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THE QUEST FOR A KNOT WITH TRIVIAL JONES POLYNOMIAL: DIAGRAM SURGERY AND THE TEMPERLEY-LIEB ALGEBRA

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ABSTRACT. This article reviews several methods of altering knot and link diagrams without changing the Jones polynomial of the underlying link. The technique, which may be called diagram surgery or generalized mutation, involves removing a part of the diagram and replacing it in an altered form. In general, the resulting knot or link is different from the original. An important possible application of this technique would be to find a nontrivial knot with trivial Jones polynomial. Our point of view involves skein theory and the Temperley-Lieb algebra, and underlines the utility of these ideas.

1 Introduction

Since V. Jones introduced his new polynomial invariant of knots, about eight years ago, nobody has answered the following very basic question: Is there a nontrivial knot with trivial Jones polynomial?

I'll take this as the main motivation for discussing various methods of producing pairs of knots (or links) which have the same Jones polynomial. They are all generalizations of Conway's concept of "mutation." By analogy with gene splicing, you take part of a picture (= diagram) of a link, remove it and replace that part in a different way. Under certain circumstances, which we'll study, the new diagram represents a link which has the same Jones polynomial.

My hope is that, with patience or cleverness, one of you will find an example which gives an answer of "yes" to that question, by finding a knot whose appropriate generalized mutant is unknotted.

Some of the ideas I'll discuss also apply to the Alexander polynomial, the so-called HOMFLY polynomial, the Kauffman polynomial, etc. For simplicity, I'll concentrate on the Jones polynomial. You can check the literature, or figure out for yourself, which methods also apply to the other polynomials. Of course, it has been known for a long time that there are plenty of knots with Alexander polynomial equal to 1; J. H. C. Whitehead gave a general method (doubling) to produce such examples. Another pair of examples will be mentioned shortly.

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Another thing I'd like to emphasize is the beauty of the ideas of skein theory. When I first learned this idea, also due to Conway, of turning the set of tangles (parts of knot diagrams) into a vector space, I thought it was a pretty wierd bit of abstract nonsense. However, the idea has proven to be extremely useful. The very well-developed ideas of linear algebra can be used to great advantage, and this algebraization of the geometry actually enables us to make explicit calculations, which in turn have concrete geometric consequences. Since vector spaces have bases, and linear transformations and pairings correspond to matrices, one can use skein theory to reduce certain questions to a simple check of a finite number of things. This is the heart of several of our arguments. The 3-manifold invariants discussed by H. Morton in this meeting give an even more convincing illustration of the power of skein theory.

2 The Jones polynomial and the Kauffmann bracket

By now, you must be familiar with this, so a quick review should suffice. Kauffman's beautiful and elementary construction of the Jones polynomial is through the bracket $\langle D\rangle$ of a planar diagram of a link. I refer you to Kauffman's book [K] for further details. Consider the ring $Z[A^{\pm 1}]$ of (Laurent) polynomials in a variable A. Let D be a diagram of a link, which as usual is a curve or set of curves in the plane with only double-point self-crossings, and an indication at each such crossing of which strand is to be the underpass in the third dimension. As usual, we indicate this by a little gap. The bracket $\langle D\rangle$ is defined by the axioms:

Axiom 1: 

Axiom 2: $\langle D \sqcup O\rangle = \delta\langle D\rangle$ where $\delta = -A^2 - A^{-2}$

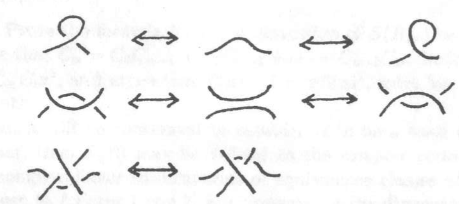
together with the stipulation that the bracket of the empty diagram equals 1. These axioms insure that $\langle D\rangle$ is invariant, not only under ambient isotopy of the plane, but also the Reidemeister moves of type II and III, which (by definition) generate the relation of regular isotopy of diagrams.

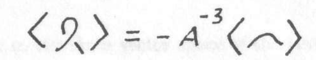
The Reidemeister moves:

I

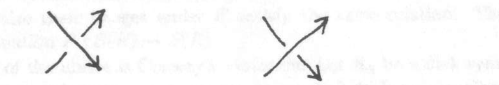
II

III



An easy calculation shows that 

so we get an invariant of all three Reidemeister moves (and hence an ambient isotopy invariant of the link L that D represents) by *orienting* the curves and calculating the writhe $w(D)$ as the algebraic sum of the crossings, counting signs ϵ according to the convention



$\epsilon = +1$ $\epsilon = -1$

and then defining $f_L(A) = (-A^{-3})^{-w(D)}\langle D\rangle$, the normalized bracket invariant. (From now on we will drop the distinction between a link L and a diagram D representing it.) Kauffmann showed that this is the same as the Jones polynomial $V_L(t)$, up to a change of variable $t = A^{-4}$ and a factor of δ. That is,

$$(-t^{1/2} - t^{-1/2})V_L(t) = f_L(t^{-1/4})$$

A key observation is that if a knot or (oriented) link diagram is modified in such a way that its bracket *and* its writhe are unchanged, then its Jones polynomial is also unchanged.

3 Skein theory, mutants and the Temperley-Lieb algebra

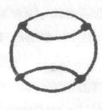
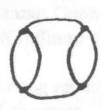
There are various skein theories for classical knots and links, corresponding to the Conway polynomial (the original version), the HOMFLY plynomial and other invariants which can be defined by "skein relations" such as the axioms defining the bracket. Moreover, the idea can be generalized to skein theory of 3-manifolds (maybe also higher dimensions) and bears a strong similarity with the topological quantum field theories which are currently being developed. Again for simplicity, I'll only discuss the skein theory corresponding to the Kauffman bracket, and planar diagrams of classical links.

A *room* R is a region of the plane (the boundary, assumed polygonal, may be empty or disconnected), together with an even number of marked points on the boundary. An *inhabitant* is a diagram in R (part of a link diagram) whose boundary is precisely the set of marked points. Two inhabitants of the room R are called *equivalent* if they are related by an isotopy of R fixed on the boundary and a finite number of Reidemeister moves of type II or III, i. e., regular isotopy within R. Let $\mathcal{M}(R)$ denote the free $Z[A^{\pm 1}]$-module generated by all equivalence classes of inhabitants of R. Define the skein module $\mathcal{S}(R)$ to be the quotient of $\mathcal{M}(R)$ modulo all equations among inhabitants, of the type stated in Axioms 1 and 2 of the previous section. Only now we imagine the brackets to be erased, and the equations asserting a relation among the inhabitants themselves, as "vectors" in $\mathcal{M}(R)$. That is, $\mathcal{S}(R) = \mathcal{M}(R)/\mathcal{I}(R)$, where $\mathcal{I}(R)$ is the 2-sided ideal generated by all elements which are differences between the left-hand and right-hand side of an equation given in Axiom 1 or 2. Axiom 1, as usual, involves inhabitants which are identical except in a neighbourhood of the crossing depicted. We interpret Axiom 2 as applying only if the unknotted curve O bounds a disk in R.

Examples: 1. If R is the entire plane, then $\mathcal{S}(R)$ is one-dimensional, with basis the empty link $\emptyset$. Any link L can be expressed $L = \langle L\rangle\emptyset$.

2. Similarly, if R_2 is a disk with 2 marked points, $\mathcal{S}(R_2)$ has a basis consisting of an arc in R connecting the two points.

3. If $R_4 =$ the disk, with 4 marked points, then $\mathcal{S}(R_4)$ is the free module with basis consisting of the inhabitants:

 and 

4. For $R_6 =$ disk with 6 marked points, $\mathcal{S}(R_6)$ is 5-dimensional with basis

1 $\quad e_1 \quad e_2 \quad e_1e_2 \quad e_2e_1$

5. The skein theory of the disk R_{2n} with $2n$ marked points has basis consisting of all (equivalence classes of) inhabitants with no crossings, and its dimension is the Catalan number $C_n = 2n(2n-1)\cdots(n+2)/n!$

6. If R is an annulus with no marked points, then $\mathcal{S}(R)$ has a countably infinite basis, the k-th basis element consisting of k parallel copies of disjoint curves which go around the hole, $k = 0, 1, 2, \ldots$

Problem: Prove the formula for $C_n =$ dimension of $\mathcal{S}(R_{2n})$ as follows. Consider $C_0 = 1$ and observe that $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$. Define a (formal) power series $f(x) = \sum_{i=0}^{\infty} C_i x^i$, and argue that $f(x) - 1 = xf(x)^2$, solve for f and deduce the form of its coefficients.

Sometimes it will be convenient to consider A to be a fixed nonzero complex number. If we do that, then $\mathcal{S}(R)$ may be defined as the *complex vector space* formed by taking all formal complex linear combinations of equivalence classes of inhabitants, modulo the relations given in Axioms 1 and 2. For "generic" A the dimensions of the above examples, as complex vector spaces, are as stated. However, there are exceptions: for example if A

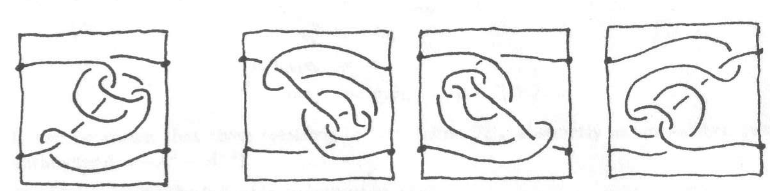
is a fourth root of -1, then $\delta = 0$ and so the skein vector space of the plane becomes zero dimensional.

Consider a room R and its complementary room R', so that $R \cup R'$ is the whole plane and $R \cap R' = \partial R = \partial R'$ and the marked points on the boundary of the rooms are the same. Then if D and D' are inhabitants of R and R', respectively, $D \cup D'$ is a link diagram. By extending linearly to the skein modules (or vector spaces), this construction defines a bilinear pairing:

$$\mathcal{S}(R) \times \mathcal{S}(R') \to \mathcal{S}(\text{plane}) = \text{scalars}$$

Again consider a room R and a function F taking inhabitants of R to inhabitants of R. Suppose the function has the property that whenever inhabitants of R satisfy a skein relation, then also their images under F satisfy the same relation. Then this induces a linear transformation $F : \mathcal{S}(R) \to \mathcal{S}(R)$.

An example of the above is Conway's *mutation*. Let R_4 be a disk symmetric under 180 degree rotations in either the x-axis, y-axis, or z-axis (which we visualize in the usual way as pointing respectively to the right, upwards, or pointing out of the page towards us); moreover, suppose the four marked points are also chosen to be setwise invariant under these rotations. If F denotes the operation of rotating an inhabitant in any one of these three senses (with crossings changed under the x- or y-rotation, as if the diagram were three-dimensional) we get a linear transformation $F : \mathcal{S}(R) \to \mathcal{S}(R)$.



Rotating a tangle: about x-axis $\quad y$-axis $\quad z$-axis.

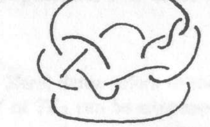
Since the two basis elements are invariant under each of these three rotations, we conclude that:

Proposition 1 *If $F : \mathcal{S}(R_4) \to \mathcal{S}(R_4)$ is induced by one of the three rotations described above, then F is the identity transformation.*

Mutation of a knot or link L consists of locating a room R_4 in a diagram for L, so that $T = L \cap R_4$ is an inhabitant of R_4 (also called a *tangle*), and replacing T in the diagram by the tangle $F(T)$, to form L'. L and L' are called *mutants*.

Example: A well-known pair of mutants are the Kinoshita-Terasaka knot and Conway's

11 crossing knot, both of which have Alexander polynomial equal to one:



The Conway knot

The Kinoshita-Terasaka knot

Their common Jones polynomial is $V(t) = t^{-6} - 2t^{-5} + 2t^{-4} - 2t^{-3} + t^{-2} + 2t - 2t^2 + 2t^3 - t^4$.

Because of proposition 1, and the fact that (when orientations are unchanged outside the room) the writhe does not change under mutation, we conclude the following.

Proposition 2 *Mutant links have the same Jones polynomial.*

As Conway observed, mutants also have the same Conway, or Alexander, polynomial as well. They also have equal HOMFLY and Kauffman polynomials, and the proof is essentially the same.

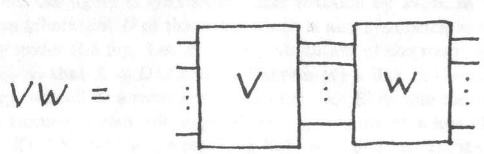
Alas, this result is useless for our strategy of altering an unknot to get a knot with trivial Jones polynomial, because of the following "folk" theorem.

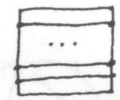
Proposition 3 *Any mutant of an unknot is itself unknotted.*

One way to see this involves considering the knot K to be in S^3, and letting M_K^3 be the two-fold branched covering of S^3 branched over K. The preimage of a thickened room R_4 enclosing a tangle of K is a solid torus upstairs in M_K. Mutation from K to K' can be lifted to a surgery on M_K, but one can check that the surgery is really a trivial surgery. Therefore, M_K and M'_K are homeomorphic if K and K' are mutants. Proposition 3 follows from this observation, together with the fact (the $Z/2$ Smith conjecture proved by Waldhausen) that K is unknotted if and only if M_K is S^3.

Questions: What happens to $\mathcal{S}(R_4)$ under a *reflection* in an appropriate plane, instead of a rotation? If we performed such an operation — a "reflective" mutation — would the Jones polynomial be invariant?

One can define a multiplicative structure on $\mathcal{S}(R_{2n})$ by the following operation:



Clearly the element $\mathbf{1} =$  is an identity for this product.

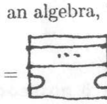
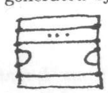
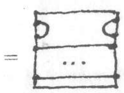
Then, for any fixed nonzero complex value of A, we have that the complex vector space $\mathcal{S}(R_{2n})$ is an ***algebra***, which is called the n^{th} Temperley-Lieb algebra, $\mathcal{TL}_n$. It's actually a family of algebras depending on the parameter A as well as the positive integer n. This version of the Temperley-Lieb algebras is due to Kauffmann (see [K]) and many of their fascinating properties are being discussed by H. Morton at this workshop. The multiplication in $\mathcal{TL}_1$ corresponds to connected sum of knots (and multiplication of their bracket invariants) and is commutative. You can also easily verify the following.

Proposition 4 *Multiplication in $\mathcal{TL}_2$ is also commutative.*

Problem: Show that $\mathcal{TL}_n, n \geq 3$ is a non-commutative algebra.

There are inclusions $\mathcal{TL}_n \subset \mathcal{TL}_{n+1}$, by adjoining an extra horizontal strand at (say) the top.

As an algebra, $\mathcal{TL}_n$ is generated by $\mathbf{1}$ and the $n-1$ elements:

$e_1 =$  $e_2 =$  $\cdots$ $e_{n-1} =$ 

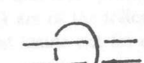
They satisfy the relations:

$$\begin{aligned} e_i^2 &= \delta e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i, \qquad |i-j| > 1 \end{aligned}$$

It can be shown that these relations in fact define $\mathcal{TL}_n$ abstractly as an algebra (with parameter $\delta = -A^2 - A^{-2}$).

Problem: Verify the following equations in $\mathcal{TL}_2$:

 $= A^2 \mathbf{1} + (1 - A^{-4})e_1$

 $= (-A^6 - A^{-6})\mathbf{1} + (1 - A^{-4})(1 - A^4)e_1$

Calculate:

 and 

as sums of products of the generators in $\mathcal{TL}_2$ and $\mathcal{TL}_3$.

Proposition 5 *A dense set of elements of $\mathcal{TL}_2$ have multiplicative inverses.*

Here, dense refers to the topology as a complex vector space. In fact, a typical element W of $\mathcal{TL}_2$ can be expressed $W = w_0\mathbf{1} + w_1\mathbf{e}_1$. If $X = x_0\mathbf{1} + x_1\mathbf{e}_1$, then we calculate

$$WX = w_0x_0\mathbf{1} + (w_0x_1 + w_1x_0 + \delta w_1x_1)\mathbf{e}_1.$$

We can solve $WX = 1$ by setting $x_0 = 1/w_0$ and $x_1 = -w_1/w_0(w_0 + \delta w_1)$. To do this, of course, one must avoid having $w_0 = 0$ or $w_0 + \delta w_1 = 0$, but those are just lines in complex 2-space.

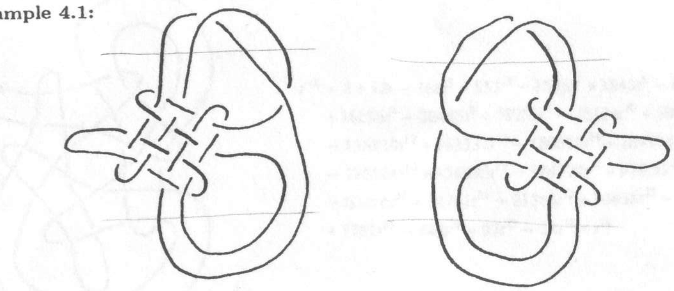
This proposition really needs the use of coefficients in a field, rather than the ring $Z[A^{\pm 1}]$, which has very few invertible elements. We can also see that, assuming $w_0 \neq 0$, the invertibility of W is assured for all but a finite number of values of the parameter A. Note also that if W is represented by an actual tangle, its inverse will probably not be a tangle, but rather will be a formal linear combination of tangles. Nevertheless, this algebraic device will have strictly geometric consequences, as we shall see later.

Question: Is Proposition 5 true for $\mathcal{TL}_n, n > 2$?

4 Rotants

I will now describe another method, a generalized mutation, discovered by R. Anstee, J. Przytycki and myself [APR], for constructing pairs of knots with the same Jones polynomial. Anstee is a combinatorist who showed us a trick of W. T. Tutte to produce different graphs with equal polynomial invariants — the chromatic and Tutte polynomials. We adapted Tutte's method as follows. Forgetting the Temperley-Lieb algebra for the moment, picture the room $R_{2n}, n \geq 3$ as a regular n-gon with a pair of marked points on each edge, so that the figure is symmetric under rotation by $2\pi/n$, as well as the dihedral flip. Consider an inhabitant D of this room which is also symmetric under the rotation, but *not* necessarily under the flip. Let E be *any* inhabitant of the room R'_{2n} which is complementary to R_{2n} so that $L = D \cup E$ is (a diagram of) a link in the plane. Borrowing Tutte's terminology, we call D a *rotor* and E a *stator*. Let E' denote the result of flipping over the stator, by turning it over 180 degrees (in 3-space) about a line of symmetry of R_{2n}. Let $L' = (L - E) \cup E'$ denote the resulting link; we call L' a *rotant* of L of order n. (Maybe "flippant" would be a better term, but that sounds too... well, flippant.) It is crucial to note that the choice of axis for flipping is immaterial, up to ambient isotopy, due to the symmetry of the rotor. Also note that we could flip over the rotor instead and the result would be ambient isotopic to L'.

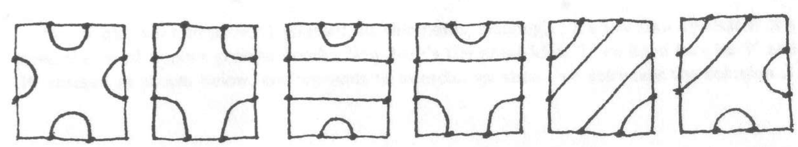
Example 4.1:



Rotants of order four

Proposition 6 *If the links L and L' are rotants of order n, $3 \leq n \leq 5$, then they have the same bracket invariant. If L and L' are oriented and have the same writhe, then their Jones polynomials agree.*

If an orientation of L orients the rotor in a rotationally invariant manner, then writhe is preserved automatically under the flip. The proof of the proposition, using skein theory, goes as follows. Fix a rotor D, and let $G : \mathcal{S}(R'_{2n}) \to \mathcal{S}(\text{plane})$ denote the linear map defined by taking an inhabitant E of R' and forming the inhabitant $G(E) = D \cup E$. Similarly define $H : \mathcal{S}(R'_{2n}) \to \mathcal{S}(\text{plane})$ by defining $H(E) = D \cup E'$, where E' is E after a dihedral flip as above. We argue that G and H are equal as linear maps, by checking on a basis of $\mathcal{S}(R'_{2n})$. Indeed, if you examine a basis for $\mathcal{S}(R'_{2n})$, you will see that every basis element has a dihedral symmetry, provided $n \leq 5$. For example, the fourteen basis elements of $\mathcal{S}(R'_8)$ are of the following six types, together with their rotated versions (the stator R'_8 is turned inside out for easier visualization):



Example 4.1, continued The rotants L and L' pictured above, which are each 2-component links, have Jones polynomial $V_L(t) = V_{L'}(t) =$

$$\begin{aligned} &-t^{-41/2} + 7t^{-39/2} - 26t^{-37/2} + 68t^{-35/2} - 139t^{-33/2} \\ &+ 237t^{-31/2} - 348t^{-29/2} + 450t^{-27/2} - 518t^{-25/2} \\ &+ 533t^{-23/2} - 494t^{-21/2} + 410t^{-19/2} - 302t^{-17/2} \\ &+ 195t^{-15/2} - 109t^{-13/2} + 50t^{-11/2} - 19t^{-9/2} + 5t^{-7/2} - t^{-5/2}. \end{aligned}$$

However, their Kauffmann polynomials are different, showing that one can not go from L to L' by a sequence of mutations in the sense of Conway (section 2).

Problem: Verify that all 42 basis elements of $\mathcal{S}(R'_{10})$ have an axis of symmetry. Find, on the other hand, a generator of $\mathcal{S}(R'_{12})$ which cannot be represented by a tangle with a symmetry axis, and so the above argument breaks down.

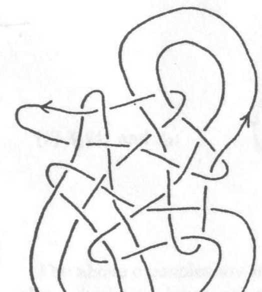
It is shown in [APR] that Proposition 6 holds for the HOMFLY polynomial if $n \leq 4$ and the Kauffmann polynomial for $n = 3$. The links of Example 4.1, and other examples presented in a paper [JR] by G. T. Jin and myself show these bounds on n are the best possible.

Problem: I don't know the answer to this one or the next — they're good questions for you graduate students to have a crack at. If L and L' are rotants of *arbitrary* order n, then do their Alexander polynomials agree?

Problem: Is a rotant of an unknot necessarily unknotted (at least for $n \leq 5$)? If so, using rotants as a strategy for producing a knot with $V(t) = 1$ is thwarted.

Here are examples of 6-rotants, from [JR], which have different Jones polynomials, although they are tantalizingly close, being:

$$\begin{aligned} &t^{-1} - 8 + 42t - 168t^2 + 552t^3 - 1555t^4 + 3846t^5 - 8481t^6 \\ &+ 16863t^7 - 30459t^8 + 50275t^9 - 76164t^{10} + 106279t^{11} \\ &- 136966t^{12} + 163352t^{13} - 180517t^{14} + 184917t^{15} \\ &- 175495t^{16} + 154062t^{17} - 124748t^{18} + 92778t^{19} \\ &- 63004t^{20} + 38756t^{21} - 21367t^{22} + 10408t^{23} - 4392t^{24} \\ &+ 1561t^{25} - 448t^{26} + 97t^{27} - 14t^{28} + t^{29} \end{aligned}$$

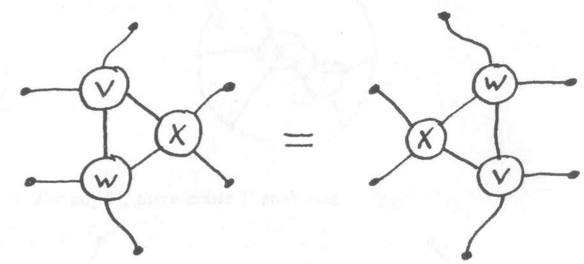


$$\begin{aligned} &t^{-1} - 8 + 42t - 168t^2 + 552t^3 - 1555t^4 + 3845t^5 - 8478t^6 \\ &+ 16856t^7 - 30445t^8 + 50253t^9 - 76134t^{10} + 106247t^{11} \\ &- 136939t^{12} + 163337t^{13} - 180519t^{14} + 184934t^{15} \\ &- 175524t^{16} + 154095t^{17} - 124778t^{18} + 92800t^{19} \\ &- 63016t^{20} + 38761t^{21} - 21368t^{22} + 10408t^{23} - 4392t^{24} \\ &+ 1561t^{25} - 448t^{26} + 97t^{27} - 14t^{28} + t^{29} \end{aligned}$$

5 Jones' trick

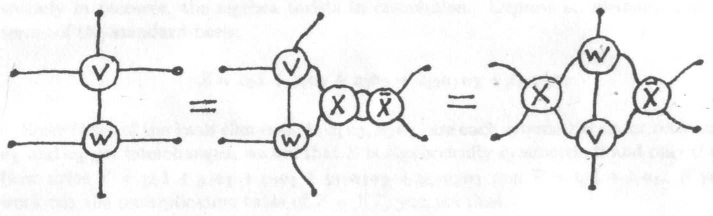
In a very recent paper [J], Jones described yet another generalization of mutation, and showed that this can be used to explain duplications in the Jones polynomial of several examples with quite low crossing number.

Proposition 7 *Let V and W be tangles, i.e., inhabitants of the room R_4. Then there exists an element X of $\mathcal{S}(R_4)$, such that the following diagrams are equal, as elements of $\mathcal{S}(R_6)$:*



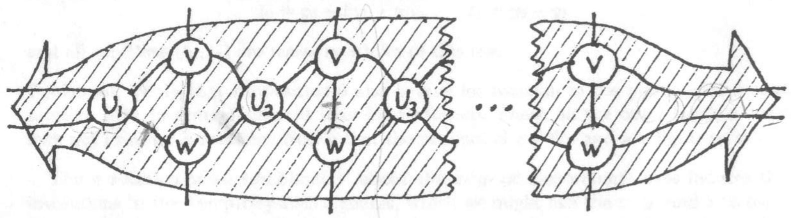
You might want to prove it yourself, at this point, although I'll show shortly that it is a special case of a more general result. Now here's the great idea. If we have tangles V and W stacked as shown below, and we want to interchange them, we *introduce* the solution X

to the above, *together with its inverse* $\overline{X}$ in $\mathcal{TL}_2$, and notice the equalities in $\mathcal{S}(R_6)$:



Now suppose the rest of the diagram is arranged so that X can slide around by an ambient isotopy around to the other side. Then it ends up next to $\overline{X}$ again and they annihilate, again not affecting the skein class. Also note that in its journey, X can pass through other V over W configurations, interchanging them as well, and that X can also pass through any tangle in its path, as multiplication in $\mathcal{TL}_2$ is commutative. This argument establishes the following.

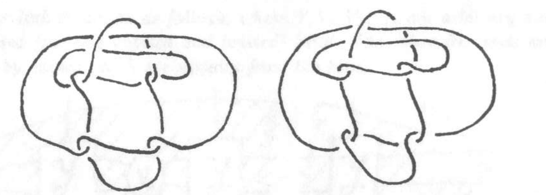
Proposition 8 (Jones) *Consider a link of the form:*



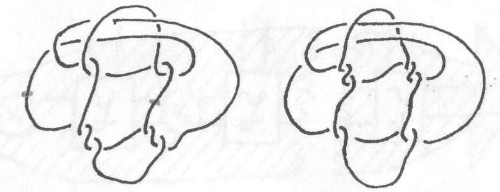
where the V, W and the U_i are arbitrary tangles. Suppose the shaded region is connected to itself, as indicate by large arrows, to form a band (which may itself be knotted) and that the other loose ends of the picture are connected in 3-space in any manner, provided that they do not pass through the band. Then one can interchange all the V tangles with all the W tangles, and the resulting link will have the same Jones polynomial as the original link.

Examples: The following pairs of knots have identical Jones polynomials:

(1) 8_8 and 10_{129}:



(2) $4_1\sharp 4_1$ and 8_9:



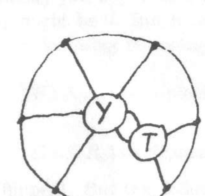
The above examples are in Jones' paper [J], along with five pairs of ten crossing knots whose duplicate Jones polynomials can be explained by this phenomenon. Notice that this type of generalized mutation can change the (minimum) crossing number and turn a composite knot into a prime one.

6 Wheel balancing and symmetries in $\mathcal{TL}_3$

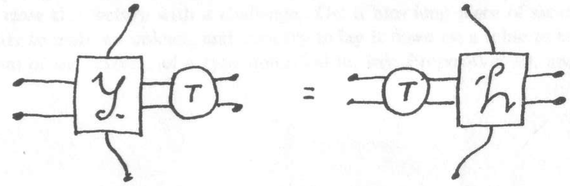
The ideas in this section are due to Hoste and Przytycki [P], although I am adopting a somewhat different approach than they use.

We will call an element Z of the skein of the room R *algebraically symmetric* with respect to a geometric motion f of the room, if $Z = f(Z)$ in $\mathcal{S}(R)$.

Proposition 9 (Wheel balancing) *Let Y be any inhabitant of the room R_6, then there exists T in $\mathcal{S}(R_4)$ such that the following diagram is algebraically symmetric with respect to 180 degree rotation (about the z-axis) of R_6 in the plane.*



Corollary 1 *For any Y, there exists T such that:*



Proof: Athough the T gets turned over in the rotation, it can right itself by mutation.

You can see that Proposition 7 follows from 9 in the same way.

Let's prove proposition 9 using the Temperley-Lieb algebra. Although one could deal entirely in pictures, the algebra assists in calculation. Express an element Z of $\mathcal{TL}_3$ in terms of the standard basis:

$$Z = z_0\mathbf{1} + z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + z_{12}\mathbf{e}_1\mathbf{e}_2 + z_{21}\mathbf{e}_2\mathbf{e}_1.$$

Since three of the basis elements, $\mathbf{1}, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_1$, are each symmetric under rotation, while $\mathbf{e}_1$ and $\mathbf{e}_2$ get interchanged, we see that Z is algebraically symmetric if and only if $z_1 = z_2$. Now write $Y = y_0\mathbf{1} + y_1\mathbf{e}_1 + y_2\mathbf{e}_1 + y_{12}\mathbf{e}_1\mathbf{e}_2 + y_{21}\mathbf{e}_2\mathbf{e}_1$ and $T = t_0\mathbf{1} + t_1\mathbf{e}_1$. If you then work out the multiplication table of $Z = YT$, you see that

$$z_1 = y_1t_0 + y_0t_1 + \delta y_1t_1 + y_{12}t_1$$

and

$$z_2 = y_2t_0.$$

Therefore Z is symmetric iff $(y_2 - y_1)t_0 = (y_0 + \delta y_1 + y_{12})t_1$. Clearly, given Y, we always have the solution

$$t_0 = y_0 + \delta y_1 + y_{12}, \qquad t_1 = y_2 - y_1$$

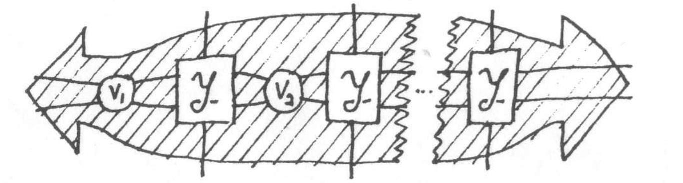
and all solutions (t_0, t_1) are scalar multiples of this one.

Problem: Verify that Proposition 9 also is true for rotation in the y-axis, and that the solution T is, in general different than for z-rotation. Show, on the other hand, solutions may not exist for x-rotation. Notice that the product of a y-flip and a z-flip is an x-flip.

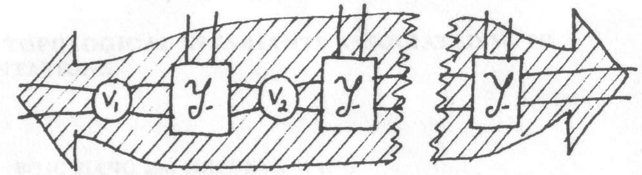
The π rotation of an inhabitant V about the x- y- or z-coordinate axes induces three involutions in the Temperley-Lieb algebras, which we might call the x- y- and z-transpose V^x, V^y, V^z. Each is a linear isomorphism and x-transposition is an algebra automorphism as well, whereas y- or z-transposition (as with transpose of matrices) is antimultiplicative. In this terminology, we see that any element of $\mathcal{TL}_3$ can be y- or z-symmetrized by multiplication (on either side) by a suitable element of $\mathcal{TL}_2 \subset \mathcal{TL}_3$. Similar results hold more generally, which is work still in progress. Notice that *all* the algebra generators of $\mathcal{TL}_n$ are y-symmetric; in general, the y-transpose of a sum of products of the $\mathbf{e}_i$ is formed by reversing the order of the products.

Proposition 10 *Consider link diagrams as follows, where $Y, V_1, V_2, \ldots$ are arbitrary and the shading denotes a closed (possibly knotted and twisted) band. The other free ends are connected in any manner by curves which are disjoint from the band:*

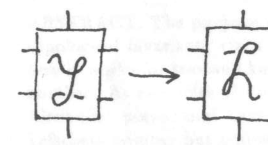
Type 1:



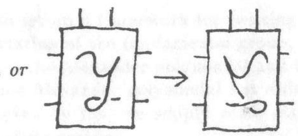
Type 2:



Then the Jones polynomial of the resulting link is the same as the original if one replaces all pictured occurences of Y as follows



for type 1, or



for type 2.

The proof is to use the Jones trick, introducing cancelling tangles T and $\overline{T}$ in the band and sending the T once around, flipping the Y's as it passes them, and finally annihilating with $\overline{T}$. The more skeptical among you might have noticed that this doesn't work if T happens to be noninvertible: t_0 might be 0. But it does work for a dense subset of Y in $\mathcal{S}(R_6)$, and that's enough, by the following reasoning. We are really trying to establish that two linear mappings:

$$F : \mathcal{S}(R_6) \to \mathcal{S}(\text{plane})$$

and

$$G : \mathcal{S}(R_6) \to \mathcal{S}(\text{plane})$$

are equal, where $G(Y) = F(Y\text{flipped})$. But this follows since we have demonstrated they agree on a dense set in the domain. Another triumph of abstract nonsense!

There are further tricks to be done, such as replacing some of the Y tangles in the original link by another Y' such that $Y'T$ and YT are simultaneously symmetric, illustrating the principle: generalized mutants are bountiful!

I'll close this lecture with a challenge. Get a nice long piece of string, fasten the ends together to make an unknot, and then try to lay it down on a table to make a complicated diagram of the unknot, of a type described in, say, Proposition 10, and with the further property that when you perform the appropriate generalized mutation, the resulting curve is really tied in a knot. It will be the desired knot with $V(t) = 1$ and will make you famous, at least among topologists!

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TWISTED TOPOLOGICAL INVARIANTS ASSOCIATED WITH REPRESENTATIONS.

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ABSTRACT. The purpose of this note is to set up a framework for twisting the classical topological invariants via a matrix representation of the fundamental group, and to show how it works for two well known invariants — the Alexander polynomial and the Lefschetz number. As examples of knots with the same Alexander polynomial but different twisted Alexander polynomial have already been given by Lin, we supply some maps with zero Lefschetz number but non-zero twisted Lefschetz number.

0. Introduction.

The purpose of this note is to set up a framework for twisting the classical topological invariants via a matrix representation of the fundamental group, and to show how it works for two well known invariants — the Alexander polynomial and the Lefschetz number.

The Alexander polynomial $A(K)$ for a knot K in S^3 was introduced by Alexander (1928). Reidemeister (1934) introduced the Reidemeister torsion invariant for manifolds. Milnor [M1] noticed a relation between the Alexander polynomial and a certain Reidemeister torsion. A systematic study of this relation is carried out by Turaev [T].

The Lefschetz number $L(f)$ for a self-map f of a manifold M was introduced by Lefschetz (1923). Weil (1949) (cf. [B]) introduced the zeta function $\zeta(f) = \exp \sum_n L(f^n)t^n/n$ and proved that $\zeta(f) = \prod_q p_q(t)^{(-1)^{q+1}}$, where each $p_q(s)$ is a polynomial closely related to the characteristic polynomial of the linear transformation $f_{*q} : H_q(M, \mathbb{Q}) \to H_q(M, \mathbb{Q})$. Milnor [M2] noticed a relation between $\zeta(f)$ and the Reidemeister torsion of the mapping torus T_f of f. Generalizing in this direction, Fried [F] introduced a twisted Lefschetz zeta function.

Turaev and Fried define their invariants via abelian coverings, because the determinant is not well defined for square matrices in a non-abelian group ring.

Some attempts have been made to obtain stronger invariants by considering non-abelian coverings. In an unpublished note [J3] of 1987, for a selfmap $f : M \to M$ the first author defined the twisted Lefschetz zeta function $\zeta_\rho(f)$ associated with a representation $\rho : \pi_1(T_f) \to \mathrm{GL}_l(R)$ where T_f is the mapping torus of f and R is a commutative ring. In a preprint [L] of 1990, for a knot K in S^3 Lin defined the twisted Alexander polynomial $A(K, \rho)$ associated with a representation $\rho : \pi_1(S^3 \setminus K) \to \mathrm{GL}_l(R)$. Two examples of knots in S^3 were given with the same classical Alexander polynomial but different twisted Alexander polynomials. Lin's definition of $A(K, \rho)$ was based on the special fact that there is a

M.E. Bozhüyük (ed.), Topics in Knot Theory, 211–227.