Calculus III (Math 233) Exam 3

December 8, 2010 Professor Ilya Kofman

Justify answers and show all work for full credit.

NAME: ____

Problem 1. Let C be the triangle in $\mathbf{R}^2$ with vertices $(0,0)$, $(1,0)$, $(1,3)$. Use Green's Theorem to evaluate

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

Problem 2. Let $F(x,y,z) = (ze^{xz},\ 0,\ xe^{xz})$. Let C be one turn of the helix,

$$C = \{\ (\cos t, \sin t, t)\ |\ 0 \le t \le 2\pi\ \}.$$

(a) Find $f(x,y,z)$ such that $F = \nabla f$.

(b) Compute $\displaystyle\int_C F \cdot d\mathbf{s}$.

Problem 3. Let $F(x,y,z) = (y^2,\ x,\ z^2)$. Let S be the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, with normal oriented upward. Verify that Stokes' Theorem is true in this case by directly evaluating both integrals.

Problem 4. Let E be the solid cylinder $x^2 + y^2 \le 1;\ 0 \le z \le 3$.
Let $F(x,y,z) = (x,\ y,\ -z)$.

(a) Directly evaluate the surface integral $\displaystyle\iint_{\partial E} F \cdot d\mathbf{S}$.

Note: ∂E consists of the cylindrical side as well as the flat top and bottom. It may help to parametrize the side by $T(\theta, z) = (\cos\theta, \sin\theta, z)$.

(b) Verify the answer above by applying one of our theorems.

Problem 5. An open bottle B lies on the xy–plane. Its volume is $750\,m\ell$. Its lip (or boundary) is the circle $\{\,x^2+(z-1)^2 = 1;\ y = 10\,\}$. Let $F(x,y,z) = (2x+y^2,\ 3,\ x^2+4)$. Compute

$$\iint_B F \cdot d\mathbf{S}.$$

Problem 6. Suppose that F is a vector field in $\mathbf{R}^3$ that is everywhere perpendicular to a surface S with boundary C. Show that

$$\iint_S (\nabla \times F) \cdot d\mathbf{S} = 0.$$

Problem 7. (Bonus) If C is the ellipse $x^2 + 4y^2 = 4$ oriented counterclockwise, compute (and justify)

$$\int_C \frac{-y\,dx + (x-1)\,dy}{(x-1)^2 + y^2}$$