Calculus 3 Final Exam
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1. Let $\vec{u} = \langle 4, 5, -2 \rangle$ and $\vec{v} = \langle 1, 0, 10 \rangle$.
   (a) $\vec{u} + \vec{v} =$
   (b) $\vec{u} - \vec{v} =$
   (c) $\vec{u} \times \vec{v} =$
   (d) $\vec{u} \cdot \vec{v} =$
   (e) The angle between $\vec{u}$ and $\vec{v}$ is:
   (f) Give a vector that is not parallel to $\vec{v}$ that is perpendicular to $\vec{u}$.

2. Let $f(x, y) = \exp(x) + (x - y)^{10}$ and $\vec{r}(t) = \langle \sin(t), \ln(t), te^{t^2} \rangle$ and $g(x, y, z) = \cos(\frac{x^2+y}{z})$.
   (a) $\nabla f(x, y) =$
   (b) $\frac{d}{dt}\vec{r}(t) =$
   (c) Give an equation for the tangent plane to $f(x, y)$ at $(x_0, y_0) = (3/2, 1/2)$.
   (d) $\int_0^1 \vec{r}(t)\, dt =$
   (e) $\nabla g(x, y, z) =$

3. Let $R$ be the region below $x = 2y^2$ and above $y = x^2$. Write the integral

$$\iint_R (x + y)\, dA$$

   as an iterated integral, and then evaluate it.

4. Let $R$ be the region $\{(x, y) \colon -1 \le x \le 1, 0 \le y \le \alpha\}$. Compute $\iint_R (x^3 + 6y^2)\, dA$.

5. (a) Give an equation (parametric or symmetric) for the line which is the intersection of the planes $2x - y + 3z = 4$ and $5x + y + 2z = 8$.
   (b) Give an equation for the plane containing the points $(1, 1, 1)$, $(2, 2, 2)$ and $(1, 2, 1)$.

6. (a) Plot the region in the $x$-$y$ plane that is above $y = 0$, below $y = \cos(x)$, with $|x| \le \pi/2$.
   (b) Express the volume of the solid defined by $0 \le z \le e^{x+y}$, with $x$ and $y$ being in the region above $y = 0$, below $y = \cos(x)$, with $|x| \le \pi/2$, as a *triple* iterated integral.

7. The integral $\oint_C -P(x, y)\, dx + Q(x, y)\, dy$ measures the flow of the field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

   across the curve $C$.
   (a) Use Gauss/Green/Stokes to express flow across as a double integral, assuming the curve $C$ is closed.
   (b) Express the flow of $\vec{F}(x, y) = \langle xy, 1 \rangle$ across the ellipse $x^2 + y^2/4 = 1$ as a contour integral.
   (c) Express the flow of $\vec{F}(x, y) = \langle xy, 1 \rangle$ across the ellipse $x^2 + y^2/4 = 1$ as a single integral.
   (d) Express the flow of $\vec{F}(x, y) = \langle xy, 1 \rangle$ across the ellipse $x^2 + y^2/4 = 1$ as a double integral over the interior of the ellipse.

8. Answer the following True/False questions. You lose one point for each incorrectly identified or unidentified statement.
   (a) ____ If $f(x, y)$ is a function of two variables defined for all $x$ and $y$, then $f(10, y)$ is a function of one variable.
   (b) ____ The plane $x + 2y - 3z = 1$ passes through the origin.

(c) ____ The vector $\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \rangle$ is a unit vector.

(d) ____ If $\vec{u} \cdot \vec{v} < 0$, then the angle between $\vec{u}$ and $\vec{v}$ is greater than $\pi/2$.

(e) ____ The plane $x + 2y + 3z = 4$ has normal vector $\langle -1, -2, -3 \rangle$.

(f) ____ $(\vec{i} \times \vec{j}) \cdot \vec{k} = \vec{i} \cdot (\vec{j} \times \vec{k})$.

(g) ____ The function $z = u\cos(v)$ satisfies the equation $\cos(v)\dfrac{\partial z}{\partial u} - \dfrac{\sin v}{u}\dfrac{\partial z}{\partial v} = 1$.

(h) ____ At the point $(3, 0)$, the function $g(x, y) = x^2 + y^2$ has the same maximal rate of increase as that of the function $h(x, y) = 2xy$.

(i) ____ If $\vec{u}$ is tangent to the level curve of $f$ at some point, then $\nabla f \cdot \vec{u} = 0$ there.

(j) ____ An equation for the tangent plane to the surface $z = x^2 + y^3$ at $(1, 1)$ is $z = 2 + 2x(x - 1) + 3y^2(y - 1)$.

(k) ____ The directional derivative $f_{\vec{u}}(a, b)$ is parallel to $\vec{u}$.

(l) ____ The iterated integral $\int_0^1 \int_5^{12} f \, dxdy$ is computed over the rectangle $0 \le x \le 1, 5 \le y \le 12$.

(m) ____ The iterated integrals $\int_{-1}^1 \int_0^1 \int_0^{1-x^2} f \, dzdydx$ and $\int_0^1 \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} f \, dxdydz$ are equal.

(n) ____ $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$, and this fact *is* hilarious.

(o) ____ The equation $\vec{r}(t) = 3t\vec{i} + (6t + 1)\vec{j}$ parameterizes a line.

(p) ____ If a particle moves with motion $\vec{r}(t) = 3t\vec{i} + 2t\vec{j} + t\vec{k}$, then the particle stops (i.e., has speed 0) at the origin.

(q) ____ The vector field $\vec{F}(x, y) = \langle y, 1 \rangle$ is a gradient field.

(r) ____ $\vec{r}'(t) \times \vec{r}(t) = \vec{0}$.

(s) ____ Both $x = -t + 1, y = 2t$ and $x = 2s, y = -4s + 2$ describe the same line in the $x$-$y$ plane.