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Examples of Homological Torsion and Volume Growth

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Abstract

We provide examples of towers of covers of cusped hyperbolic 3-manifolds whose exponential homological torsion growth is explicitly computed in terms of volume growth. These examples arise from abelian covers of alternating links in the thickened torus. A corollary is that the spanning tree entropy for each regular planar lattice is given by the volume of a hyperbolic polyhedron.

Keywords Homological torsion · Hyperbolic volume · Biperiodic link · Alexander polynomial · Mahler measure

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1 Introduction

For a manifold M, a *tower of covers* is a sequence of finite covers

$$\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = M.$$

If M is a 3-manifold, the homology groups $H_1(M_n;\mathbb{Z})$ can have arbitrarily large torsion subgroups, denoted here by $TH_1(M_n)$. For a tower of covers of M, the growth rate of the order of $TH_1(M_n)$ has attracted recent interest in the context of Lück's far-reaching Approximation Conjecture in L^2-torsion theory (see [13]). As a special case ([1, Section 7.5], [12], [13, Example 10.5]), if M is a hyperbolic 3-manifold with empty or toroidal boundary then, conjecturally, for any tower of regular covers M_n such that $\bigcap_n \pi_1 M_n = \{1\}$

$$\lim_{n\to\infty} \frac{\log|TH_1(M_n)|}{\mathrm{vol}(M_n)} = \frac{1}{6\pi}. \tag{1}$$

With more relaxed conditions on M and $\{M_n\}$, Le [11] proved $\limsup\limits_{n\to\infty} \frac{\log|TH_1(M_n)|}{\mathrm{vol}(M_n)} \le \frac{1}{6\pi}$.

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For a knot complement $M = S^3 - K$ and finite cyclic covers X_n given by $\pi_1(M) \to \mathbb{Z}/n\mathbb{Z}$, the L^2-torsion is well understood. If $\mathrm{m}(\Delta_K(t))$ denotes the logarithmic Mahler measure of the Alexander polynomial of K, then the following result proved in [17] has been interpreted (see [10, 15]) as a special case of the Approximation Conjecture for $\{X_n\}$

$$\lim_{n\to\infty} \frac{\log |TH_1(X_n)|}{n} = \mathrm{m}(\Delta_K(t)). \tag{2}$$

Our main result, Theorem 1 below, is a pair of towers of abelian covers of cusped arithmetic hyperbolic 3-manifolds, which have totally geodesic surfaces in every cover, such that

$$\lim_{n\to\infty} \frac{\log |TH_1(M_n)|}{\mathrm{vol}(M_n)} = \frac{1}{4\pi}.$$

As far as we know, these are the first examples of towers of non-cyclic covers of hyperbolic 3-manifolds whose exponential homological torsion growth can be computed exactly in terms of volume growth. These towers of abelian covers are not cofinal as in (1). They are derived from previous results [4–7] on alternating links in the thickened torus.

After discussing these two examples, we present a broader context and a related conjecture for infinite families of links. Finally, we prove as a corollary that the spanning tree entropy for each regular planar lattice is given by the volume of a hyperbolic polyhedron.

2 Main Results

2.1 Biperiodic Links

A *biperiodic alternating link* is a link in $\mathbb{R}^2 \times I$, with $I = (-1, 1)$, with a diagram on $\mathbb{R}^2 \times \{0\}$ which can be isotoped to be invariant under $\mathbb{Z}^2$ acting by independent translations of $\mathbb{R}^2$, such that the quotient link diagram is alternating on the torus $T^2 \times \{0\}$. Figure 1**a** shows the biperiodic alternating link $\mathcal{W}$, called the square weave, with the fundamental domain in $T^2 \times I$ shown in Fig. 1**b**. Let W_n be the toroidally alternating link in $T^2 \times I$, which is the quotient of $\mathcal{W}$ by the $n\mathbb{Z} \times n\mathbb{Z}$ action, so that the crossing number $c(W_n) = 4n^2$. Figure 1**c** shows another biperiodic alternating link, the triaxial link $\mathcal{Q}$. The fundamental domain for its toroidally alternating quotient link Q_1 in $T^2 \times I$ is shown in Fig. 1**d**. Taking the quotient by $n\mathbb{Z} \times n\mathbb{Z}$, we get the links Q_n in $T^2 \times I$ with $c(Q_n) = 3n^2$.

For any link L in $T^2 \times I$, we have $T^2 \times I - L \cong S^3 - \ell$, where ℓ has a Hopf sublink given by the cores of the two Heegaard tori. In the case of W_1, $\ell = L12n2256$ (see [6, Figure 4]). In the case of Q_1, $\ell = L12n2232$ (see [7, Figure 12]), whose complement is homeomorphic to that of the minimally twisted 5-chain link. The complements of both W_1 and Q_1 are principal congruence manifolds [2].

More generally, let $\mathcal{L}$ be any biperiodic alternating link, with a toroidally alternating quotient link L_1 in the fundamental domain $T^2 \times I$. For $n \geq 1$, taking the quotient by $n\mathbb{Z} \times n\mathbb{Z}$ as above, we get a sequence of toroidally alternating links L_n in $T^2 \times I$. Let $X(L_n)$ be the 2-fold cyclic cover of $T^2 \times I - L_n$, which is the cover associated with the kernel of $\pi_1(T^2 \times I - L_n) \to \mathbb{Z}/2\mathbb{Z}$, with each meridian of L_n mapped to 1, and the two Hopf link meridians mapped to 0. Thus, $\{X(L_n)\}$ are finite covers of $T^2 \times I - L_1$, corresponding to finite index subgroups of $\mathbb{Z}^2$. Moreover, $\mathbb{R}^2 \times I - \mathcal{L}$ and, hence, all $T^2 \times I - L_n$ and $X(L_n)$ are often hyperbolic 3-manifolds (see [7]). In particular, $\mathcal{W}$ and $\mathcal{Q}$ are hyperbolic.

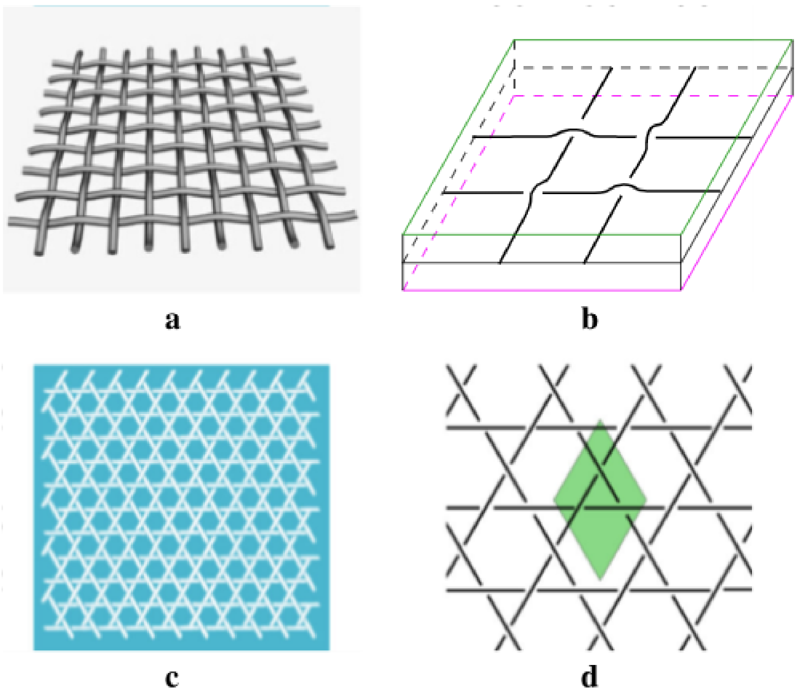

Fig. 1 **a** Square weave $\mathcal{W}$ in $\mathbb{R}^2 \times I$. **b** Link W_1 in $T^2 \times I$. **c** Triaxial link $\mathcal{Q}$ in $\mathbb{R}^2 \times I$. **d** Fundamental domain for the link Q_1 in $T^2 \times I$

2.2 Torsion and Alexander Polynomials

Suppose $T^2 \times I - L_1 \cong S^3 - \ell$. The Hopf link meridians acquire an orientation from the action by $\mathbb{Z}^2$ on the universal cover of $T^2 \times \{0\}$. Following [18], we can define the Alexander polynomial $\Delta_\ell(-1, \ldots, -1, x, y)$ via Fox calculus, with $\mathbb{Z}[\pi_1(S^3 - \ell)] \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ given by mapping the oriented Hopf link meridians to x, y, and the other meridians to -1.

Using results in [18] and [8], we now extend (2) to these non-cyclic covers.

Lemma 1 *For any biperiodic alternating link $\mathcal{L}$, the manifolds $\{X(L_n)\}$ form a sequence of covers over the 3-manifold $X(L_1)$, such that*

$$\lim_{n\to\infty} \frac{\log |TH_1(X(L_n))|}{n^2} = \mathrm{m}(\Delta_\ell(-1, \ldots, -1, x, y)).$$

Passing to the subsequence $n = 2^j$ for $j \geq 0$, we get a tower of covers with the same limit.

Proof Let $G_{\mathcal{L}}$ be the Tait graph (checkerboard graph) of $\mathcal{L}$, and let $G_{L_1} = G_{\mathcal{L}}/\mathbb{Z}^2$. The spanning tree entropy of $G_{\mathcal{L}}$ is given by the logarithmic Mahler measure of the Laplacian determinant polynomial $\mathrm{m}(D_{G_{L_1}}(x, y))$ (see [14, 18]). As a consequence of the Dehn relations from the link diagram, the Laplacian matrix is a presentation matrix (Goeritz matrix) for the homology of the branched 2-fold cyclic cover, which has the same torsion subgroup $TH_1(X(L_n))$. Extending the argument in [18, Theorem 5.1] to two variables, we obtain

$$\mathrm{m}\left(D_{G_{L_1}}(x, y)\right) = \mathrm{m}\left(\Delta_\ell(-1, \ldots, -1, x, y)\right).$$

The limit as claimed is established as lim sup in [18], and the actual limit is proved in [8] for cubical sublattices, which in this case are $n\mathbb{Z} \times n\mathbb{Z} \subset \mathbb{Z}^2$. □

For general abelian covers, Le [10, 11] proved the following result. Let $p\colon \tilde{X} \to X$ be any regular cover of a finite CW-complex with $\mathrm{Aut}(p) = \mathbb{Z}^N$. Let $\mathrm{m}(\Delta_j(H_j(\tilde{X})))$ denote the logarithmic Mahler measure of the first non-zero Alexander polynomial of $H_j(\tilde{X};\mathbb{Z})$. For $\Gamma \subset \mathbb{Z}^N$ of finite index, let X_Γ be the corresponding finite cover, and let $\langle\Gamma\rangle = \min\{||\mathbf{x}||, \mathbf{x} \in \Gamma - \{0\}\}$. In [10, 11], Le showed

$$\limsup_{\langle\Gamma\rangle\to\infty} \frac{\log|TH_1(X_\Gamma)|}{[\mathbb{Z}^N : \Gamma]} = \mathrm{m}\left(\Delta_j\left(H_j(\tilde{X})\right)\right).$$

If $N = 1$, then Le showed that lim sup can be replaced by lim. For $N > 1$ and cubical sublattices $\Gamma = (n\mathbb{Z})^N$, Dimitrov [8] showed that lim sup can be replaced by lim. Lemma 1 provides a version of the latter result for $N = 2$.

To prove Theorem 1, it will be useful to reinterpret Lemma 1 using the toroidal dimer model. For a biperiodic alternating link $\mathcal{L}$, the planar balanced bipartite *overlaid graph* $G^b_{\mathcal{L}}$ was defined in [4, 5]. The black vertices of $G^b_{\mathcal{L}}$ are vertices of the Tait graph $G_{\mathcal{L}}$ and its planar dual graph $G^*_{\mathcal{L}}$, the white vertices of $G^b_{\mathcal{L}}$ are crossings of $\mathcal{L}$, and the edges join each black vertex to every white vertex incident to that face of $\mathcal{L}$. Let $p_{L_1}(z,w)$ be the characteristic polynomial of the toroidal dimer model on $G^b_{L_1} = G^b_{\mathcal{L}}/\mathbb{Z}^2$, as in [4, 5].

Lemma 2

$$\lim_{n\to\infty} \frac{\log|TH_1(X(L_n))|}{n^2} = \mathrm{m}\left(p_{L_1}(z,w)\right).$$

Passing to the subsequence $n = 2^j$ for $j \geq 0$, we get a tower of covers with the same limit

$$\cdots \to X(L_{2n}) \to X(L_n) \to \cdots \to X(L_1).$$

Proof By [9, Theorem 3.5], the asymptotic growth rate of the toroidal dimer model on $G^b_{\mathcal{L}}/(n\mathbb{Z})^2$ is given by the logarithmic Mahler measure, $\mathrm{m}(p_{L_1}(z,w))$. By [4, Proposition 5.3], the spanning tree entropy of $G_{\mathcal{L}}$ equals the growth rate of the toroidal dimer model on $G^b_{\mathcal{L}}$. Hence

$$\mathrm{m}\left(p_{L_1}(z,w)\right) = \mathrm{m}\left(D_{G_{L_1}}(x,y)\right) = \mathrm{m}\left(\Delta_\ell(-1,\ldots,-1,x,y)\right).$$

The claim now follows by Lemma 1. □

2.3 Torsion and Volume

Theorem 1 *The manifolds $\{X(W_n)\}$ and $\{X(Q_n)\}$ each form a sequence of covers over the cusped arithmetic hyperbolic 3-manifolds $X(W_1)$ and $X(Q_1)$, respectively, such that*

$$\lim_{n\to\infty} \frac{\log|TH_1(X(W_n))|}{\mathrm{vol}(X(W_n))} = \lim_{n\to\infty} \frac{\log|TH_1(X(Q_n))|}{\mathrm{vol}(X(Q_n))} = \frac{1}{4\pi}.$$

Passing to the subsequence $n = 2^j$ for $j \geq 0$, we get towers of covers with the same limit. Moreover, $\{X(W_n)\}$ and $\{X(Q_n)\}$ have embedded totally geodesic surfaces for all n.

Proof Let $\mathcal{L}$ be any hyperbolic biperiodic alternating link, with toroidally alternating quotient links L_n as above. Since each $X(L_n)$ is a finite cover of $T^2 \times I - L_1$

$$\mathrm{vol}(X(L_n)) = 2\,\mathrm{vol}(T^2 \times I - L_n) = 2n^2\,\mathrm{vol}(T^2 \times I - L_1).$$

Hence, the limit in Lemma 2 is equivalent to

$$\lim_{n\to\infty} \frac{\log |TH_1(X(L_n))|}{\mathrm{vol}(X(L_n))} = \frac{\mathrm{m}(p_{L_1}(z,w))}{2\,\mathrm{vol}(T^2\times I - L_1)}. \tag{3}$$

Specializing to $\mathcal{W}$ and $\mathcal{Q}$, let $v_{\mathrm{tet}} \approx 1.01494$ and $v_{\mathrm{oct}} \approx 3.66386$ be the hyperbolic volumes of the regular ideal tetrahedron and the regular ideal octahedron, respectively. By [5, Theorems 12, 13], and adjusting for the fundamental domain in Fig. 1**b**

$$2\pi\,\mathrm{m}(p_{W_1}) = 4\,v_{\mathrm{oct}} \quad \text{and} \quad 2\pi\,\mathrm{m}(p_{Q_1}) = 10\,v_{\mathrm{tet}}.$$

By [7, Theorem 3.5]

$$\mathrm{vol}(T^2\times I - W_1) = 4\,v_{\mathrm{oct}} \quad \text{and} \quad \mathrm{vol}(T^2\times I - Q_1) = 10\,v_{\mathrm{tet}}.$$

The limits now follow as claimed.

We now describe $X(L_n)$ geometrically. We first decompose $T^2\times I - L_n$ into two ideal *torihedra* $\mathcal{T}^+$ and $\mathcal{T}^-$ as in [7, Theorem 2.2]. Each torihedron is homeomorphic to $T^2\times[0,1)$ with finitely many points removed from $T^2\times\{0\}$. If we glue by the identity map along all faces on $T^2\times\{0\}$, we have $T^2\times I = \mathcal{T}^+\cup_{\mathrm{id}}\mathcal{T}^-$. On the other hand, if we glue along the checkerboard-colored faces of the torihedra on $T^2\times\{0\}$ by homeomorhisms ϕ and ψ, which rotate each shaded face clockwise (ϕ), and each white face counterclockwise (ψ), we obtain $T^2\times I - L_n = \mathcal{T}^+\cup_{\phi\circ\psi}\mathcal{T}^-$. Let $\mathcal{T}_1^{\pm}$ and $\mathcal{T}_2^{\pm}$ be two copies of each of these torihedra. Then $X(L_n)$ is obtained as follows

$$X(L_n) = \left(\mathcal{T}_1^+\cup_{\phi}\mathcal{T}_2^-\right)\cup_{\mathrm{id}}\left(\mathcal{T}_2^+\cup_{\phi}\mathcal{T}_1^-\right),$$

where we glue by ϕ along shaded faces, and glue by the identity map along white faces.

By [7, Theorem 5.1], right-angled torihedra give the complete hyperbolic structure for $T^2\times I - W_n$ and $T^2\times I - Q_n$. Hence, $X(W_n)$ and $X(Q_n)$ are each obtained by gluing right-angled torihedra. Thus, by the proof of [7, Theorem 5.1], $X(W_n)$ and $X(Q_n)$ have totally geodesic checkerboard surfaces. Finally, arithmeticity follows by [7, Theorem 4.1]. □

For any hyperbolic biperiodic alternating link $\mathcal{L}$, we can construct the infinite-volume 2-fold cyclic cover $X(\mathcal{L})$ of $\mathbb{R}^2\times I - \mathcal{L}$ by the same construction as above, replacing each torihedron $\mathcal{T}^{\pm}$ used in the decomposition of $T^2\times I - L_1$ with its $\mathbb{Z}^2$-cover, which is homeomorphic to $\mathbb{R}^2\times[0,1)$, and gluing faces in the same local pattern as above. Since $\{X(L_n)\}$ is a sequence of covers, it follows from the definition that its geometric limit is $X(\mathcal{L})$ (see [3]).

The RHS of the conjectured equation (1) comes from L^2-torsion theory [13]; namely, for any closed or 1-cusped hyperbolic 3-manifold M, the analytic L^2-torsion of the corresponding covering transformations of $\mathbb{H}^3$ is

$$\rho^{(2)}(M) = -\frac{1}{6\pi}\mathrm{vol}(M).$$

Question 1 Similarly for Theorem 1, can $\frac{1}{4\pi}$ be explained in terms of L^2-torsion of covering transformations of $X(\mathcal{W})$ and of $X(\mathcal{Q})$?

2.4 Semi-regular Links

The square weave $\mathcal{W}$ and the triaxial link $\mathcal{Q}$ are two examples of a particularly nice infinite family of hyperbolic *semi-regular links*, studied in [7]. If they do not have bigons, their planar projections are semi-regular Euclidean tilings, as is the case for $\mathcal{W}$ and $\mathcal{Q}$. As a special case of equation (3), if $\mathcal{L}$ is a semi-regular biperiodic alternating link with no bigons, and the fundamental domain for L_1 contains H hexagons and S squares, then

$$\lim_{n\to\infty} \frac{\log |TH_1(X(L_n))|}{\mathrm{vol}(X(L_n))} = \frac{\mathrm{m}(p_{L_1}(z,w))}{20H\, v_{\mathrm{tet}} + 2S\, v_{\mathrm{oct}}}. \tag{4}$$

Moreover, except for the square weave, $\{X(L_n)\}$ are arithmetic if and only if the semi-regular Euclidean tiling for $\mathcal{L}$ contains only triangles and hexagons [7, Theorem 4.1].

For any hyperbolic biperiodic alternating link $\mathcal{L}$ with toroidally alternating quotient link L_1, conjecturally $\mathrm{vol}(T^2 \times I - L_1) \le 2\pi\, \mathrm{m}(p_{L_1}(z,w))$ [5, Conjecture 1]. We know of equality occurring only for $\mathcal{W}$ and $\mathcal{Q}$, which are also the only semi-regular links with totally geodesic checkerboard surfaces [7, Theorem 5.1]. Combining the inequality in [5, Conjecture 1] with this geometric characterization, we propose the following:

Conjecture 1 *For any hyperbolic biperiodic alternating link $\mathcal{L}$, with $\{X(L_n)\}$ as above*

$$\lim_{n\to\infty} \frac{\log |TH_1(X(L_n))|}{\mathrm{vol}(X(L_n))} \ge \frac{1}{4\pi},$$

with equality if and only if $\{X(L_n)\}$ have embedded totally geodesic surfaces for all n.

For example, for the Rhombitrihexagonal link $\mathcal{R}$, with fundamental domain for R_1 as shown in Fig. 2, by (4) and [5, Corollary 14] we have

$$\lim_{n\to\infty} \frac{\log |TH_1(X(R_n))|}{\mathrm{vol}(X(R_n))} = \frac{1}{4\pi}\left(\frac{10\, v_{\mathrm{tet}} + 2\pi\ \log(6)}{10\, v_{\mathrm{tet}} + 3\, v_{\mathrm{oct}}}\right) \approx \frac{1.0126}{4\pi}.$$

In [5], rigorously computed Mahler measures for several other biperiodic alternating links also imply that their corresponding limits are slightly greater than $1/4\pi$.

3 Spanning Tree Entropy for Regular Planar Lattices

Theorem 1 also provides insight into the surprising fact that each regular planar lattice has a spanning tree entropy given by the volume of a hyperbolic polyhedron (cf. [16]). Let $\tau(G)$ be the number of spanning trees of a graph G. Let $v(G)$ be the number of its vertices.

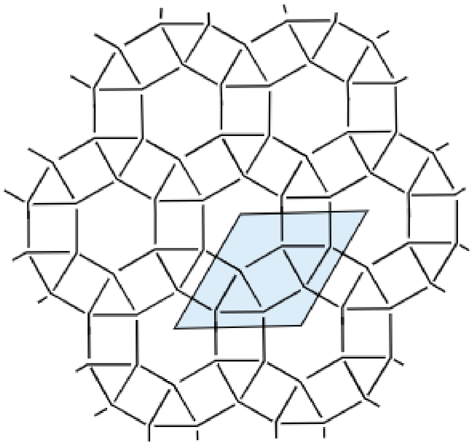

Fig. 2 Fundamental domain for the Rhombitrihexagonal link

For a biperiodic lattice $\mathcal{G}$, let G_n denote the exhaustive nested sequence of finite planar graphs $\mathcal{G} \cap (n\mathbb{Z} \times n\mathbb{Z})$. The spanning tree entropy of $\mathcal{G}$ is defined as

$$T(\mathcal{G}) = \lim_{n\to\infty} \frac{\log \tau(G_n)}{v(G_n)}.$$

Corollary 1 *The spanning tree entropies for the regular triangular, square and hexagonal lattices, $\mathcal{G}^{\triangle}$, $\mathcal{G}^{\square}$ and $\mathcal{G}^{\hexagon}$, are as follows*

$$T\left(\mathcal{G}^{\triangle}\right) = 10\, v_{tet}/2\pi, \quad T\left(\mathcal{G}^{\square}\right) = 2\, v_{oct}/2\pi, \quad T\left(\mathcal{G}^{\hexagon}\right) = 5\, v_{tet}/2\pi.$$

Proof Let H_n denote the toroidal graphs $\mathcal{G}/(n\mathbb{Z} \times n\mathbb{Z})$, such that $v(H_1) = v(G_1)$. The medial graph of H_n is the projection graph of a toroidally alternating link L_n whose Tait graph is H_n. By Lemma 2 and its proof, the spanning tree entropy of $\mathcal{G}$ satisfies

$$T(\mathcal{G}) = \lim_{n\to\infty} \frac{\log \tau(G_n)}{v(G_1)\, n^2} = \lim_{n\to\infty} \frac{\log |T H_1(X(L_n))|}{v(H_1)\, n^2}.$$

For the square lattice $\mathcal{G}^{\square}$, we use the square weave $\mathcal{W}$ with W_1 as above, so that $v(H_1^{\square}) = 2$. For the regular triangular and hexagonal lattices, $\mathcal{G}^{\triangle}$ and $\mathcal{G}^{\hexagon}$, we use the triaxial link $\mathcal{Q}$ with Q_1 as above, so that $v(H_1^{\triangle}) = 1$ and $v(H_1^{\hexagon}) = 2$. By Theorem 1 and its proof, their spanning tree entropies are as follows

$$\begin{aligned}
T\left(\mathcal{G}^{\triangle}\right) &= \lim_{n\to\infty} \frac{\log |T H_1(X(Q_n))|}{n^2} = \mathrm{m}\left(p_{Q_1}(z, w)\right) = \frac{10\, v_{\text{tet}}}{2\pi}.\\
T\left(\mathcal{G}^{\square}\right) &= \lim_{n\to\infty} \frac{\log |T H_1(X(W_n))|}{2n^2} = \frac{\mathrm{m}(p_{W_1}(z, w))}{2} = \frac{2\, v_{\text{oct}}}{2\pi}.\\
T\left(\mathcal{G}^{\hexagon}\right) &= \lim_{n\to\infty} \frac{\log |T H_1(X(Q_n))|}{2n^2} = \frac{\mathrm{m}(p_{Q_1}(z, w))}{2} = \frac{5\, v_{\text{tet}}}{2\pi}.
\end{aligned}$$

□

The mysterious 2π factor in Corollary 1 seems closely related to Question 1.

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