## Sample problems for Linear Algebra, Spring 2016, Exam 2

**Problem 1.**

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & -1 & 1 & 3 & 1 \\ 2 & -1 & 3 & 5 & 9 \end{bmatrix}$$

**(a)** Find the reduced row echelon form for $A$.

**(b)** Find the dimensions of the four fundamental spaces of $A$.

**(c)** Find a basis for each of the four fundamental spaces of $A$.

**(d)** Find the complete solution to $A\mathbf{x} = [1, 1, 1]^T$.

**Problem 2.**

$$S = \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$$

**(a)** Show that $S$ is a basis for $\mathbf{R}^2$.

**(b)** Express $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of these basis vectors.

**Problem 3.** Let $X$ and $Y$ be the following sets of vectors:

$$X = \{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \}, \qquad Y = \{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \}$$

**(a)** Are the vectors in $X$ linearly independent? Justify.

**(b)** Do the vectors in $X$ span $\mathbf{R}^3$? Justify.

**(c)** Find a subset of $X$ which is a basis for span($X$). Justify.

**(d)** Are the vectors in $Y$ linearly independent? Justify.

**(e)** Do the vectors in $Y$ span $\mathbf{R}^3$? Justify.

**(f)** Find a subset of $Y$ which is a basis for span($Y$). Justify.

**Problem 4.** **(a)** Let $S$ be a spanning set for $\mathbf{R}^{100}$ which is not a basis of $\mathbf{R}^{100}$. How many vectors can $S$ contain?

**(b)** Let $S$ be a set of linearly independent vectors in $\mathbf{R}^{100}$ which is not a basis of $\mathbf{R}^{100}$. How many vectors can $S$ contain?

**(c)** Let $\mathbf{v}$ be a vector in $\mathbf{R}^{100}$. Show that the set of all vectors perpendicular to $\mathbf{v}$ forms a subspace of $\mathbf{R}^{100}$.

**Problem 5.** Suppose $A$ is an $m \times n$ matrix such that

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has no solutions, and } A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has a unique solution.}$$

**(a)** Find the possible values of $m$, $n$, and the rank $r$ of $A$.

**(b)** Find all solutions to $A\mathbf{x} = 0$. Justify.

**(c)** Give an example of such a matrix $A$.

**Problem 6.** Suppose $A$ can be reduced to

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**(a)** Find the dimensions of the four fundamental spaces of $A$.

**(b)** Find a basis for each of the four fundamental spaces of $A$. State if there is not enough information to answer.

**Problem 7.** The following matrix depends on $c$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

**(a)** For each $c$ find a basis for the column space of $A$.

**(b)** For each $c$ find a basis for the nullspace of $A$.

**(c)** For each $c$ find the complete solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

**Problem 8.** Suppose $A$ is an $m \times n$ matrix with rank $r$. How are $m$, $n$, and $r$ related, and what is the nullspace of $A$ in the following situations:

**(a)** $A\mathbf{x} = \mathbf{b}$ has a unique solution.

**(b)** There are no solutions.

**(c)** There are infinitely many solutions.

**(d)** All solutions to $A\mathbf{x} = \mathbf{b}$ have the form $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

**Problem 9.** Suppose the columns of a $5 \times 5$ matrix $A$ are a basis for $\mathbf{R}^5$. Explain why the following are true:

**(a)** The only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$.

**(b)** $A\mathbf{x} = \mathbf{b}$ always has a solution.

**(c)** The rows of $A$ are also a basis for $\mathbf{R}^5$.

**Problem 10.** Suppose $A$ is an $m \times n$ matrix. Explain why the following are impossible:

**(a)** The column space of $A$ has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and the nullspace of $A$ has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

**(b)** The basis for both the row space and the column space of $A$ is $S = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \}$

**(c)** $A$ has a row $\mathbf{v} = (1, 0, -1)$ and $\mathbf{v}$ is in the nullspace of $A$.

**(d)** $A\mathbf{x} = \mathbf{b}$ has no solutions, and $A^T\mathbf{y} = 0$ has a unique solution $\mathbf{y} = 0$.