/2 > 1$ for some $a < 2 - 3/\beta$, i.e. $(3-3/\beta)h/2 > 1$. Thus we will have h-variation 0 when $h > (2/3)\beta'$. This completes the proof of Theorem 2.

4 The Doob-Meyer decomposition

Proof of Theorem 3: It is easy to check that the proofs of Theorems 1 and 2 go through with X replaced by Y and in that case

$$(4.1) \qquad \gamma'_{\infty} = \lim_{\epsilon \to 0} \gamma'_{\epsilon,\infty} \text{ and } \gamma'_t = \lim_{\epsilon \to 0} \gamma'_{\epsilon,t}$$

converge a.s. and in all L^p spaces. Let

$$(4.2) \qquad U^1 F_{\epsilon,t}(y) = -\int_0^t \int u^1(x - (y - Y_r)) f'_{\epsilon}(x) \, dx \, dr.$$

The proof of Theorem 3 then follows from (4.1) and the next Lemma.

Lemma 3 *Let* $\{Y_s \, ; \, s \in R^1_+\}$ *be the exponentially killed symmetric stable process of index* $\beta > 3/2$ *in* R^1. *Then for any* $\epsilon > 0$

$$(4.3) \qquad U^1 F_{\epsilon,t}(Y_t) = E^x(\gamma'_{\epsilon,\infty} \,|\, \mathcal{F}_t) - \gamma'_{\epsilon,t}$$

Proof of Lemma 3: We have

(4.4)
$$\gamma'_{\epsilon,t} = -\left\{ \int_0^t \int_0^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\}$$

so that

(4.5)
$$\begin{aligned}
\gamma'_{\epsilon,\infty} &= -\left\{ \int_0^\infty \int_0^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\} \\
&= \gamma'_{\epsilon,t} - \left\{ \int_t^\infty \int_t^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\} \\
&\quad - \left\{ \int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\}.
\end{aligned}$$

Now, using the fact that $\{Y_s\,;\, s \in R^1_+\}$ has independent increments

(4.6)
$$\begin{aligned}
E&\left(\left\{ \int_t^\infty \int_t^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\} \Big| \mathcal{F}_t \right) \\
&= E\left(\int_t^\infty \int_t^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \Big| \mathcal{F}_t \right) \\
&\quad - E\left(\int_t^\infty \int_t^s f'_\epsilon(Y_s - Y_r)\, dr\, ds \right) = 0
\end{aligned}$$

and

(4.7)
$$\begin{aligned}
E&\left(\left\{ \int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \right\} \Big| \mathcal{F}_t \right) \\
&= E\left(\int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \Big| \mathcal{F}_t \right) \\
&\quad - E\left(\int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \right).
\end{aligned}$$

We have

(4.8)
$$\begin{aligned}
E&\left(\int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \Big| \mathcal{F}_t \right) \\
&= E\left(\int_t^\infty \int_0^t f'_\epsilon((Y_s - Y_t) + (Y_t - Y_r))\, dr\, ds \Big| \mathcal{F}_t \right) \\
&= \bar{E}\left(\int_0^\infty \int_0^t f'_\epsilon(\bar{Y}_s + (Y_t - Y_r))\, dr\, ds \right)
\end{aligned}$$

where $\{\bar{Y}_s\,;\, s \in R^1_+\}$ is an independent copy of $\{Y_s\,;\, s \in R^1_+\}$ and $\bar{E}$ denotes expectation with respect to $\{\bar{Y}_s\,;\, s \in R^1_+\}$. Hence

(4.9)
$$E\left(\int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\, dr\, ds \Big| \mathcal{F}_t \right)$$

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$$= \int_0^t \int f'_\epsilon(x + Y_t - Y_r)u^1(x)\,dx\,dr$$

$$= \int_0^t \int u^1(x - (Y_t - Y_r))f'_\epsilon(x)\,dx\,dr.$$

The same sort of argument shows that

(4.10) $\quad E\left(\int_t^\infty \int_0^t f'_\epsilon(Y_s - Y_r)\,dr\,ds\right) = \int_0^t \int f'_\epsilon(x + y)u^1(x)p_r(y)\,dx\,dy\,dr$

which is zero by symmetry. This concludes the proof of Lemma 3.

5　Proof of Lemma 2

Proof of Lemma 2: We have

(5.1) $\qquad E\left(\alpha'_{\epsilon,t}(x)\right) = -\int_0^t \int_0^s \left(\int f'_\epsilon(y - x)p_{s-r}(y)\,dy\right)dr\,ds$

We first consider the case $1 < \beta < 2$. Using Plancherel and then Fubini

(5.2) $\qquad -\int_0^t \int_0^s \left(\int f'_\epsilon(y - x)p_{s-r}(y)\,dy\right)dr\,ds$

$$= (2\pi i)^{-1}\int_0^t \int_0^s \left(\int e^{ipx}p\widehat{f}(\epsilon p)e^{-(s-r)p^\beta}\,dp\right)dr\,ds$$

$$= (2\pi i)^{-1}\int e^{ipx}p\widehat{f}(\epsilon p)\left(\int_0^t \int_0^s e^{-(s-r)p^\beta}\,dr\,ds\right)dp$$

and

(5.3) $\displaystyle\int_0^t \int_0^s e^{-(s-r)p^\beta}\,dr\,ds$

$$= \int_0^t \int_r^t e^{-(s-r)p^\beta}\,ds\,dr = \int_0^t \left(\int_0^{t-r} e^{-sp^\beta}\,ds\right)dr = \int_0^t \int_0^r e^{-sp^\beta}\,ds\,dr$$

$$= t\int_0^\infty e^{-sp^\beta}\,ds - \int_0^t \int_r^\infty e^{-sp^\beta}\,ds\,dr$$

$$= tp^{-\beta} - \int_0^t e^{-rp^\beta}p^{-\beta}\,dr = tp^{-\beta} - (1 - e^{-tp^\beta})p^{-2\beta}.$$

Hence

(5.4) $\quad E\left(\alpha'_{\epsilon,t}(x)\right) = (2\pi i)^{-1}t\int e^{ipx}\mathrm{sgn}(p)p^{1-\beta}\widehat{f}(\epsilon p)\,dp$

$$-(2\pi i)^{-1}\int e^{ipx}\mathrm{sgn}(p)(1 - e^{-tp^\beta})p^{1-2\beta}\widehat{f}(\epsilon p)\,dp.$$

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It is easily checked that $(1 - e^{-tp^\beta})p^{1-2\beta} \in L^1(R^1)$ so that the last term converges uniformly as $\epsilon \to 0$ to a continuous limit. On the other hand, it follows from [8, formula (13), page 173] that

(5.5) $\qquad (2\pi i)^{-1} \int e^{ipx} \text{sgn}(p) p^{1-\beta} \widehat{f}(\epsilon p) \, dp$

$$= \pi^{-1} \Gamma(2 - \beta) \cos((1 - \beta)\pi/2) \int \text{sgn}(y) y^{\beta-2} f_\epsilon(y - x) \, dy.$$

Since $\text{sgn}(y)y^{\beta-2}$ is locally in L^1 and continuous away from 0, this last term converges locally uniformly away from 0 and locally in L^1 to $c(\beta)\text{sgn}(x)x^{\beta-2}$. On the other hand, when $x = 0$, (5.1) is 0 by symmetry. This completes the proof of Lemma 2 for $\beta \neq 2$.

For Brownian motion, we proceed differently. We first write

(5.6) $\qquad -\int_0^t \int_0^s \left(\int f_\epsilon'(y - x) p_{s-r}(y) \, dy \right) dr \, ds$

$$= \int_0^t \int_0^s \left(\int f_\epsilon(y - x) p_{s-r}'(y) \, dy \right) dr \, ds.$$

Recall the formula for the potential of three dimensional Brownian motion

(5.7) $\qquad \int_0^\infty \frac{1}{(2\pi s)^{3/2}} e^{-y^2/2s} \, ds = \frac{1}{2\pi|y|}, \qquad y \neq 0.$

We have $p_s'(y) = \frac{-1}{(2\pi)^{1/2} s^{3/2}} y e^{-y^2/2s}$ so that

(5.8) $\qquad \int_0^\infty |p_s'(y)| \, ds = 1, \qquad y \neq 0.$

This justifies the use of Fubini

(5.9) $\qquad \int_0^t \int_0^s \left(\int f_\epsilon(y - x) p_{s-r}'(y) \, dy \right) dr \, ds$

$$= \int f_\epsilon(y - x) \left(\int_0^t \int_0^s p_{s-r}'(y) \, dr \, ds \right) dy$$

and then, just as in (5.3)

(5.10) $\int_0^t \int_0^s p_{s-r}'(y) \, dr \, ds = t \int_0^\infty p_s'(y) \, ds - \int_0^t \int_r^\infty p_s'(y) \, ds \, dr, \qquad y \neq 0.$

Just as in (5.8) we see that the first integral is $\text{sgn}(y)$ for $y \neq 0$. Now $|p_s'(y)| \leq \frac{1}{(2\pi)^{1/2} s^{3/2}}|y|$, so that by the dominated convergence theorem the second integral is continuous in y. As before, when $y = 0$, the left hand side of (5.6)is 0 by symmetry. Lemma 2 now follows.

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