$$X - P = \langle X - P, N\rangle N + \langle X - P, v\rangle v + \langle X - P, w\rangle w,$$

which shows that $X - P$ lies in $[v, w]$ if and only if $\langle X - P, N\rangle = 0$. Thus, $\{X | \langle X - P, N\rangle = 0\}$ is a plane. □

Incidence geometry of the sphere

The sphere $\mathbf{S}^2$ on whose geometry we will be concentrating is determined by the familiar condition

$$\mathbf{S}^2 = \{x \in \mathbf{E}^3 | \ |x| = 1\}.$$

88 If one begins at a point of $\mathbf{S}^2$ and travels straight ahead on the surface, one will trace out a great circle. Viewed as a set in $\mathbf{E}^3$, this is the intersection of

$\mathbf{S}^2$ with a plane through the origin. However, from the point of view of our bug on $\mathbf{S}^2$ it is more appropriate to call this path a line. This motivates the following definition.

Definition. *Let ξ be a unit vector. Then*

$$\ell = \{x \in \mathbf{S}^2 | \langle \xi, x \rangle = 0\}$$

is called the line with pole ξ. *We also call ℓ the* polar line *of ξ.*

Remark: Spherical geometry is non-Euclidean. This means that whenever we represent a figure by a diagram, distortions are inevitable. Diagrams that faithfully represent one aspect (e.g., straightness of lines) will distort some other aspect (e.g., lengths and angles). You are cautioned against basing arguments on a diagram, but you are encouraged to use them to suggest facts that can then be verified rigorously. Often it is desirable to have more than one diagram of the same situation, each providing insight, yet containing some misleading information. Figures 4.3 and 4.4 show two ways of thinking about a point and its polar line.

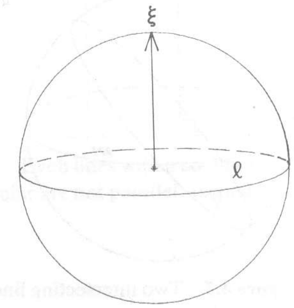

Figure 4.3 A point ξ and its polar line ℓ, first view.

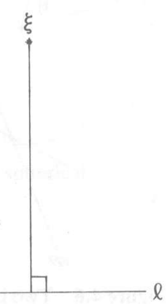

Figure 4.4 A point ξ and its polar line ℓ, second view.

Two points P and Q of $\mathbf{S}^2$ are said to be *antipodal* if $P = -Q$. Lines of $\mathbf{S}^2$ cannot be parallel, and two lines intersect not in just one point but in a pair of antipodal points. We assert the following facts that you may verify as exercises (Exercise 5).

Theorem 5.
i. *If ξ is a pole of ℓ, so is its antipode $-\xi$.*
ii. *If P lies on ℓ, so does its antipode $-P$.*

However, once these facts are noticed, there are no further anomalies, and we get the following analogues of the Euclidean results.

Theorem 6. *Let P and Q be distinct points of $\mathbf{S}^2$ that are not antipodal. Then there is a unique line containing P and Q, which we denote by $\overleftrightarrow{PQ}$.*

Proof: In order to determine a candidate for $\overleftrightarrow{PQ}$, we need a pole ξ. This must be a unit vector orthogonal to both P and Q. Because P and Q are not antipodal, we may choose ξ equal to $(P \times Q)/|P \times Q|$. Clearly, the line with pole ξ passes through P and Q.

We now consider uniqueness. If η is a pole of any line through P and Q, we must have

$$\langle \eta, P \rangle = \langle \eta, Q \rangle = 0.$$

Thus, by the triple product formula, in Theorem 1,

$$\eta \times (P \times Q) = 0,$$

and, hence, η is a multiple of the nonzero vector $P \times Q$. Because $|\eta| = 1$, we must have $\eta = \pm\xi$. Thus, $\overleftrightarrow{PQ}$ is uniquely determined. □

Theorem 7. *Let ℓ and m be distinct lines of $\mathbf{S}^2$. Then ℓ and m have exactly two points of intersection, and these points are antipodal. (See Figures 4.5 and 4.6.)*

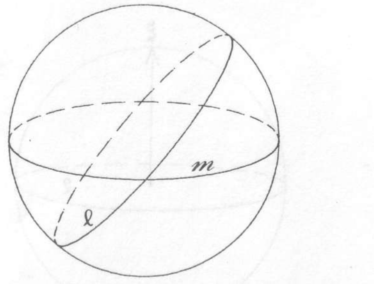

Figure 4.5 Two intersecting lines ℓ and m, first view.

Proof: Suppose ξ and η are poles of ℓ and m, respectively. Because ℓ and m are distinct, $\xi \neq \pm\eta$, and, hence, $\xi \times \eta \neq 0$. But clearly, both points $\pm(\xi \times \eta)/|\xi \times \eta|$ lie in the intersection. Any third point, however, could lie on at most one of ℓ and m by the uniqueness part of the previous theorem. □

Corollary. *No two lines of $\mathbf{S}^2$ can be parallel.*

Remark: Even lines that have a common perpendicular will intersect. See Figures 4.7 and 4.8 for two views of this situation.

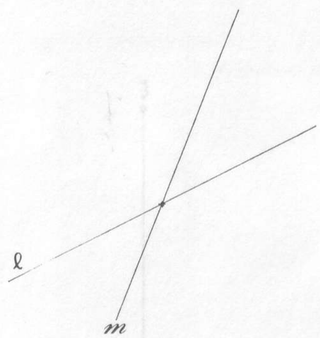

Figure 4.6 Two intersecting lines ℓ and m, second view.

Distance and the triangle inequality

The distance between two points P and Q of $\mathbf{S}^2$ is defined by the equation

$$d(P, Q) = \cos^{-1}\langle P, Q\rangle.$$

This definition reflects the idea that the measure of the angle subtended at the center of the sphere by the arc PQ should be numerically equal to the length of the arc. See Figures 4.9 and 4.10. The following theorem should be compared with Theorem 5 of Chapter 1.

Theorem 8. *If P, Q, and R are points of $\mathbf{S}^2$, then*

i. $d(P, Q) \geqslant 0$.
ii. $d(P, Q) = 0$ *if and only if* $P = Q$.
iii. $d(P, Q) = d(Q, P)$.
iv. $d(P, Q) + d(Q, R) \geqslant d(P, R)$ *(the triangle inequality)*.

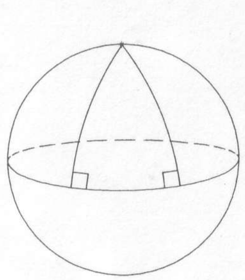

Figure 4.7 Even lines with a common perpendicular are not parallel, first view.

Proof: Properties (i)–(iii) follow from the Cauchy–Schwarz inequality and the properties of the $\cos^{-1}$ function. (See Appendix F.) The details are left to the reader as exercises. We concentrate our attention on the triangle inequality.

Let $r = d(P, Q)$, $p = d(Q, R)$, and $q = d(P, R)$. By the Cauchy–Schwarz inequality we have