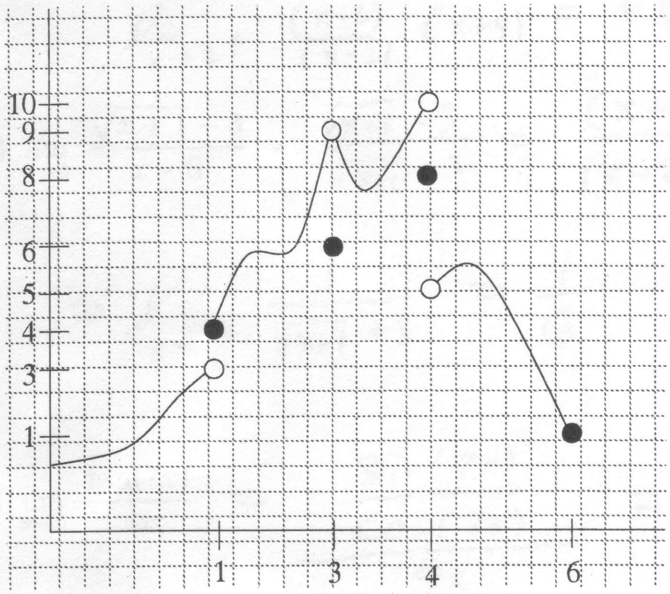
# Calculus I (Math 231) Exam 1

September 29, 2009 — Professor Ilya Kofman

Justify answers and show all work for full credit. No calculators allowed.

NAME: Key




**Problem 1** (21pts). *The graph of $y = f(x)$ is shown above. Evaluate each limit, or write DNE if the limit does not exist. No justifications are necessary.*

(a) $\lim_{x\to 1^-} f(x) =$ 3

(b) $\lim_{x\to 1^+} f(x) =$ 4

(c) $\lim_{x\to 3} f(x) =$ 9

(d) $\lim_{x\to 4^-} f(x) =$ 10

(e) $\lim_{x\to 4^+} f(x) =$ 5

(f) $\lim_{x\to 6} f(x) =$ DNE [Since $f(x)$ not defined for $x > 6$.]

(g) *For which $x$ does $f(x)$ have a removable discontinuity?*

$x = 3$

**Problem 2** (50pts). *Evaluate these limits. For an infinite limit, write $+\infty$ or $-\infty$. Otherwise, if a limit does not exist (DNE), you must justify.* **Show all work!**

**(a)** $\lim_{x\to 2} \dfrac{x^2+2x-8}{|x-2|} = \dfrac{(x-2)(x+4)}{|x-2|}$

$$\left.\begin{array}{l} \lim_{x\to 2^-} \dfrac{(x-2)}{|x-2|}(x+4) = -(2+4) = -6 \\ \lim_{x\to 2^+} \dfrac{(x-2)}{|x-2|}(x+4) = +(2+4) = +6 \end{array}\right\} \Rightarrow \underline{DNE}$$

**(b)** $\lim_{x\to 8} \dfrac{\sqrt{x+1}-3}{x-8} \cdot \dfrac{\sqrt{x+1}+3}{\sqrt{x+1}+3} = \dfrac{(x+1)-9}{(x-8)(\sqrt{x+1}+3)} = \dfrac{x-8}{x-8} \cdot \dfrac{1}{\sqrt{x+1}+3}$

$$= \lim_{x\to 8} \frac{1}{\sqrt{x+1}+3} = \frac{1}{6}$$

**(c)** $\lim_{x\to 0} \dfrac{\tan(7x)}{4x} = \dfrac{\frac{\sin(7x)}{\cos(7x)}}{4x} = \underbrace{\dfrac{1}{\cos(7x)}}_{\text{as } x\to 0 \ \downarrow\ 1} \cdot \underbrace{\dfrac{\sin(7x)}{7x}}_{\downarrow\ 1} \cdot \dfrac{7}{4}$

$$\lim_{x\to 0} \frac{\tan(7x)}{4x} = \frac{7}{4}$$

**(d)** $\lim_{x\to 1} (x-1)^2 \sin\left(\dfrac{\pi}{x-1}\right)$

$$\underbrace{-(x-1)^2}_{\text{as } x\to 1\ \downarrow\ 0} \le (x-1)^2 \sin\left(\tfrac{\pi}{x-1}\right) \le \underbrace{(x-1)^2}_{\text{as } x\to 1\ \downarrow\ 0}$$

By Squeeze Thm, $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{\pi}{x-1}\right) = 0$

**(e)** $\lim_{x\to 0^+} \left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x^2+x}}\right) = \dfrac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}}$

$$\frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{(x+1)-1}{\sqrt{x}\sqrt{x+1}(\sqrt{x+1}+1)} = \frac{\sqrt{x}}{\sqrt{x+1}(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x+1}(\sqrt{x+1}+1)} = \frac{0}{1\cdot 2} = 0$$

**Problem 3** (12pts). *For what value of c (if any) is the function $f(x)$ continuous at $x = 1$? Justify your answer.*

$$f(x) = \begin{cases} x + \dfrac{3}{x-2} & x < 1 \\ c & x = 1 \\ \dfrac{\cos(\pi x)}{x} & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(x + \frac{3}{x-2}\right) = 1 + \frac{3}{-1} = -2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{\cos(\pi x)}{x} = \frac{\cos(\pi)}{1} = -1$$

Jump discontinuity at $x=1$, so no such $c$ exists.

**Problem 4** (8pts). *For a sphere of radius R, its surface area $S = 4\pi R^2$. What is the average ROC of the surface area when the radius is increasing from $R = 1$ to $R = 3$?*

$$\text{avg ROC} = \frac{S(3) - S(1)}{3-1} = \frac{4\pi \cdot 9 - 4\pi \cdot 1}{2}$$

$$= \frac{32\pi}{2} = 16\pi$$

**Problem 5** (12pts). *Show that $e^x = 2\cos(x)$ has a solution for some $x > 0$. You do not need to find this solution.* ($e \approx 2.718$)

Let $g(x) = e^x - 2\cos(x)$

Solution exists if $g(x) = 0$ for some $x > 0$.

1. $g(x)$ is continuous (difference of continuous functions)
2. $g(0) = e^0 - 2\cos(0) = 1 - 2 = -1 < 0$
3. $g(1) = e^1 - 2\cos(1) \geq e - 2 > 0$

   [or $g(\pi) = e^\pi - 2\cos(\pi) = e^\pi + 2 > 0$]

By Intermediate Value Thm, $g(x) = 0$ for some $0 < x < 1$.