Generic embeddings associated to an indestructibly weakly compact cardinal*

Gunter Fuchs
Westfälische Wilhelms-Universität Münster
gfuchs@uni-muenster.de

December 4, 2008

Abstract

I use generic embeddings induced by generic normal measures on $\mathcal{P}_\kappa(\lambda)$ that can be forced to exist if κ is an indestructibly weakly compact cardinal. These embeddings can be used in order to obtain the forcing axioms $\mathsf{MA}^{++}(<\mu\text{-closed})$ in forcing extensions. This has consequences in V: The singular cardinal hypothesis holds above κ, and κ has a useful Jónsson-like property. This, in turn, implies that the countable tower $\mathbb{Q}_{<\kappa}$ works much like it does when κ is a Woodin limit of Woodin cardinals. One consequence is that every set of reals in the Chang model satisfies the regularity properties. So indestructibly weak compactness has effects on the cardinal arithmetic high up and also on the structure of the sets of real numbers, down low, similar to supercompactness.

1 Introduction

A weakly compact cardinal κ is indestructibly weakly compact if it stays weakly compact after any forcing which is $<\kappa$-closed. I came across the concept of indestructible weak compactness for the first time when working on Maximality Principles for $<\kappa$-closed forcings. The lightface version of this principle, $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$, is the scheme of formulae (in the language with a constant symbol for κ) expressing that whenever $\varphi(\kappa)$ is a formula that can be forced to be true by a $<\kappa$-closed forcing in such a way that it stays true in every further forcing extension by $<\kappa$-closed forcing, then $\varphi(\kappa)$ is true already. I analyzed the consistency strength of

*2000 Mathematics Subject Classification: 03E55, 03E40, 03E65 this principle, together with various large cardinal properties of κ. Concerning weak compactness, the strength is given by the following:

Lemma 1.1 ([Fuc08, Lemma 3.14]). *The following theories in the language of set theory with an additional constant symbol κ are equiconsistent.*

1. $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\text{closed}}(\{\kappa\})+$ *"κ is weakly compact",*
2. $\mathsf{ZFC}+$ *"κ is indestructibly weakly compact."*

Writing $\mathsf{MP}_\Gamma(\{\kappa\})$ for the maximality principle for all forcings in Γ, with κ as a parameter, the proof in fact shows that also the theory $\mathsf{ZFC} + \mathsf{MP}_\Gamma(\{\kappa\})+$ "κ is indestructibly weakly compact" is equiconsistent with the theories 1. and 2. from the lemma above, where Γ is the class of forcings of the form $\mathrm{Col}(\kappa,\xi)$ or $\mathrm{Col}(\kappa,<\xi)$, or the class of all $<\kappa$-directed closed forcings.

So indestructible weak compactness occurs naturally in the context of maximality principles. Unfortunately, the consistency strength of indestructible weak compactness, in turn, is not known. It is known that (something slightly stronger than) the $\mathsf{AD}_\mathbb{R}$ hypothesis is a lower bound (see [JSSS07]). The only consequence of an indestructibly weakly compact κ that's needed in order to run this argument is that in a forcing extension, κ is weakly compact and $(\kappa^+)^{\mathsf{HOD}} < \kappa^+$. This can be achieved by forcing with $\mathrm{Col}(\kappa,(\kappa^+)^{\mathsf{HOD}})$, since this forcing is homogeneous and hence, HOD of the forcing extension is contained in the HOD of the ground model. So this argument, which can be viewed as a weak covering theorem at weakly compact cardinals for HOD, does not need the full power of indestructible weak compactness, but just that the indestructibility degree $\mathsf{ID}(\kappa)$ which I introduce in section 2 is greater than $(\kappa^+)^{\mathsf{HOD}}$. In the other direction, a supercompact cardinal is an upper bound: In [Fuc08, Lemma 3.12, plus the following remark] it is shown that the consistency strength of $\mathsf{ZFC} + \mathsf{MP}_{\mathrm{Col}(\dot\kappa)}(\{\kappa\})+$"$\kappa$ is weakly compact", which is the same as an indestructibly weakly compact, is at most a supercompact cardinal. See also section 2 for another way to prove this.

There is also a result by Apter and Hamkins which connects indestructible weak compactness to supercompactness: If κ is indestructibly weakly compact, and if the universe is the forcing extension of a ground model by a forcing which has a closure point less than κ, then κ is supercompact in that inner model (see [AH01]). The latter argument uses certain generic embeddings that indestructible weak compactness gives rise to.

In this paper, I am using generic embeddings of a similar kind without the hypothesis on closure point forcing.

In section 2, I develop the properties of generic normal measures on $\mathcal{P}_\kappa(\lambda)$ in a general setting and show that they exist assuming κ is indestructibly weakly compact.

In section 3, I turn to forcing axioms. I show among other things that one can force $\mathsf{MA}^{++}(\sigma\text{-closed})$ over a model in which there is an indestructibly weakly compact cardinal, using the embeddings the properties of which were developed in section 2. This is a forcing axiom that has many of the consequences that MM has, some of which are not known to have consistency strength less than a supercompact cardinal. A consequence of this is that the singular cardinal hypothesis holds above an indestructibly weakly compact cardinal, which is reminiscent of the classical result due to Solovay that SCH holds above a strongly compact cardinal. Another fact is that indestructibly weakly compact cardinals are countably completely ω_1-Jónsson, a large cardinal property that I introduce because of its usefulness in connection with the countable tower. I also introduce versions of $\mathsf{MA}^{++}(\sigma\text{-closed})$ for more highly closed forcings.

The fact that indestructibly weakly compact cardinals are countably completely ω_1-Jónsson is made use of in section 4. I show that if κ is indestructibly weakly compact, then the generic embeddings obtained from forcing with $\mathbb{Q}_{<\kappa}$, the countable stationary tower at κ, are well-founded, and ultimately that every set of reals in the Chang model has the regularity properties. This is just an example, the main point being that the machinery used in the context of $\mathbb{Q}_{<\kappa}$ works if κ is indestructibly weakly compact.

The status of indestructible weak compactness as a large cardinal axiom is somewhat ambiguous. On the one hand, the concept behaves like supercompactness or strong compactness in many ways, as the results above show. On the other hand, indestructibly weakly compact cardinals have only very weak reflection properties (I elaborate on this in section 3, one known relevant fact in this context being that the least weakly compact cardinal may be indestructible). Another key difference to customary large cardinal concepts is that it is not preserved by small forcing, as was shown by Hamkins - see the end of section 4. So it is a very subtle large cardinal concept.

Altogether, the results of this article support Conjecture 1 of [AH01], stating that the existence of an indestructibly weakly compact cardinal is equiconsistent over ZFC with a supercompact cardinal. Many other applications of indestructible weak compactness are thinkable.

2 Generic Weak Compactness Measures

In this section, I first introduce the concepts of the weak compactness indestructibility degree of a cardinal and of indestructible weak compactness. After that, I develop abstractly the properties of external supercompactness measures, in particular of generic supercompactness measures that arise from indestructible weak compactness.

2.1 Indestructible Weak Compactness

Definition 2.1. Let κ be an ordinal. Let the *weak compactness indestructibility degree of* κ be:

$$\mathsf{ID}(\kappa) = \sup\{\alpha \mid \Vdash_{\mathrm{Col}(\kappa,\alpha)} \text{“}\kappa \text{ is weakly compact”}\}.$$

Let's say that an ordinal $\alpha > 0$ is *$<\kappa$-closed* if for all $\gamma < \alpha$, $\gamma^{<\kappa} < \alpha$. Since $(\beta^{<\kappa})^{<\kappa} = \beta^{<\kappa}$ in general, α is $<\kappa$-closed if and only if α is a limit of ordinals γ such that $\gamma^{<\kappa} = \gamma$. The phenomenon underlying the following observation was noted by Thomas Johnstone in his dissertation.

Observation 2.2. *Let α be $<\kappa$-closed. Then following are equivalent:*

1. $\mathsf{ID}(\kappa) \geq \alpha$.
2. *κ is weakly compact in every forcing extension obtained by forcing with a $<\kappa$-closed poset of size less than α.*

Proof. $1 \Longrightarrow 2$: Let a $<\kappa$-closed forcing $\mathbb{P}$ of size less than α be given. Note that since $\mathsf{ID}(\kappa) > 0$, it follows that κ is regular. Let $\delta < \alpha$ be such that $\overline{\overline{\mathbb{P}}} \leq \delta = \delta^{<\kappa}$. Then $\mathbb{P} \times \mathrm{Col}(\kappa, \delta)$ has size δ and is hence forcing equivalent to $\mathrm{Col}(\kappa, \delta)$ – see [Fuc08, Lemma 2.2] for a proof. Let G be $\mathbb{P}$-generic. To see that κ is weakly compact in $\mathrm{V}[G]$, pick G' $\mathrm{Col}(\kappa, \delta)$-generic over $\mathrm{V}[G]$ and H $\mathrm{Col}(\kappa, \delta)$-generic over V in such a way that $\mathrm{V}[G][G'] = \mathrm{V}[H]$. Since $\delta < \alpha \leq \mathsf{ID}(\kappa)$, κ is weakly compact in $\mathrm{V}[H]$. Suppose it were not weakly compact in $\mathrm{V}[G]$. This is a $\Sigma^1_2(\kappa)$-property true in $\mathrm{V}[G]$: *There is* a κ-tree $T \subseteq \kappa \times \kappa$ such that *for all* $b \subseteq \kappa$, b is not a cofinal branch of T. Pick a witness $T \subseteq \kappa$ of which the latter $\Pi^1_1(\kappa)$-statement is true in $\mathrm{V}[G]$. Since $\mathrm{V}[H] = \mathrm{V}[G][G']$ is a $<\kappa$-closed generic extension of $\mathrm{V}[G]$, it follows from $<\kappa$-closed-generic $\Pi^1_1(\kappa)$-absoluteness (due to Silver; cf. [Kun80, p. 298, (I6)]) that the same statement is true in $\mathrm{V}[H]$, so κ is not weakly compact in $\mathrm{V}[H]$ after all, a contradiction.

$2 \Longrightarrow 1$: Let $\gamma < \alpha$. Then $\mathrm{Col}(\kappa, \gamma)$ has size $\gamma^{<\kappa} < \alpha$, so by assumption, κ is weakly compact in $\mathrm{Col}(\kappa, \gamma)$-generic extensions of V. $\square$

Note that the proof of this observation also shows that if $\gamma < \delta$ and κ is weakly compact in $\mathrm{Col}(\kappa, \delta)$-generic extensions, then it is also weakly compact in $\mathrm{Col}(\kappa, \gamma)$-generic extensions.

Definition 2.3. A cardinal κ is *indestructibly weakly compact* if $\mathsf{ID}(\kappa) = \infty$.

So if κ is indestructibly weakly compact, then the weak compactness of κ is preserved by arbitrary $<\kappa$-closed forcing.

Note that the supercompactness of a supercompact cardinal κ can always be forced to be indestructible under $<\kappa$-*directed* closed forcing, using the Laver preparation [Lav78]. Since the forcings $\mathrm{Col}(\kappa, \lambda)$ are $<\kappa$-directed closed, it follows that after the Laver preparation, κ is weakly compact with indestructibility degree $\mathsf{ID}(\kappa) = \infty$, so that κ's *weak* compactness is indestructible under arbitrary $<\kappa$-closed forcing.

2.2 External Supercompactness Ultrapowers

I shall now state a very general lemma on external supercompactness ultrapowers of a transitive model N by a fine, N-normal measure $\mathcal{F}$ on $\mathcal{P}_\kappa(\lambda)^N$, where κ is an infinite cardinal in N. An ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))^N$ is *fine* here if for every $\alpha < \lambda$, the set of all $x \in \mathcal{P}_\kappa(\lambda)^N$ with $\alpha \in x$ has $\mathcal{F}$-measure 1, i.e., is a member of $\mathcal{F}$. $\mathcal{F}$ is *very fine* if for every $a \in \mathcal{P}_\kappa(\lambda)^N$, the set of all $x \in \mathcal{P}_\kappa(\lambda)^N$ with $a \subseteq x$ is in $\mathcal{F}$. Note that if $\mathcal{F}$ is $<\kappa$-closed over N, then fineness implies very fineness. $\mathcal{F}$ is *N-normal* if it has the property that whenever $A \in \mathcal{F}$ and $f : A \longrightarrow \lambda$ is a function in N such that $f(x) \in x$ for every $x \in A$, then f is constant on a set of $\mathcal{F}$-measure 1.

I will apply the following lemma in $\mathrm{V}[G]$ to $N = \mathrm{V}$ later, where G is generic over V for a $<\kappa$-distributive or a $<\kappa$-closed forcing. The gaps in the proof can easily be filled by consulting standard treatments of supercompactness measures (or normal, fine ultrafilters on $\mathcal{P}_\kappa(\lambda)$) like [Jec03] or [Kan03].

Lemma 2.4. *Let N be an inner model of* ZFC*, and let $\mathcal{F}$ be a fine N-normal measure on $\mathcal{P}_\kappa(\lambda)^N$ which is σ-complete, meaning that the intersection of countably many $\mathcal{F}$-measure 1 sets is non-empty. Let $j : N \longrightarrow_{\mathcal{F}} M$ be the ultrapower and embedding given by $\mathcal{F}$. Then:*

1. *M is well-founded, and hence can in the following be assumed to be transitive.*
2. *Łoś's theorem holds:*
$$M \models \varphi([\vec{f}]_{\mathcal{F}}) \iff \{x \in \mathcal{P}_\kappa(\lambda)^N \mid N \models \varphi(\vec{f}(x))\} \in \mathcal{F}.$$[1]
3. *If $\vec{f} = \langle f_\alpha \mid \alpha < \lambda\rangle \in N$, where each f_α is a function with domain $\mathcal{P}_\kappa(\lambda)^N$, then the set $\{[f_\alpha]_{\mathcal{F}} \mid \alpha < \lambda\}$ is a member of M, i.e., there is a $g : \mathcal{P}_\kappa(\lambda)^N \longrightarrow N$ in N such that for any $f : \mathcal{P}_\kappa(\lambda)^N \longrightarrow N$ in N, $[f]_{\mathcal{F}} \in [g]_{\mathcal{F}}$ iff there is an $\alpha < \lambda$ such that $[f]_{\mathcal{F}} = [f_\alpha]_{\mathcal{F}}$.*[2]

[1] This is true in general whenever $\mathcal{F}$ is an ultrafilter on some set in N.

[2] This statement is weaker than the assertion that ${}^\lambda M \subseteq M$. For if $\vec{x} \in \cap^\lambda M$, while it is true that each x_α is of the form $[f_\alpha]_{\mathcal{F}}$, and such a sequence of functions exists in V (where $\mathcal{F}$ exists), it is unclear that such a sequence of representing functions exists in N. If ${}^\lambda N \subseteq N$, then that stronger assertion follows.

4. $j``\lambda = [\mathrm{id}]_{\mathcal{F}} \in M$.[3]

5. $\alpha = [x \mapsto \mathrm{otp}(\alpha \cap x)]_{\mathcal{F}}$, *for* $\alpha < \lambda$.

6. $[f]_{\mathcal{F}} = j(f)(j``\lambda)$.

7. *For* $X \in \mathcal{P}(\mathcal{P}_\kappa(\lambda))^N$, $X \in \mathcal{F} \iff j``\lambda \in j(X)$.

8. *The critical point of* j *is at most* κ, *and* $j(\kappa) \geq \lambda$.

9. *If* $\mathcal{F}$ *is very fine, then* κ *is the critical point of* j.

Proof. The usual proofs work. As an example, using "Łoś's theorem" 2, 3 is obvious by the usual argument: Given a sequence $\vec{f}$ as in 3, let g on $\mathcal{P}_\kappa(\lambda)^N$ be defined in N by setting $g(x) = \{f_\alpha(x) \mid \alpha \in x\}$. To check that g is as wished, two things have to be verified: Firstly that $[f_\alpha]_{\mathcal{F}} \in [g]_{\mathcal{F}}$, which is equivalent to showing that X:=$\{x \in \mathcal{P}_\kappa(\lambda)^N \mid f_\alpha(x) \in g(x)\} \in \mathcal{F}$. But this is the case, since $\{x \in \mathcal{P}_\kappa(\lambda)^N \mid \alpha \in x\}$ is a measure one subset of X, by fineness of $\mathcal{F}$. And vice versa, if $[f]_{\mathcal{F}} \in [g]_{\mathcal{F}}$, then this means that the set $A = \{x \in \mathcal{P}_\kappa(\lambda)^N \mid f(x) \in g(x)\}$ has $\mathcal{F}$-measure one. By definition of g, for every $x \in A$, there is some $h(x) \in x$ with $f(x) = f_{h(x)}(x)$, where h can be chosen in N. So by N-normality, h is constant on a measure one subset of A. Letting α_0 be this constant value, this means that $[f]_{\mathcal{F}} = [f_{\alpha_0}]_{\mathcal{F}}$. □

Let's now look at the special case that an external supercompactness measure on $\mathcal{P}_\kappa(\lambda)$ is added by a $<\kappa$-distributive forcing.

Corollary 2.5. *Let* κ *be a regular cardinal, and assume the existence of a* $<\kappa$-*distributive notion of forcing* $\mathbb{P}$ *such that if* G *is* V-*generic for* $\mathbb{P}$, *then there is a* V-*normal fine measure* $\mathcal{F}$ *on* $\mathcal{P}_\kappa(\lambda)$, *where* $\kappa \leq \lambda$. *Note that* $\mathcal{P}_\kappa(\lambda)^{\mathrm{V}} = \mathcal{P}_\kappa(\lambda)^{\mathrm{V}[G]}$, *so there's no need to distinguish between the two.*

Let's subsume this assumption by saying that there is a $<\kappa$-*distributive generic* V-*normal fine measure on* $\mathcal{P}_\kappa(\lambda)$. *Analogously, if the forcing which adds the measure is* $<\kappa$-*closed, I'll refer to it as a* $<\kappa$-*closed generic* V-*normal fine measure on* $\mathcal{P}_\kappa(\lambda)$.

Then the ultrapower of V *by* $\mathcal{F}$ *is well-founded. Let* $j : \mathrm{V} \longrightarrow_{\mathcal{F}} M$ *be the corresponding embedding and transitivized ultrapower. Then the following assertions hold:*

1. $\mathrm{V}[G] \cap {}^{<\kappa}M \subseteq M$.

2. *If* $T \in \mathrm{V}$ *is a transitive set of* V-*cardinality at most* λ *and* $a \subseteq T$ *is a member of* V, *then* $j \restriction a \in M$. *This is true, in particular, for* $a \subseteq \lambda$.

[3]Here, id denotes the restriction of the identity function to $\mathcal{P}_\kappa(\lambda)^N$.

3. *If $\mathcal{F}$ is very fine, then κ is inaccessible and* $\mathrm{V}_\kappa^{\mathrm{V}} = \mathrm{V}_\kappa^M$.

Proof. I shall apply Lemma 2.4 in $\mathrm{V}[G]$ here, where V will play the role of the model N in the statement of that lemma. Note that 1 implies that M is well-founded, so this doesn't need to be proved separately.

For 1, if $\vec{x} = \langle x_\alpha \mid \alpha < \gamma \rangle \in {}^\gamma M$, $\vec{x} \in \mathrm{V}[G]$ and $\gamma < \kappa$, then there is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \gamma \rangle$ in $\mathrm{V}[G]$ such that every f_α is a function in V with domain $\mathcal{P}_\kappa(\lambda)$ and $[f_\alpha]_{\mathcal{F}} = x_\alpha$. Since $\mathbb{P}$ is $<\kappa$-distributive, it follows that $\vec{f} \in \mathrm{V}$, and from this it follows by Lemma 2.4, item 3 that $\{[f_\alpha]_{\mathcal{F}} \mid \alpha < \kappa\} \in M$. So in particular, M is well-founded and can hence, a posteriori, be assumed to be transitive.

For 2, $j``a = \{[\mathrm{const}_x]_{\mathcal{F}} \mid x \in a\} \in M$, by Lemma 2.4, item 3. For the same reason, $j``T \in M$. Since $j \upharpoonright T$ is the inverse of the Mostowski collapse of the set $j``T$, which is in $M$, it follows that $k := j \upharpoonright T \in M$, as well. But then $a = k^{-1}``(j``a)$, so that $a \in M$. So $j \upharpoonright a = k \upharpoonright a \in M$.

Finally, let's prove 3. If $\mathcal{F}$ is very fine, then by Lemma 2.4, item 9, κ is the critical point of j. Since moreover, $\mathrm{V}_\kappa^M \subseteq \mathrm{V}_\kappa^{\mathrm{V}[G]} = \mathrm{V}_\kappa^{\mathrm{V}}$ by the $<\kappa$-distributivity of $\mathbb{P}$, it follows that κ is a strong limit cardinal in V: Otherwise there would be a surjective function $f : \mathcal{P}(\alpha) \twoheadrightarrow \kappa$, for some $\alpha < \kappa$. But $\mathcal{P}(\alpha)^{\mathrm{V}} = \mathcal{P}(\alpha)^M = j(\mathcal{P}(\alpha)^{\mathrm{V}})$. So $j(f) : \mathcal{P}(\alpha) \twoheadrightarrow j(\kappa)$. But for $x \subseteq \alpha$, $j(f)(x) = j(f)(j(x)) = j(f(x)) = f(x)$, so that $\mathrm{ran}(j(f)) \subseteq \kappa < j(\kappa)$, a contradiction. So since κ is regular, it is inaccessible in V. It follows that $j \upharpoonright \mathrm{V}_\kappa = \mathrm{id}$, and hence, $\mathrm{V}_\kappa^{\mathrm{V}} = j``\mathrm{V}_\kappa^{\mathrm{V}} \subseteq \mathrm{V}_\kappa^M \subseteq \mathrm{V}_\kappa^{V[G]} = \mathrm{V}_\kappa^{\mathrm{V}}$. □

It turns out that weakly compact cardinals of a certain indestructibility degree give rise to external V-normal supercompactness measures. I'll apply the following very useful characterization of weak compactness, which is folklore, but there is a proof outline in [Lar04].

Fact 2.6. *Let κ be an inaccessible cardinal. Then the following are equivalent:*

1. *κ is weakly compact,*
2. *For every transitive model $M = \langle |M|, \in, \ldots \rangle$ with $\overline{\overline{M}} = \kappa$ of a language which extends the language of set theory, such that $\kappa \in |M|$ and $\dot{\in}^M = \in \upharpoonright |M|$, there is a function π and another model N of that language, again transitive with $\dot{\in}^N = \in \upharpoonright |N|$, such that $\pi : M \longrightarrow N$ is elementary and κ is the critical point of π. Call $\pi : M \longrightarrow N$ a* weakly compact embedding.

The following is implicit in [AH01, Thm. 3] as well.

Theorem 2.7. *Let λ be an ordinal greater than or equal to κ. Set $\Omega = \Omega(\lambda) := 2^{(\lambda^{<\kappa})}$ and assume that κ is weakly compact with $\mathsf{ID}(\kappa) > \Omega$.*

Then there is a $<\kappa$-closed-generic V-normal, very fine measure on $\mathcal{P}_\kappa(\lambda)$. This is witnessed by $\mathrm{Col}(\kappa, \Omega)$: If G is V-generic for that partial order, then there is a

V-*normal very fine ultrafilter* $\mathcal{F} \in \mathrm{V}[G]$ *on* $\mathcal{P}_\kappa(\lambda)$. *This ultrafilter is* $<\kappa$-*complete. I shall refer to such* $\mathcal{F}$ *as an indestructible weak compactness measure on* $\mathcal{P}_\kappa(\lambda)$.

Proof. Let G be $\mathrm{Col}(\kappa, \Omega)$-generic over V. In $\mathrm{V}[G]$, $\mathcal{P}(\mathcal{P}_\kappa(\lambda))^{\mathrm{V}}$ has size κ, and κ is still weakly compact. So I can pick a model $N \in \mathrm{V}[G]$ which has the following properties:

1. N is a transitive ZFC^- model of size κ,
2. $({}^{<\kappa}N) \cap \mathrm{V}[G] \subseteq N$,
3. $\mathcal{P}(\mathcal{P}_\kappa(\lambda)) \cap \mathrm{V} \subseteq N$,
4. $N \models$ "λ has cardinality κ".

Note that $\mathcal{P}_\kappa(\lambda)$ is the same in V, $\mathrm{V}[G]$ and N, by the closedness of the forcing and the closedness of N. Since κ is weakly compact in $\mathrm{V}[G]$, I can pick $\pi : N \longrightarrow N'$ to be a weakly compact embedding. So N' is transitive, π is elementary, and $\kappa = \mathrm{crit}(\pi)$. Note that $\pi``\lambda \in N'$. This is because if $f : \kappa \longrightarrow \lambda$ is a surjection with $f \in N$ (and such an $f$ exists, because $\lambda$ has cardinality $\kappa$ in $N$), then $\pi``\lambda = \pi``(f``\kappa) = \pi(f)``\kappa \in N'$. Moreover, this argument shows that $\pi``\lambda$ has size κ in N' and is hence a member of $\mathcal{P}_{\pi(\kappa)}(\pi(\lambda))^{N'}$. So it is possible to derive an ultrafilter $\mathcal{F}'$ on $\mathcal{P}_\kappa(\lambda)^N$ from π by setting:

$$\mathcal{F}' = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid X \in N \wedge \pi``\lambda \in \pi(X)\}.$$

Let $\mathcal{F} = \mathcal{F}' \cap \mathrm{V}$. I claim that $\mathcal{F}$ is a very fine V-normal measure on $\mathcal{P}_\kappa(\lambda)$.

To see that $\mathcal{F}$ is an ultrafilter, let $X \subseteq \mathcal{P}_\kappa(\lambda)$, $X \in \mathrm{V}$ be such that $X \notin \mathcal{F}$. Let $Y = \mathcal{P}_\kappa(\lambda) \setminus X$. Since $\mathcal{P}_\kappa(\lambda)$ is the same in V and in N, it is also true in N that $Y = \mathcal{P}_\kappa(\lambda) \setminus X$. So $\pi(Y) = \mathcal{P}_{\pi(\kappa)}(\pi(\lambda))^{N'} \setminus \pi(X)$, since π is fully elementary. That $X \notin \mathcal{F}$ means that $\pi``\lambda \notin \pi(X)$. But since $\pi``\lambda \in \mathcal{P}_{\pi(\kappa)}(\pi(\lambda))^{N'}$, it follows that $\pi``\lambda \in \pi(Y)$, the relative complement. So by definition, $Y \in \mathcal{F}$.

Turning to $<\kappa$-completeness, let $\delta < \kappa$ and $\langle X_\alpha \mid \alpha < \delta\rangle \in ({}^{\delta}\mathcal{F})$. Then $\vec{X} \in N$, by the closedness of N. By definition of $\mathcal{F}$, $\pi``\lambda \in \pi(X_\alpha)$, for each α. Since $\kappa = \mathrm{crit}(\pi)$, $\pi(\langle X_\alpha \mid \alpha < \delta\rangle) = \langle \pi(X_\alpha) \mid \alpha < \delta\rangle$. So

$$\pi``\lambda \in \bigcap_{\alpha<\delta} \pi(X_\alpha) = \pi(\bigcap_{\alpha<\delta} X_\alpha),$$

which means that $\bigcap_{\alpha<\delta} X_\alpha \in \mathcal{F}'$. Note that $\vec{X} \in \mathrm{V}$, since each X_α is in $\mathcal{F}$ and hence in V. So $\bigcap_{\alpha<\delta} X_\alpha \in \mathcal{F}' \cap \mathrm{V} = \mathcal{F}$.

Let's now check that $\mathcal{F}$ is very fine. So let $x \in \mathcal{P}_\kappa(\lambda)$, and set $\hat{x} = \{y \in \mathcal{P}_\kappa(\lambda) \mid x \subseteq y\}$. It has to be shown that $\pi``\lambda \in \pi(\hat{x})$. It is now crucial again that $\mathcal{P}_\kappa(\lambda)$ is the same in V and in N. For as a consequence, $\hat{x}$ is the same when computed in V and in N. Now $\pi(\hat{x})$ consists of those $y \in \mathcal{P}_{\pi(\kappa)}(\pi(\lambda))^{N'}$ with $\pi(x) \subseteq y$. So it has to be shown that $\pi(x) \subseteq \pi``\lambda$. But this is clear, because $x$ has cardinality less than $\kappa$, so that $\pi(x) = \pi``x \subseteq \pi``\lambda$.

Finally, let's check V-normality. Let $X \in \mathcal{F}$ and $f : X \longrightarrow \cup X$ be regressive, $f \in$ V. For $\alpha < \lambda$, let

$$Z_\alpha = \{x \in X \mid f(x) = \alpha\}.$$

It has to be shown that $\pi``\lambda \in \pi(Z_{\alpha_0})$, for some $\alpha_0 < \lambda$. This is equivalent to saying that $\pi(f)(\pi``\lambda) = \pi(\alpha_0)$ (for trivially, $\pi``\lambda \in X$, as $X \in \mathcal{F}$). And such an $\alpha_0$ clearly exists, as $\pi(f)(\pi``\lambda) \in \pi``\lambda$, since $\pi(f)$ is regressive. □

Lemma 2.8. *Suppose that $\lambda \geq \kappa$ and there is a $<\kappa$-closed generic* V*-normal very fine measure on $\mathcal{P}_\kappa(\lambda)$. Then κ is weakly compact.*

Proof. Let G be generic over V for a $<\kappa$-closed forcing which adds a $<\kappa$-closed generic V-normal measure on $\mathcal{P}_\kappa(\lambda)$. Let $j : \mathrm{V} \longrightarrow M$ be the corresponding embedding. Then κ is inaccessible: It is regular by fiat,[4] and it is a strong limit cardinal in V by Corollary 2.5.3. Also, κ is the critical point of j by Lemma 2.4.9.

In order to verify that κ is weakly compact, it now suffices to show that it has the tree property. So let $T \in$ V be a κ-tree on κ whose nodes are ordinals below κ. Then $j(T)$ is a $j(\kappa)$-tree in M. Pick a node x on level κ of $j(T)$. Then the set b of predecessors of x in $j(T)$ is a cofinal branch of T which exists in V$[G]$. So the statement that T has a cofinal branch is true in V$[G]$. This is a $\Sigma^1_1(\kappa)$ statement about T, so that by $<\kappa$-closed-generic $\underset{\sim}{\Sigma}^1_1(\kappa)$ absoluteness, it is true in V as well. T was an arbitrary κ-tree in V, so κ is indeed weakly compact in V. □

3 Forcing Axioms

The aim in this section is to try to run the argument used to force a model of Martin's Maximum or PFA starting in a model with a supercompact cardinal, but this time replacing supercompactness with indestructible weak compactness.

One is immediately faced with a problem: There are no sufficient Laver functions available for indestructible weak compactness. The Laver functions one gets from weak compactness as in [Ham] don't seem to be strong enough.

At first sight, the way out seems to be the use of Hamkins' method of lottery sums as in [Apt05]. However, in order for these constructions to work, one would need that $\mathrm{V}_\kappa \prec_{\Sigma_2} \mathrm{V}$, where κ is indestructibly weakly compact. This is because one wants to reflect the statement "There is a poset $\mathbb{P}$ which is proper (or stationary set preserving) and there is an ω_1-sequence $\vec{D}$ of dense subsets of $\mathbb{P}$ for which there

[4] When talking about $<\kappa$-closed generic measures, it is tacitly assumed that κ is regular.

is no $\vec{D}$-generic filter" down to V_κ, and this statement can be expressed in a Σ_2 fashion. If κ is supercompact or even just strong, then this is no problem, but the following fact, which is based on the work [AH99] of Apter and Hamkins, shows that this is not true in general for indestructibly weakly compact cardinals.

Fact 3.1. *([Fuc, Thm. 3.10]) If it is consistent that there is a supercompact cardinal, then it is consistent that the least weakly compact cardinal is indestructible.*

Of course, the least weakly compact cardinal κ can never be Σ_2-correct in V, because the existence of a weakly compact cardinal is a Σ_2-truth in V which is false in V_κ. In more detail, the problem is the following. Suppose κ is indestructibly weakly compact and $\mathbb{P} = \mathbb{P}_\kappa$ is an iteration designed to force PFA. Let G be $\mathbb{P}$-generic over V and $j : V \longrightarrow_{\mathcal{F}} M$ be an ultrapower of V by a $<\kappa$-closed generic weak compactness measure $\mathcal{F} \in V[X]$ (X being generic over V for some collapse to κ) and $\mathbb{Q}$ is a forcing which is proper in $V[G]$ and a member of $M[G]$. Then it's not clear that $\mathbb{Q}$ is also proper in $M[G]$.

So instead of shooting for Martin's Maximum or PFA, I aim at a type of forcing axioms which are a little weaker but still very useful. In order to formulate them, and also in the whole section 4, I shall need some basics on generalized stationary sets. What I refer to as "stationary" is sometimes called "weakly stationary". Correspondingly, the notion of club I use is sometimes referred to as "strong club".

Definition 3.2. Let $X \neq \emptyset$ be a set. An *algebra* on X is a structure $\langle X, \langle f_n \mid n < \omega \rangle \rangle$, such that for each $n < \omega$, there is a nonzero $m < \omega$ such that $f_n : X^m \longrightarrow X$ is partial function, and the collection $\{f_n \mid n < \omega\}$ of functions is closed under compositions. If $\mathfrak{A} = \langle X, \langle f_n \mid n < \omega \rangle \rangle$ is an algebra on X, then a set $Y \subseteq X$ is $\mathfrak{A}$-*closed* if for every $n < \omega$, if m is the arity of f_n and $\langle x_0, \ldots, x_{m-1} \rangle \in \mathrm{dom}(f_n)$, then $f_n(x_0, \ldots, x_{m-1}) \in Y$. A collection $a \subseteq \mathcal{P}(X)$ of nonempty sets is *club* (or closed and unbounded) in X if there is an algebra $\mathfrak{A}$ on X such that a is the collection of nonempty subsets of X which are $\mathfrak{A}$-closed. A collection $a \subseteq \mathcal{P}(X)$ of nonempty sets is *stationary* in X if it intersects every set which is club in X, or, equivalently, if for every algebra $\mathfrak{A}$ on X, there is an $x \in a$ which is $\mathfrak{A}$-closed. a is stationary (without further qualification) if it is stationary in $\cup a$.[5]

Definition 3.3. Let $X \subseteq Y$. If $a \subseteq \mathcal{P}(X)$, then set

$$a \uparrow Y := \{y \subseteq Y \mid y \cap X \in a \text{ and } \overline{\overline{y}} \leq \omega\}.$$

This is the *(countable) lift of a to Y*. Vice versa, if $b \subseteq \mathcal{P}(Y)$, then I write

$$b \downarrow X := \{y \cap X \mid y \in b\}.$$

This is the *projection* of b onto X.

[5] This makes sense because if there is an X such that a is stationary in X, then $X = \cup a$.

Fact 3.4.

1. *If $a \subseteq [X]^\omega$ is stationary in X and $X \subseteq Y$, then $a\!\uparrow\! Y$ is stationary in Y.*
2. *If $b \subseteq [Y]^\omega$ is stationary and $X \subseteq Y$, then $b\!\downarrow\! X$ is stationary in X.*
3. *If $a \subseteq \mathcal{P}(\cup a)$ is stationary and $f : a \longrightarrow \cup a$ is a choice function, then f is constant on a stationary subset of a.*

Proof. Just to be on the safe side, I prove the first point: Let $\mathfrak{A} = \langle Y, f_0, f_1, \ldots \rangle$ be an algebra on Y. Let $\mathfrak{A}|X$ be its reduction to X. Let $x \in a$ be closed under $\mathfrak{A}|X$. Let y be the closure of x under $\mathfrak{A}$. Then y is countable, since x was, and since the $\vec{f}$'s are closed under composition, it follows that $y \cap X = x$, so that $y \in a\!\uparrow\! Y$. □

I shall be particularly interested in stationary sets which are preserved by certain closed forcings. To this end, I'll use terminology introduced in [For].

Definition 3.5 ([For, Def. 8.26])**.** Let μ be a regular cardinal. A stationary set is *μ-robust* if it stays stationary in every forcing extension by a $<\!\mu$-closed forcing notion.

Note that σ-closed forcings, being proper, preserve arbitrary stationary sets consisting of countable sets, so every such set is $\aleph_1$-robust by fiat. This is not true for $\mu > \aleph_1$. Note also that if S is μ-robust and G is $\mathbb{P}$-generic for a $<\!\mu$-closed forcing, then S is not only stationary in $\mathrm{V}[G]$ but also μ-robust.

I shall now introduce a generalization of the forcing axiom $\mathsf{MA}^+(\sigma\text{-closed})$ which first appears in the literature in [FMS88]. In its original form, it says that whenever $\mathbb{P}$ is a σ-closed forcing, $\vec{D}$ is an ω_1-sequence of dense subsets of $\mathbb{P}$ and $\dot{S}$ is a $\mathbb{P}$-name for a stationary subset of ω_1, then there is a $\vec{D}$-generic filter F such that $\dot{S}^F$ is stationary. If one generalizes this notion to $<\!\mu$-closed forcings in the obvious way, some of the powerful consequences of the ω_1 case are lost. The right generalization seems to be the one given in the next definition.

Definition 3.6. Let μ be a regular cardinal, and let Γ be a class of $<\!\mu$-closed forcings. Let $\mathsf{MA}^+(\Gamma, \mu)$, the *strong Martin Axiom for forcings in* Γ *at* μ*,* say that whenever $\mathbb{P}$ is a forcing in Γ, $\langle D_\alpha \mid \alpha < \mu \rangle$ is a sequence of dense subsets of $\mathbb{P}$ and $\dot{S}$ is a $\mathbb{P}$-name such that $\mathbb{P}$ forces that $\dot{S}$ is a μ-robust subset of $\mathcal{P}_\mu(\mu)$, then there is a filter F in $\mathbb{P}$ such that $F \cap D_\alpha \neq \emptyset$ for every $\alpha < \mu$, and the set

$$\dot{S}^F = \{x \in \mathcal{P}_\mu(\mu) \mid \exists p \in F \quad p \Vdash \check{x} \in \dot{S}\}$$

is stationary. If Γ is the class of all $<\!\mu$-closed forcings, I just write $\mathsf{MA}^+(<\!\mu\text{-closed})$ for $\mathsf{MA}^+(\Gamma, \mu)$. In the case $\mu = \omega_1$, I'll write $\mathsf{MA}^+(\sigma\text{-closed})$ for the corresponding axiom.

Note that the existence of a filter F intersecting μ many given dense subsets of a $<\mu$-closed poset is provable in ZFC. It is the stationarity of $\dot{S}^F$ which makes $\mathsf{MA}^+(<\mu\text{-closed})$ strong. Also, $\mathcal{P}_\mu(\mu)^{\mathrm{V}} = \mathcal{P}_\mu(\mu)^{\mathrm{V}[G]}$, if G is V-generic for a forcing which is $<\mu$-closed. Finally, if $\mu = \aleph_1$, then the present version of $\mathsf{MA}^+(\sigma\text{-closed})$ is equivalent to the original one, since every stationary subset of ω_1 is also a stationary subset of $\mathcal{P}_{\aleph_1}(\omega_1)$, and vice versa, if $S \subseteq \mathcal{P}_{\aleph_1}(\omega_1)$ is stationary, then $S \cap \omega_1$ is a stationary subset of ω_1.

I give the proof of the following lemma in some detail, because it is the key point that makes it possible to work without Laver functions when forcing $\mathsf{MA}^+(\mu\text{-closed})$ to hold.

Lemma 3.7. *Let μ be a regular cardinal. Then*

$$\mathsf{MA}^+(<\mu\textit{-closed}) \iff \mathsf{MA}^+(\{\mathrm{Col}(\mu,\lambda) \mid \lambda = \lambda^{<\mu}\},\mu).$$

Proof. For the nontrivial direction, let $\mathbb{P}$ be $<\mu$-closed, $\vec{D} = \langle D_\alpha \mid \alpha < \mu\rangle$ a sequence of dense open subsets of $\mathbb{P}$, and $\dot{S}$ be a $\mathbb{P}$-name for a μ-robust subset of $\mathcal{P}_\mu(\mu)$. Pick λ such that $\mathbb{P} \times \mathrm{Col}(\mu,\lambda)$ is forcing equivalent to $\mathrm{Col}(\mu,\lambda)$. Let Δ be dense in $\mathrm{Col}(\mu,\lambda)$, D dense in $\mathbb{P} \times \mathrm{Col}(\mu,\lambda)$ and $\pi : (\mathbb{P} \times \mathrm{Col}(\mu,\lambda)) \restriction D \overset{\sim}{\longleftrightarrow} \mathrm{Col}(\mu,\lambda) \restriction \Delta$; see [Fuc08, Lemma 2.2]. In fact, $\Delta = \{p \in \mathrm{Col}(\mu,\lambda) \mid \exists\gamma < \mu(\mathrm{dom}(p) = \gamma+1)\}$. I want to translate $\langle D_\alpha \mid \alpha < \mu\rangle$ into a sequence $\langle \tilde{D}_\alpha \mid \alpha < \mu\rangle$ of dense subsets of $\mathrm{Col}(\mu,\lambda)$ and $\dot{S}$ into a $\mathrm{Col}(\mu,\lambda)$-name for a μ-robust subset of $\mathcal{P}_\mu(\mu)$.

For $\alpha < \mu$, let $D'_\alpha = D \cap (D_\alpha \times \mathrm{Col}(\mu,\lambda))$. Then D'_α is a dense subset of $(\mathbb{P} \times \mathrm{Col}(\mu,\lambda)) \restriction D$: Given $\langle p,q\rangle \in (\mathbb{P} \times \mathrm{Col}(\mu,\lambda)) \restriction D$, pick $p' \leq_{\mathbb{P}} p$, $p' \in D_\alpha$, by density of D_α. Then pick $\langle p'',q''\rangle \leq_{\mathbb{P}\times\mathrm{Col}(\mu,\lambda)} \langle p',q\rangle$ such that $\langle p'',q''\rangle \in D$, which is possible, since D is a dense subset of $\mathbb{P} \times \mathrm{Col}(\mu,\lambda)$. Then $p'' \leq_{\mathbb{P}} p' \in D_\alpha$, so that $p'' \in D_\alpha$ also, as D_α is open. So $\langle p,q\rangle \geq_{(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D} \langle p'',q''\rangle \in D'_\alpha$, showing that D'_α is dense.

Set $\tilde{D}_\alpha = \pi``D'_\alpha$, for $\alpha < \mu$. Clearly, $\tilde{D}_\alpha$ is dense in $\mathrm{Col}(\mu,\lambda)$, since it is dense in $\mathrm{Col}(\mu,\lambda) \restriction \Delta$ and Δ is dense in $\mathrm{Col}(\mu,\lambda)$.

Turning to translating $\dot{S}$, observe that $p_0[D]$, the projection of D onto the $\mathbb{P}$-coordinate, is dense in $\mathbb{P}$. So one may assume that $\dot{S}$ is a $\mathbb{P} \restriction (p_0[D])$ name, since there is such a name $\dot{S}'$ such that $\mathbb{P} \Vdash \dot{S} = \dot{S}'$.[6] Let $\dot{T}$ be the canonical $(\mathbb{P} \times \mathrm{Col}(\mu,\lambda))$-name such that if $G \times H$ is $\mathbb{P} \times \mathrm{Col}(\mu,\lambda)$-generic, then $\dot{T}^{G\times H} = \dot{S}^G$. I.e., $\dot{T} = i_0(\dot{S})$, where i_0 is the canonical injection from $\mathbb{P}$ into $\mathbb{P} \times \mathrm{Col}(\mu,\lambda)$.

[6]In general, if $\mathbb{Q}$ is a notion of forcing and $B \subseteq \mathbb{Q}$ is dense, then there is a way of recursively translating any $\mathbb{Q}$-name τ to a $\mathbb{Q} \restriction B$-name $\tau{\downarrow}B$: One can define

$$\tau{\downarrow}B = \{\langle \sigma{\downarrow}B, q\rangle \mid \exists p(\langle \sigma,p\rangle \in \tau \wedge p \geq_{\mathbb{P}} q \in B)\}.$$

It is easy to check that $\mathbb{P} \Vdash \tau = \tau{\downarrow}B$.

Actually, we may pick $\dot{T}$ in such a way that it is a $(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D$-name, again using the translation described in footnote 6.

$\mathrm{Col}(\mu,\lambda)$ forces that $\pi(\dot{T})$ is μ-robust, where I use π also to denote the canonical transformation of names it induces: If G is generic for $\mathrm{Col}(\mu,\lambda)$, then $\mathrm{V}[G]=\mathrm{V}[H]$, where $H=\pi^{-1}``G$ is generic for $(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D$. Let H' be the filter in $\mathbb{P}\times\mathrm{Col}(\mu,\lambda)$ which is generated by H. Then H' is generic for $\mathbb{P}\times\mathrm{Col}(\mu,\lambda)$, because given a dense open subset E of $\mathbb{P}\times\mathrm{Col}(\mu,\lambda)$, $D\cap E$ is dense in $(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D$, hence E has nonempty intersection with H. So H' is of the form $H'_0\times H'_1$. Now $\pi(\dot{T})^G=\pi(\dot{T})^{G\cap\Delta}=\dot{T}^H=\dot{T}^{H'}=\dot{S}^{H'_0}$. The latter is μ-robust in $\mathrm{V}[H'_0]$, by assumption. So since $\mathrm{Col}(\mu,\lambda)$ is $<\!\mu$-closed in $\mathrm{V}[H'_0]$, it follows that $\dot{S}^{H'_0}=\pi(\dot{T})^G$ is μ-robust in $\mathrm{V}[H'_0][H'_1]=\mathrm{V}[H]=\mathrm{V}[G]$, as claimed (see the remark after Definition 3.5).

Now I apply the assumption to $\mathrm{Col}(\mu,\lambda)$, $\langle\tilde{D}_\alpha\mid\alpha<\mu\rangle$ and $\pi(\dot{T})$. It is unproblematic to add the dense sets $\langle\Delta_\alpha\mid\alpha<\mu\rangle$, where Δ_α consists of those conditions $p\in\mathrm{Col}(\mu,\lambda)$ with $\alpha\subseteq\mathrm{dom}(p)$. This gives a filter F intersecting each $\tilde{D}_\alpha$ and Δ_α, such that $(\pi(\dot{T}))^F$ is stationary in μ. Let $G=\pi^{-1}``F$. Then G is a filter in $(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D$: First note that $F':=F\cap\Delta$ is a filter in $\mathrm{Col}(\mu,\lambda)\restriction\Delta$. It is clearly nonempty, as F intersects the Δ_α's, and it is clearly upward closed. To see that it is a filter, note that if $p,q\in F'$, then p and q have to be compatible, since they both are in F. But then one of them must extend the other, since the domains of the conditions in F' are linearly ordered by inclusion. Now it follows immediately that $G=\pi^{-1}``F=\pi^{-1}``F'$ is a filter in $(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D$.

Let G' be the filter generated by G in $\mathbb{P}\times\mathrm{Col}(\mu,\lambda)$, and let $H=p_0[G']$. Then H is a filter in $\mathbb{P}$. I claim that H has the desired properties.

H intersects every D_α, for $\alpha<\mu$: By assumption, $F\cap\tilde{D}_\alpha\neq\emptyset$. Since $\tilde{D}_\alpha=\pi``D'_\alpha$, this implies that $G\cap D'_\alpha\neq\emptyset$, so in particular that $G'\cap D'_\alpha\neq\emptyset$. Since $D'_\alpha=(D_\alpha\times\mathrm{Col}(\mu,\lambda))\cap D$ and $H=p_0[G']$, this implies that $H\cap D_\alpha\neq\emptyset$.

Finally, I have to verify that $\dot{S}^H$ is a stationary subset of $\mathcal{P}_\mu(\mu)$. For this, it suffices to prove that $\pi(\dot{T})^F\subseteq\dot{S}^H$, as the former set is stationary, by the choice of F. So let $x\in\pi(\dot{T})^F$. Let $q\in F$ force that $\check{x}\in\pi(\dot{T})$. Pick $\beta<\mu$ such that $\mathrm{dom}(q)\subseteq\beta$. Choose $q'\in F\cap\Delta_{\beta+1}$. It follows that $\tilde{q}:=q'\restriction(\beta+1)$ is an extension of q, and moreover that $\tilde{q}\Vdash_{\mathrm{Col}(\mu,\lambda)\restriction\Delta}\check{x}\in\pi(\dot{T})$, the point being that $\tilde{q}\in F'$.[7] So $p:=\pi^{-1}(\tilde{q})\Vdash_{(\mathbb{P}\times\mathrm{Col}(\mu,\lambda))\restriction D}\check{x}\in\dot{T}$. But then it also follows that $p\Vdash_{\mathbb{P}\times\mathrm{Col}(\mu,\lambda)}\check{x}\in\dot{T}$ (again by footnote 7), and this means that $p_0(p)\Vdash_{\mathbb{P}}\check{\alpha}\in\dot{S}$, by the properties of $\dot{T}$. Since $\tilde{q}\in F'$, it follows that $p\in G\subseteq G'$, so that $p_0(p)\in H=p_0[G']$. So it follows that $x\in\dot{S}^H$, as wished. $\square$

This lemma makes it possible to work without any Laver function in the proof of the following theorem.

[7]It is generally true that if $E\subseteq\mathbb{Q}$ is dense, τ is a $\mathbb{Q}\restriction E$-name, $p\in E$ and $\varphi(v)$ is a formula, then $p\Vdash_{\mathbb{Q}}\varphi(\tau)$ if and only if $p\Vdash_{\mathbb{Q}\restriction E}\varphi(\tau)$, as the reader will verify without difficulty.

Theorem 3.8. *Let κ be an indestructibly weakly compact cardinal, and let $\mu < \kappa$ be a regular uncountable cardinal. Let G be $\mathrm{Col}(\mu, {<}\kappa)$-generic over V. Then*

$$\mathrm{V}[G] \models \mathsf{MA}^+(<\mu\text{-}closed).$$

Proof. Let G be generic for $\mathrm{Col}(\mu, {<}\kappa)$ over V. In order to verify that $\mathsf{MA}^+({<}\mu\text{-closed})$ holds in $\mathrm{V}[G]$, it suffices by the previous lemma to consider forcings $\mathbb{Q} \in \mathrm{V}[G]$ of the form $\mathrm{Col}(\mu, \lambda)$, for λ with $\lambda = \lambda^{<\mu}$. Fix such λ and $\mathbb{Q}$, let $\langle D_\alpha \mid \alpha < \mu\rangle$ be a sequence of dense subsets of $\mathbb{Q}$ in $\mathrm{V}[G]$, and let $\dot{S} \in \mathrm{V}[G]$ be a $\mathbb{Q}$-name such that $\mathbb{Q}$ forces over $\mathrm{V}[G]$ that $\dot{S}$ is a μ-robust subset of $\mathcal{P}_\mu(\mu)$. Let $\Omega \geq \Omega(\lambda)$, as computed in V, and let X be $\mathrm{Col}(\kappa, \Omega)^{\mathrm{V}}$-generic over $\mathrm{V}[G]$. Note that of course, $\mathrm{Col}(\kappa, \Omega)^{\mathrm{V}}$ is not the same as $\mathrm{Col}(\kappa, \Omega)^{\mathrm{V}[G]}$. Let $j : \mathrm{V} \longrightarrow_{\mathcal{F}} M$ be the elementary embedding induced by a suitable generic λ-weak compactness measure $\mathcal{F}$ on $\mathcal{P}_\kappa(\lambda)$. So j and M are defined in $\mathrm{V}[X]$. Observe that $j(\kappa) = [\mathrm{const}_\kappa]_{\mathcal{F}} > [x \mapsto \mathrm{otp}(x)]_{\mathcal{F}} = \lambda$.

Note also that

$$j(\mathrm{Col}(\mu, {<}\kappa)^{\mathrm{V}}) = \mathrm{Col}(\mu, {<}j(\kappa))^M = \mathrm{Col}(\mu, {<}j(\kappa))^{\mathrm{V}[X]} = \mathrm{Col}(\mu, {<}j(\kappa))^{\mathrm{V}},$$

because M is closed under ${<}\kappa$-sequences in $\mathrm{V}[X]$, and because $\mathrm{Col}(\kappa, \Omega)$ is more than sufficiently closed. Now let H be a $\mathrm{Col}(\mu, [\kappa, j(\kappa)))$-generic filter over $\mathrm{V}[X][G]$. Standard arguments show that j can be extended in $\mathrm{V}[X][G][H]$ to an embedding

$$j' : \mathrm{V}[G] \longrightarrow M[G][H],$$

the point being that $j``G = G \subseteq G \times H$, in the appropriate sense.

Since $\mathrm{Col}(\mu, \lambda)$ is forcing equivalent to $\mathrm{Col}(\mu, [\kappa, \lambda])$, which is witnessed by a dense subset $D_0 \subseteq \mathrm{Col}(\mu, \lambda)$, a dense subset $D_1 \subseteq \mathrm{Col}(\mu, [\kappa, \lambda])$ and an isomorphism $\pi : \mathrm{Col}(\mu, \lambda) \restriction D_0 \overset{\sim}{\longleftrightarrow} \mathrm{Col}(\mu, [\kappa, \lambda]) \restriction D_1$ in V, it follows that there are filters G' and H' which are definable from H in any model containing π and $\mathrm{Col}(\mu, \lambda)$, such that $G \times G' \times H'$ is $\mathrm{Col}(\mu, {<}\kappa) \times \mathrm{Col}(\mu, \lambda) \times \mathrm{Col}(\mu, (\lambda, j(\kappa)))$-generic over $\mathrm{V}[X]$ and $\mathrm{V}[X][G][G'][H'] = \mathrm{V}[X][G][H]$.

Note that $\lambda^{<\mu} = \lambda$ in V, so that the transitive closure of $\mathrm{Col}(\mu, \lambda)$ has size λ in V. It follows from point 2 of Corollary 2.5 that $j \restriction \mathrm{Col}(\mu, \lambda) \in M$. Actually, it follows that $\mathrm{Col}(\mu, \lambda)$ has size at most λ in M, since any bijection between λ and $\mathrm{Col}(\mu, \lambda)$ that exists in V is also in M. For the same reason, the isomorphism $\pi : \mathrm{Col}(\mu, \lambda) \restriction D_0 \overset{\sim}{\longleftrightarrow} \mathrm{Col}(\mu, [\kappa, \lambda]) \restriction D_1$ is in M. So $G' \in M[G][H]$, and it follows that $\bar{F} = j``G' \in M[G][H]$. $\bar{F}$ generates a filter in $\mathrm{Col}(\mu, j(\lambda))$, call it F. Let's verify the following points in $M[G][H]$:

1. For every $\alpha < \mu$, $F \cap j'(\vec{D})_\alpha \neq \emptyset$,

2. $j'(\dot{S})^F$ is a stationary subset of $\mathcal{P}_\mu(\mu)$.

Note that $\mathcal{P}_\mu(\mu)$ is the same in each of the models at hand, because M is $<\kappa$-closed in $\mathrm{V}[X]$ and all the forcings considered are $<\mu$-closed.

The first point follows, since $j'(\vec{D})_\alpha = j'(D_\alpha)$ (as $\mu < \kappa = \mathrm{crit}(j')$), and $G' \cap D_\alpha \neq \emptyset$, as G' is $\mathrm{Col}(\mu, \lambda)$-generic over $\mathrm{V}[G]$, where $\vec{D}$ lives.

For the second one: Let $C \in M[G][H]$ be a club subset of $\mathcal{P}_\mu(\mu)$. Let $S = \dot{S}^{G'}$. Then S is a stationary subset of μ in $\mathrm{V}[G][G']$ by assumption, hence S is stationary in $\mathrm{V}[G][G'][X]$, because $\mathrm{Col}(\kappa, \Omega)^{\mathrm{V}}$ is still $<\mu$-closed in $\mathrm{V}[G][G']$ and hence preserves stationary subsets of μ.

($*$) $S \in M[G][G']$.

Proof of ($*$). There is a nice $\mathrm{Col}(\mu, \lambda)$-name $\dot{R} \in \mathrm{V}[G]$ for a subset of $\mathcal{P}_\mu(\mu)$ such that $\dot{R}^{G'} = \dot{S}^{G'}$. Note that in V, $\mathcal{P}_\mu(\mu)$ has size $\mu^{<\mu} \leq \lambda^{<\mu} = \lambda$. So using a bijection between $\mathrm{Col}(\mu, \lambda)$ and λ, and an injection from $\mathcal{P}_\mu(\mu)$ into λ, which exist in V and hence in M, $\dot{R}$ can be viewed as a subset of λ. So there is a $\mathrm{Col}(\mu, <\kappa)$-name $\dot{R}'$ for $\dot{R}$ in V. Again, $\dot{R}'$ can be chosen to be a nice $\mathrm{Col}(\mu, <\kappa)$-name for a subset of λ, and so, $\dot{R}'$ can be viewed as a subset of λ, again using a bijection between $\mathrm{Col}(\mu, <\kappa) \times \lambda$ and λ which exists in V and hence in M. It follows that $\dot{R}' \in M$. So $\dot{R} = (\dot{R}')^G \in M[G]$ (more precisely, the subset of λ coding $\dot{R}$ is in $M[G]$. But the bijections used to encode $\dot{R}$ are in M, and so, the subset of λ can be decoded in $M[G]$, so that $\dot{R} \in M[G]$). So $S = \dot{R}^{G'} \in M[G][G']$. $\square_{(*)}$

Since $M \subseteq \mathrm{V}[X]$ and hence $M[G][G'] \subseteq \mathrm{V}[G][G'][X]$, where S is stationary, it follows that S is stationary in $M[G][G']$, as well. Moreover, S is μ-robust in $\mathrm{V}[G][G'][X]$ by assumption, which implies that S is μ-robust also in $M[G][G']$. It is easiest to see this by realizing that it suffices to show that the stationarity of S is preserved by forcings of the form $\mathrm{Col}(\mu, \theta)$ over $M[G][G']$. These forcings are the same in all of the models considered, and in particular, the stationarity of S is preserved by forcing with $\mathrm{Col}(\mu, \theta)$ over $\mathrm{V}[G][G'][X]$, which contains $M[G][G']$.

So since H' is generic over $M[G][G']$ for a $<\mu$-closed forcing, S remains stationary in $M[G][G'][H'] = M[G][H]$. Since $C \in M[G][H]$, there is some $x \in S \cap C$. Continuing in $\mathrm{V}[G]$, and remembering that $S = \dot{S}^{G'}$, let $p \in G'$ now be such that p forces over $\mathrm{V}[G]$ with respect to $\mathrm{Col}(\mu, \lambda)$ that $\check{x} \in \dot{S}$. Then $j'(p) \in F$ forces over $M[G][H]$ that $\check{x} \in j'(\dot{S})$. So $x \in C \cap j'(\dot{S})^F$, which proves the second point.

So in $M[G][H]$, the statement that there exists a filter F in $j'(\mathrm{Col}(\mu, \lambda))$ satisfying the above points is true. This is a statement about the parameters $j'(\mathrm{Col}(\mu, \lambda))$, $\mu = j'(\mu)$, $j'(\vec{D})$ and $j'(\dot{S})$. Hence, by elementarity of j', the same statement is true in $\mathrm{V}[G]$ of $\mathrm{Col}(\mu, \lambda)$, μ, $\vec{D}$ and $\dot{S}$, showing that there is a $\vec{D}$-generic filter $F' \subseteq \mathrm{Col}(\mu, \lambda)$ in $\mathrm{V}[G]$ such that $\dot{S}^{F'}$ is stationary in μ. $\square$

Remark 3.9. The proof of the previous theorem goes through if κ's indestructible weak compactness is replaced by the assumption that there are arbitrarily large α such that there is a $<\kappa$-closed very fine V-normal measure on $\mathcal{P}_\kappa(\alpha)$.

There are natural strengthenings of the axiom $\mathsf{MA}^+(\Gamma,\mu)$, called $\mathsf{MA}^{++}(\Gamma,\mu)$, stating that given a poset $\mathbb{P}\in\Gamma$, a sequence $\langle D_\alpha \mid \alpha<\mu\rangle$ of dense subsets of $\mathbb{P}$ and now a *sequence* of names $\langle \dot{S}_\alpha \mid \alpha<\mu\rangle$ for μ-robust subsets of $\mathcal{P}_\mu(\mu)$, there is a filter F in $\mathbb{P}$ which is $\vec{D}$-generic and has the property that *for all* $\alpha<\mu$, the set $\dot{S}^F_\alpha$ is stationary in μ. Using the same notational simplifications as before, a straightforward modification of the proof of Lemma 3.7 shows the following.

Lemma 3.10. *Let μ be a regular cardinal. Then*

$$\mathsf{MA}^{++}(<\!\mu\text{-}closed) \iff \mathsf{MA}^{++}(\{\mathrm{Col}(\mu,\lambda)\mid \lambda=\lambda^{<\kappa}\},\mu).$$

Using this, the proof of Theorem 3.8 is easily adapted to yield:

Theorem 3.11. *Let κ be an indestructibly weakly compact cardinal, and let $\mu<\kappa$ be a regular uncountable cardinal. Let G be $\mathrm{Col}(\mu,<\!\kappa)$-generic over* V*. Then*

$$\mathrm{V}[G]\models \mathsf{MA}^{++}(<\!\mu\text{-}closed).$$

Proof. As before. Instead of $\dot{S}$, one has to work with a sequence $\vec{S}=\langle \dot{S}_\alpha \mid \alpha<\mu\rangle$ this time. Running the proof as before, one now has to replace point 2 with the following:

2.' For all $\alpha<\mu$, $(j'(\vec{S})_\alpha)^F$ is stationary in $\mathcal{P}_\mu(\mu)$.

The point is that fixing $\alpha<\mu$, $j'(\vec{S})_\alpha=j'(\dot{S}_\alpha)$, as $\mu<\kappa=\mathrm{crit}(j')$. The original proof shows that $j'(\dot{S}_\alpha)^F$ is stationary in $M[G][H]$. Pulling back to $\mathrm{V}[G]$ finishes the proof. □

Definition 3.12 ([For, Def. 8.22])**.** If S is a stationary subset of $\mathcal{P}(H_\theta)$, then it *reflects to a set of size* μ if there is a set $Y\subseteq H_\theta$ with $\mu\subseteq Y$ of cardinality μ such that $S\cap\mathcal{P}(Y)$ is stationary in Y.

The following lemma is a generalization of an observation in [FMS88, p. 20]. It suggests that MA^+($<\!\mu$-closed) seems to be the right generalization of MA^+(σ-closed) - see Theorem 3.16 for some consequences.

Lemma 3.13. *Assume MA^+($<\!\mu$-closed), where μ is a regular cardinal. If $\lambda>\mu$ and $S\subseteq\mathcal{P}_\mu(H_\lambda)$ is μ-robust, then S reflects to a set of size μ.*

Proof. Let $\mathbb{P}=\mathrm{Col}(\mu,\overline{\overline{H_\lambda}})$. Let $\dot{f}$ be a $\mathbb{P}$-name for a bijection between μ and H^{V}_λ. Since S is μ-robust, it is still stationary in $\mathrm{V}[G]$, whenever G is $\mathbb{P}$-generic. So the set

$$\{x\in\mathcal{P}_\mu(\mu)\mid (\dot{f}^G)``x\in S\}$$

is a stationary subset of $\mathcal{P}_\mu(\mu)$ in $\mathrm{V}[G]$. Let $\dot{T}$ be a name for this stationary set. Apply MA^+($<\!\mu$-closed), to $\mathbb{P}$, $\dot{T}$, and the collection $\mathcal{D}=\{D_\alpha\mid\alpha<\mu\}$ of dense sets, where D_α consists of those conditions that decide the value of $\dot{f}(\check{\alpha})$ and that force that α is in the range of $\dot{f}$. Working below a condition that forces that $\dot{T}$ is the set of all $x \in \mathcal{P}_\mu(\mu)$ such that $\dot{f}``x \in \check{S}$ and that $\dot{f}$ is injective, this gives a $\mathcal{D}$-generic filter $F \in \mathrm{V}$ such that $\dot{T}^F = \{\alpha < \mu \mid \bar{f}``\alpha \in S\}$ is stationary. Let $\bar{f}(\alpha) = (\dot{f}(\check{\alpha}))^F$. It follows that S reflects to $X := \bar{f}``\mu$: Let $\bar{h} : [X]^{<\omega} \longrightarrow X$. Let $h : [\mu]^{<\omega} \longrightarrow \mu$ be induced by $\bar{f}$, i.e., let $\bar{h}(s) = \bar{f}^{-1}(h(\bar{f}``s))$. Since $\dot{T}^F$ is stationary in $\mathcal{P}_\mu(\mu)$, there is an $x \in \dot{T}^F$ which is closed under $\bar{h}$. It follows that $\bar{f}``x$ is closed under $h$, and by the choice of $F$, $\bar{f}``x \in S$. Moreover, $\mu \subseteq X$, by the choice of $\mathcal{D}$. So X is as wished. □

Let's concentrate on the case $\mu = \aleph_1$ for a while. Since every stationary set is $\aleph_1$-robust, the previous lemma shows that under $\mathsf{MA}^+(\sigma\text{-closed})$, every stationary subset of $\mathcal{P}_{\aleph_1}(X)$ reflects to a set of size $\aleph_1$. This property is sometimes referred to as the *reflection principle* (RP), and it is this special case of the previous lemma that is contained in [FMS88]. (RP), in turn, implies the principle (†) of [FMS88] which says that a forcing notion preserves stationary subsets of ω_1 if and only if it is semi-proper, see [Jec03, Ex. 37.13]. In [FMS88, p. 31, Thm. 26] it was shown that (†) implies that the nonstationary ideal on ω_1 is precipitous, and in Thm. 25 (on p. 29) of the same paper it was shown that $\mathsf{MA}^+(\sigma\text{-closed})$ implies that the nonstationary ideal on ω_1 is pre-saturated. So this gives yet another way to produce generic elementary embeddings. Abstracting from [Vel92, Section 3], let's say that a stationary set $S \subseteq [H_\theta]^\omega$ *strongly reflects* if there exists an elementary chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ of countable elementary submodels of H_θ such that the set $\{\alpha < \omega_1 \mid M_\alpha \in S\}$ is stationary in ω_1. It is shown in [Jec03, Exercise 37.23] that $\mathsf{MA}^+(<\aleph_1\text{-closed})$ implies that every stationary subset $[H_\lambda]^\omega$ (where λ has uncountable cofinality) strongly reflects, and [Vel92, Theorem 3.2] shows that if every stationary subset of $[H_\lambda]^\omega$ strongly reflects, where λ is regular, then $\lambda^\omega = \lambda$. So since under $\mathsf{MA}^+(<\aleph_1\text{-closed})$, this holds for every regular $\lambda > \aleph_1$, this implies the singular cardinal hypothesis: It suffices to prove for singular λ of countable cofinality with $2^\lambda < \lambda$, that $\lambda^\omega = \lambda^+$. This follows since $\lambda^+ \le \lambda^\omega \le (\lambda^+)^\omega = \lambda^+$. So putting these known results together with Theorem 3.8 results in the following.

Corollary 3.14. *If κ is indestructibly weakly compact, and G is $\mathrm{Col}(\omega_1, <\kappa)$-generic over* V*, then in* V$[G]$*, the following hold:*

1. MA^+*(σ-closed),*
2. *the reflection principle* (RP)*,*
3. *the principle* (†)*,*
4. *the nonstationary ideal on ω_1 is pre-saturated,*
5. SCH.

In fact, these points follow from MA^+*(σ-closed).*

That SCH holds in $V^{\mathrm{Col}(\omega_1,<\kappa)}$ implies that a certain amount of SCH holds already in V:

Corollary 3.15. SCH *holds above an indestructibly weakly compact cardinal, in the following sense: If λ is a singular cardinal larger than κ such that $2^{\mathrm{cf}(\lambda)} < \lambda$, then $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+$.*

Proof. Fix such λ, let $\bar{\lambda} = \mathrm{cf}(\lambda)$, and let G be $\mathrm{Col}(\omega_1, <\kappa)$-generic over V. Since the forcing is $<\kappa$-c.c., it suffices to show that $(\lambda^{\bar{\lambda}})^{V[G]} = (\lambda^+)^{V[G]}$. For if $\lambda^{\bar{\lambda}}$ were greater than $\lambda^+ = (\lambda^+)^{V[G]}$, then this would mean that $\lambda^{\bar{\lambda}}$ is collapsed. And to see that $(\lambda^{\bar{\lambda}})^{V[G]} = (\lambda^+)^{V[G]}$, it suffices to show that in $V[G]$, $2^{\mathrm{cf}(\lambda)} < \lambda$, since SCH holds in $V[G]$.

If $\bar{\lambda} \geq \kappa$, then $\mathrm{cf}(\lambda)^{V[G]} = \bar{\lambda}$, because $\bar{\lambda}$ is preserved as a regular cardinal. By the $<\kappa$-c.c., it follows that $(2^{\mathrm{cf}(\lambda)})^{V[G]} \leq \kappa^{\bar{\lambda}} \leq (2^\kappa)^{\bar{\lambda}} = 2^{\bar{\lambda}} < \lambda$, as wished. If $\bar{\lambda} = \omega$, then since $\mathrm{Col}(\omega_1, <\kappa)$ is σ-closed, it follows that $(2^\omega)^{V[G]} \leq 2^\omega < \kappa < \lambda$, so there is also no problem. If $\bar{\lambda} \in [\omega_1, \kappa)$, then $\bar{\lambda}$ has cardinality ω_1 in $V[G]$, so $\mathrm{cf}(\lambda)^{V[G]} = \omega_1$. It follows that in this case also, $(2^{\mathrm{cf}(\lambda)})^{V[G]} = (2^{\omega_1})^{V[G]} = \kappa < \lambda$. $\square$

This is a striking parallel to strongly compact cardinals. The following version of [For, Theorem 8.37] highlights the relevance of reflection of robust sets.

Theorem 3.16. *If μ is a regular cardinal less than κ and for every $\lambda > \mu$, every μ-robust subset of $\mathcal{P}_\mu(H_\lambda)$ reflects to a set of size μ, then the nonstationary ideal, restricted to $\mathcal{P}_\mu(\mu)$, is precipitous. In particular,*

1. $\mathsf{NS} \upharpoonright \mu$ *is precipitous,*
2. $\mathsf{NS} \upharpoonright \mathcal{P}_\gamma(\mu)$ *is precipitous, for every regular uncountable $\gamma \leq \mu$.*

So by Lemma 3.13, these are consequences of MA^+*($<\mu$-closed), and hence true in $V^{\mathrm{Col}(\mu,<\kappa)}$, if $\kappa > \mu$ is indestructibly weakly compact.*

Proof. The proof of Theorem 8.37 of [For] shows that the conclusion holds in $V^{\mathrm{Col}(\mu,<\kappa)}$, where κ is supercompact. But it uses only the fact that every μ-robust subset of $\mathcal{P}_\mu(H_\lambda)$ reflects to a set of size μ there. $\square$

There is another consequence of indestructible weak compactness that will be of importance in connection with the countable stationary tower, in section 4. Here is a weak version of a completely Jónsson cardinal that's sufficiently strong to guarantee for the countable tower what completely Jónsson cardinals guaranteed for the full stationary tower; see Theorem 4.8.

Definition 3.17. Let κ be inaccessible. Then κ is *countably completely ω_1-Jónsson* if for every nonempty, stationary set $a \in V_\kappa$ which consists of countable sets, the set of $X \in V_\kappa$ with $X \cap (\cup a) \in a$ and $\mathrm{otp}(X \cap \kappa) \geq \omega_1$ is stationary.

Observation 3.18. *If κ is a weakly compact cardinal which is countably completely ω_1-Jónsson, then the set of $\bar{\kappa} < \kappa$ which are countably completely ω_1-Jónsson is stationary in κ.*

Proof. The fact that κ is countably completely ω_1-Jónsson is expressible as a Π^1_1 statement about κ, so it reflects to a stationary set, by κ's weak compactness. $\square$

Observation 3.19. *If κ is regular and the set of countably completely ω_1-Jónsson cardinals below κ is stationary in κ, then κ is countably completely ω_1-Jónsson.*

Proof. First observe that κ is inaccessible. So given a member a of $\mathbb{Q}_{<\kappa}$ and an algebra $\mathfrak{A}$ on V_κ, the set of $\bar{\kappa} < \kappa$ such that $\mathrm{V}_{\bar{\kappa}}$ is closed under $\mathfrak{A}$ is club in κ. So pick such a $\bar{\kappa}$ which is completely ω_1-Jónsson and which is large enough that $a \in \mathrm{V}_{\bar{\kappa}}$. Pick $X \in \mathrm{V}_{\bar{\kappa}}$ such that $\mathrm{otp}(X \cap \bar{\kappa}) \geq \omega_1$, X is closed under $\mathfrak{A}|\mathrm{V}_{\bar{\kappa}}$ and $X \cap (\cup a) \in a$, which is possible by the choice of $\bar{\kappa}$. But then X is also $\mathfrak{A}$-closed, as $\mathrm{V}_{\bar{\kappa}}$ is. $\square$

So in fact, for weakly compact κ, κ is countably completely ω_1-Jónsson iff the set of countably completely ω_1-Jónsson cardinals below κ is stationary. I shall prove that if κ is indestructibly weakly compact, then it is also countably completely ω_1-Jónsson. To this end, I shall use the following concept, introduced by Shelah.

Definition 3.20. The *strong Chang Conjecture (SCC)* says that for all large enough λ ($\lambda > 2^{\aleph_2}$ will suffice), all models M with universe H_λ, all countable $N \prec M$ and all $\alpha < \aleph_2$, there is a $\beta \in (\alpha, \aleph_2)$ and a model N' such that

$$N \subseteq N' \prec M,\ \beta \in N' \text{ and } N \cap \omega_1 = N' \cap \omega_1.$$

The following is due to Shelah:

Theorem 3.21 ([She98, Theorem 1.3])**.** *If Namba forcing is semi-proper, then (SCC) holds.*

So since (†) holds in $\mathrm{V}^{\mathrm{Col}(\omega_1,<\kappa)}$ if κ is indestructibly weakly compact, we get:

Corollary 3.22. *If κ is indestructibly weakly compact, then* $\mathrm{V}^{\mathrm{Col}(\omega_1,<\kappa)} \models (SCC)$.

This gives another consequence of indestructible weak compactness that will be of importance in section 4. The proof of the following theorem shows that if $\mathrm{Col}(\omega_1, <\kappa)$ forces (SCC) to be true, then κ is countably completely ω_1-Jónsson.

Theorem 3.23. *If (SCC) holds in $\mathrm{V}^{\mathrm{Col}(\omega_1,<\kappa)}$, then κ is countably completely ω_1-Jónsson. So this is true, in particular, if κ is indestructibly weakly compact.*

Proof. Let $a \in \mathbb{Q}_{<\kappa}$ be given. Fix an algebra $\mathfrak{A} = \langle \mathrm{V}_\kappa, \langle f_n \mid n < \omega\rangle\rangle$. To show that κ is countably completely ω_1-Jónsson, an $\mathfrak{A}$-closed set X with $\mathrm{otp}(X \cap \kappa) \geq \omega_1$ and $X \cap (\cup a) \in a$ is needed.

To this end, let G be $\mathrm{Col}(\omega_1, {<}\kappa)$-generic over V. Since a consists of countable sets, it follows that a is stationary in $\mathrm{V}[G]$. Since $a \in \mathrm{V}_\kappa^{\mathrm{V}}$, it follows that $\cup a$ has size at most $\aleph_1$ in $\mathrm{V}[G]$.

Work in $\mathrm{V}[G]$. Consider the model $M = \langle H_\lambda, \in, <^*, \mathfrak{A}\rangle$, where $<^*$ is a well-order of H_λ (that one can do without). Since a is stationary, so is its countable lift $b := a \uparrow H_\lambda$. So let $\cup a \in N_{-1} \prec M$ with $N_{-1} \in b$. This means that N_{-1} is countable and $N_{-1} \cap (\cup a) \in a$. Applying (SCC) in $\mathrm{V}[G]$ $\aleph_1$ many times gives sequences $\vec{N} = \langle N_\alpha \mid -1 \leq \alpha < \omega_1\rangle$ and $\vec{\theta} = \langle \theta_\alpha \mid \alpha < \omega_1\rangle$ such that for $-1 \leq \alpha < \beta < \omega_1$, the following conditions hold:

1. For $\alpha \geq 0$, $\theta_\alpha < \theta_\beta < \omega_2$,
2. $N_{-1} \subseteq N_\alpha \subseteq N_\beta \prec M$,
3. $N_\alpha \cap \omega_1 = N_{-1} \cap \omega_1$,
4. N_α is countable,
5. $\theta_\beta \in N_\beta$.

It follows that

$$N_\alpha \cap (\cup a) = N_{-1} \cap (\cup a).$$

To see this, only the inclusion from left to right is substantial. So let $x \in N_\alpha \cap (\cup a)$. Let $\xi = \overline{\overline{\cup a}}^{\mathrm{V}[G]}$, so either $\xi = \aleph_1$ or $\xi = \aleph_0$. Since ξ is the cardinality of $\cup a$ in M, the same is true in N_{-1}, so there is a g such that N_{-1} thinks that $g : \xi \rightarrowtail\!\!\!\twoheadrightarrow \cup a$ is a bijection. g is then really a bijection, since $N_{-1} \prec M$, and since $N_{-1} \subseteq N_\alpha$, $g \in N_\alpha$ as well, and g is a bijection between ξ and $\cup a$ from the point of view of N_α, as well. It follows that $\gamma := g^{-1}(x) \in N_\alpha$. This is a countable ordinal, so since $N_\alpha \cap \omega_1 = N_{-1} \cap \omega_1$, it follows that $\gamma \in N_{-1}$. But $g \in N_{-1}$ as well, so that $x = g(\gamma) \in N_{-1}$.

Define $x_0 = N_{-1} \cap (\cup a)$.

Now let $M_\alpha = N_\alpha \cap \mathrm{V}_\kappa^{\mathrm{V}}$, for $\alpha < \omega_1$ (including $\alpha = -1$. Noting that $\kappa = (\aleph_2)^{\mathrm{V}[G]}$, it follows that $\langle M_\alpha \mid \alpha < \omega_1\rangle$ has the corresponding properties (for $-1 \leq \alpha < \beta < \omega_1$):

1. For $\alpha \geq 0$, $\theta_\alpha < \theta_\beta < \kappa$,
2. $M_{-1} \subseteq M_\alpha \subseteq M_\beta$, and M_α is $\mathfrak{A}$-closed,
3. $M_\alpha \cap (\cup a) = M_{-1} \cap (\cup a) = x_0$,

4. M_α is countable,

5. $\theta_\alpha \in M_\alpha$.

Note that $M_\alpha \cap (\cup a) = (N_\alpha \cap \mathrm{V}_\kappa^\mathrm{V}) \cap (\cup a) = N_\alpha \cap (\mathrm{V}_\kappa^\mathrm{V} \cap (\cup a)) = N_\alpha \cap (\cup a) = x_0$. That M_α is $\mathfrak{A}$-closed is a standard argument: Fix $\vec{a} \in M_\alpha$, $\vec{a}$ having the arity of f_n, the n-th function in the algebra $\mathfrak{A}$. Note that since $\mathfrak{A} \in N_\alpha$, it follows that $f_n \in N_\alpha$. Also, $\vec{a} \in N_\alpha$, and so, $f_n(\vec{a}) \in N_\alpha$. Of course, $f_n(\vec{a}) \in \mathrm{V}_\kappa^\mathrm{V}$, so that $f_n(\vec{a}) \in M_\alpha$.

Since M_α is a countable subset of V and $\mathrm{Col}(\omega_1, {<}\kappa)$ is σ-closed, it follows that $M_\alpha \in \mathrm{V}$, for every $\alpha < \omega_1$. And trivially, $x_0 \in a \in \mathrm{V}$.

Now work in V. Pick a name $\dot{\vec{M}}$ for the sequence $\vec{M}$ and a name $\dot{\vec{\theta}}$ for the sequence $\vec{\theta}$. Pick a condition $p \in \mathrm{Col}(\omega, {<}\kappa)$ which forces the properties 1.-5. to hold of these names. Let D_α be the set of conditions below p in $\mathrm{Col}(\omega_1, {<}\kappa)$ which decide the value of $\dot{M}_\alpha$ and $\dot{\theta}_\alpha$. Let $\bar{G} \subseteq \mathrm{Col}(\omega_1, {<}\kappa)$ be $\{D_\alpha \mid \alpha < \omega_1\}$-generic. Let $\bar{M}_\alpha = (\dot{\vec{M}})^{\bar{G}}$ and $\bar{\theta}_\alpha = (\dot{\vec{\theta}})^{\bar{G}}$. Then $\langle \bar{M}_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle \bar{\theta}_\alpha \mid \alpha < \omega_1 \rangle$ have properties 1.-5. in V. So setting $\bar{M} := \bigcup_{\alpha < \omega_1} \bar{M}_\alpha$ gives the desired model in V. For $\bar{M} \cap (\cup a) = x_0 \in a$, and $\{\bar{\theta}_\alpha \mid \alpha < \omega_1\} \subseteq \bar{M} \cap \kappa$, so $\mathrm{otp}(\bar{M} \cap \kappa) \geq \omega_1$. $\square$

This gives the answer to a question I had at one point:

Question 3.24. Is there a weakly compact cardinal below every countably completely ω_1-Jónsson cardinal?

The answer is *no*, since it is consistent that the least weakly compact cardinal κ is indestructible (see [Fuc, Thm. 3.11]). By the previous theorem, it follows that κ is also countably completely ω_1-Jónsson. By Observation 3.18, there are many countably completely ω_1-Jónsson cardinals below κ, each of which has the property that there is no weakly compact cardinal below it.

4 The Countable Tower

In this section, I shall presuppose a certain acquaintance with stationary tower forcing and in particular with the countable tower. The monograph [Lar04] serves as my basic reference on this method. I introduced some notions and notations that will be needed in the present section already in Definition 3.2. I will recall some additional, relevant definitions (in the form that's most convenient) and facts when I need them. Since I will work only with the countable tower, in this section a stationary set a will always be a subset of $[\cup a]^\omega$.

Definition 4.1. Let κ be an inaccessible cardinal. Then the countable tower (below κ) is the partial ordering $\mathbb{Q}_{<\kappa} = \langle |\mathbb{Q}_{<\kappa}|, \leq \rangle$, consisting of non-empty stationary sets which are members of V_κ and which consist of countable sets. The ordering is

$$b \leq a \iff \cup a \subseteq \cup b \text{ and } b{\downarrow}(\cup a) \subseteq a.$$

The projection $b{\downarrow}(\cup a)$ was introduced in Definition 3.2.

Definition 4.2. Let x and y be sets. Then y *end-extends* x, $x <_{\text{end}} y$, if $x = y \cap V_{\text{rnk}(x)}$, where $\text{rnk}(x)$ is the rank of x, that is, the least α such that $x \subseteq V_\alpha$.

Definition 4.3. Let D be a predense subset of $\mathbb{Q}_{<\kappa}$. Then $\mathsf{sp}_{\text{countable}}(D)$ is the set of countable $X \prec V_{\kappa+1}$ such that there exists a countable $Y \prec V_{\kappa+1}$ with the following properties:

1. $X \subseteq Y$,

2. $X \cap V_\kappa <_{\text{end}} Y \cap V_\kappa$,

3. Y *captures* D, i.e., there is an $a \in Y \cap D$ such that $Y \cap (\cup a) \in a$.

The set D is said to be be (countably) *semi-proper* if $\mathsf{sp}_{\text{countable}}(D)$ contains a club subset of $[V_{\kappa+1}]^\omega$.

The following is the crucial technical lemma on countable semi-properness, extracted from [Lar04], see the proof of Lemma 2.5.6 there.

Lemma 4.4. *Let κ be inaccessible, $a \in \mathbb{Q}_{<\kappa}$, $\eta < \kappa$ and $\langle D_\alpha \mid \alpha < \eta\rangle$ be a sequence of predense, countably semi-proper subsets of $\mathbb{Q}_{<\kappa}$. Let a be the set of $X \prec V_{\kappa+1}$ such that*

1. $\overline{\overline{X}} = \omega$,

2. $X \cap (\cup a_0) \in a_0$,

3. for all $\alpha \in X \cap \eta$, X *captures* D_α.

Then a is stationary in $V_{\kappa+1}$.

Lemma 4.5. *Assume there is a $<\kappa$-closed-generic V-normal very fine measure on $\mathcal{P}_\kappa(2^\kappa)$. Then every predense subset of $\mathbb{Q}_{<\kappa}$ is semi-proper. In particular, this is true if κ is weakly compact with $\mathsf{ID}(\kappa) > \Omega(2^\kappa)$.*

Proof. Let $\mathbb{P}$ be an adequate $<\kappa$-closed partial order, G be V-generic for $\mathbb{P}$, and let $\mathcal{F} \in V[G]$ be a V-normal very fine measure on $\mathcal{P}_\kappa(2^\kappa)$. Let $j : V \longrightarrow_{\mathcal{F}} M$ be the ultrapower and corresponding embedding. Let $D \subseteq \mathbb{Q}_{<\kappa}$ be predense. Assuming that D is not countably semi-proper, it follows that $a := ([V_{\kappa+1}]^\omega \setminus \mathsf{sp}_{\text{countable}}(D))^V$ is stationary.

By Corollary 2.5, item 2 (take $T = \mathrm{V}_{\kappa+1} \cup [\mathrm{V}_{\kappa+1}]^\omega$, which is allowed as T has size 2^κ), it's clear that $j \restriction \mathrm{V}_{\kappa+1} \in M$ and that $j \restriction a \in M$. In particular, $a \in M$. Moreover, a is stationary in M: Since $\mathbb{P}$ is σ-closed and hence proper, it follows that a is stationary in $\mathrm{V}[G]$, and stationarity obviously is downward absolute, so that a is also stationary in M. Also, note that by item 3 of the same corollary, $\mathrm{V}^{\mathrm{V}}_\kappa = \mathrm{V}^M_\kappa = \mathrm{V}^{\mathrm{V}[G]}_\kappa$, so that I'll just write V_κ for any of the three.

Since a is stationary in M, it follows that $a \in j(\mathbb{Q}_{<\kappa})$. So since $j(D)$ is a predense subset of $j(\mathbb{Q}_{<\kappa})$ in M, there is some $b \in j(D)$ such that a and b are compatible in $j(\mathbb{Q}_{<\kappa})$. Work in M now, where b is stationary, since $b \in j(D) \subseteq (\mathbb{Q}_{<j(\kappa)})^M$. Fix η such that $j \restriction \mathrm{V}^{\mathrm{V}}_{\kappa+1} \in \mathrm{V}^M_\eta$. That a and b are compatible means that there is some countable $X \prec \mathrm{V}^M_\eta$ with

1. $\{a, b, j \restriction \mathrm{V}^{\mathrm{V}}_{\kappa+1}, \mathrm{V}^M_{j(\kappa)+1}\} \subseteq X$,

2. $X \cap (\cup a) \in a$ and $X \cap (\cup b) \in b$.

Let $Y = X \cap \mathrm{V}^{\mathrm{V}}_{\kappa+1}$. Note that $Y \prec \mathrm{V}^{\mathrm{V}}_{\kappa+1}$, because $\mathrm{V}^{\mathrm{V}}_{\kappa+1} \in X \prec \mathrm{V}^M_\eta$. Since $\cup a = \mathrm{V}^{\mathrm{V}}_{\kappa+1}$, it follows by 2. that $Y \in a$. So $j(Y) \in j(a)$. Since in M, $j(a) = [\mathrm{V}_{j(\kappa)+1}]^\omega \setminus \mathsf{sp}_{\mathrm{countable}}(j(D))$, this means that $j(Y) \notin \mathsf{sp}_{\mathrm{countable}}(j(D))^M$. This will yield a contradiction, since

$$\bar{X} := X \cap j(\mathrm{V}^{\mathrm{V}}_{\kappa+1}) = X \cap \mathrm{V}^M_{j(\kappa)+1} \text{ witnesses that } j(Y) \in \mathsf{sp}_{\mathrm{countable}}(j(D))^M,$$

which I will verify in the rest of the proof. First note that $j(Y) \prec \mathrm{V}^M_{j(\kappa)+1}$, because $Y \prec \mathrm{V}^{\mathrm{V}}_{\kappa+1}$. Also, since by 1., $\mathrm{V}^M_{j(\kappa)+1} \in X$, and since $X \prec \mathrm{V}^M_\eta$, it follows that $\bar{X} \prec \mathrm{V}^M_{j(\kappa)+1}$. Now let's go through the points that need verification, according to Definition 4.3:

1) It must be verified that $j(Y) \subseteq \bar{X}$. Since $\overline{\overline{Y}} < \kappa$ (it's even countable), it follows that $j(Y) = j``Y$. So since $Y \subseteq X$ and $j \restriction \mathrm{V}^{\mathrm{V}}_{\kappa+1} \in X$, as a consequence, $j(Y) = j``Y \subseteq X$. Of course, $Y \subseteq \mathrm{V}^{\mathrm{V}}_{\kappa+1}$, so $j(Y) \subseteq \mathrm{V}^M_{j(\kappa)+1}$, so that $j(Y) \subseteq \bar{X}$.

2) It has to be shown that $\bar{X} \cap \mathrm{V}^M_{j(\kappa)+1}$ end-extends $j(Y) \cap j(\mathrm{V}_\kappa)$. Remembering that V_κ is the same, no matter which of the three models at hand it is computed in, first note that

$$j(Y) \cap j(\mathrm{V}_\kappa) = j``Y \cap j(\mathrm{V}_\kappa) = Y \cap \mathrm{V}_\kappa.$$

But continuing this,

$$Y \cap \mathrm{V}_\kappa = (X \cap \mathrm{V}^{\mathrm{V}}_{\kappa+1}) \cap \mathrm{V}_\kappa = X \cap \mathrm{V}_\kappa.$$

Putting this together gives the desired conclusion.

3) D is captured by $\bar{X}$: The set b is a witness. For $b \in X \cap \mathrm{V}^M_{j(\kappa)+1} \cap j(D) = \bar{X} \cap j(D)$ and $\bar{X} \cap (\cup b) = X \cap \mathrm{V}^M_{j(\kappa)+1} \cap (\cup b) = X \cap (\cup b) \in b$, by the choice of X.

So this shows that $j(Y) \in \mathsf{sp}_{\text{countable}}(j(D))$ after all, which is a contradiction. $\square$

The following is the version of [Lar04, Lemma 2.5.15] for the countable tower. For completeness, I give at least a proof sketch here, which is organized a bit differently. Note that the assumption of the lemma is satisfied if κ is weakly compact with $\mathsf{ID}(\kappa) > \Omega(2^\kappa)$, by the previous lemma.

Lemma 4.6. *If κ is weakly compact and every predense subset of $\mathbb{Q}_{<\kappa}$ is semi-proper, then the $\mathbb{Q}_{<\kappa}$-generic ultrapower is $<\kappa$-closed in the generic extension.*

Proof. If γ is inaccessible, $\eta < \gamma$, $\vec{D} = \langle D_\alpha \mid \alpha < \eta \rangle$ is a sequence of predense subsets of $\mathbb{Q}_{<\gamma}$ and $a_0 \in \mathbb{Q}_{<\gamma}$, then define the set $S_{a_0,\vec{D},\gamma}$ to consist of all countable $X \prec \mathrm{V}_{\gamma+1}$ such that $X \cap (\cup a_0) \in a_0$ and for every $\xi \in X \cap \eta$, X captures D_ξ.

Under the assumption that every predense subset of $\mathbb{Q}_{<\kappa}$ is semi-proper, the following statement is a consequence of κ's weak compactness:

(1) *If $a_0 \in \mathbb{Q}_{<\kappa}$, $\eta < \kappa$, and $\vec{D} = \langle D_\alpha \mid \alpha < \eta \rangle$ is a sequence of predense subsets of $\mathbb{Q}_{<\kappa}$, then there is an inaccessible $\gamma < \kappa$ such that $a_0, \eta \in \mathrm{V}_\gamma$, for all $\alpha < \eta$, $D_\alpha \cap \mathrm{V}_\gamma$ is predense in $\mathbb{Q}_{<\gamma}$, and $S_{a_0,\vec{D}\cap\mathrm{V}_\gamma,\gamma}$ is stationary in $\mathrm{V}_{\gamma+1}$.*[8]

In fact, it follows that the set of γ as in (1) is stationary in κ, by the reflection properties of κ. To see this, let $C \subseteq \kappa$ be club, and let a_0, η and $\vec{D}$ be as in (1). The point is that by Lemma 4.4, the set $S_{a_0,\vec{D},\kappa}$ is stationary in $\mathrm{V}_{\kappa+1}$, since each D_α is (countably) semiproper, by assumption. Now let λ be a regular cardinal greater than κ, and let N be the transitive collapse of an elementary submodel of V_λ of size κ which is closed under $<\kappa$-sequences and contains $\vec{D}$ as an element. If $j : N \longrightarrow N'$ is a weakly compact embedding which is an ultrapower embedding (so that N' is also closed under $<\kappa$-sequences), it follows that N' believes that $j(\vec{D}) \cap \mathrm{V}_\kappa$ is a sequence of predense subsets of $\mathbb{Q}_{<\kappa}$ and that $S_{j(a_0),j(\vec{D})\cap\mathrm{V}_\kappa,\kappa}$ is a stationary subset of $[\mathrm{V}_{\kappa+1}]^\omega$: Let $\mathfrak{A}$ be an algebra on $\mathrm{V}^{N'}_{\kappa+1}$ such that whenever a subset of $\mathrm{V}^{N'}_{\kappa+1}$ is $\mathfrak{A}$-closed, it is an elementary submodel of $\mathrm{V}^{N'}_{\kappa+1}$. Since $S_{a_0,\vec{D},\kappa}$ is stationary in V, there is an $X \prec \mathrm{V}_{\kappa+1}$ which is closed under $\mathfrak{A}$ such that $X \in S_{a_0,\vec{D},\kappa}$, in particular, X is countable. So $X' := X \cap N' \in N'$, by the closedness of N', and X' is countable in N'. X' is $\mathfrak{A}$-closed, so that $X' \prec \mathrm{V}^{N'}_{\kappa+1}$, and since $\mathrm{V}_\kappa = \mathrm{V}^N_\kappa = \mathrm{V}^{N'}_\kappa$, it follows that $X' \cap \mathrm{V}^{N'}_\kappa = X \cap \mathrm{V}_\kappa$. In particular, $X' \cap (\cup a_0) \in a_0 = j(a_0)$ and X' captures $D_\xi = j(D_\xi) \cap \mathrm{V}_\kappa$ whenever $\xi \in X' \cap \eta$, and hence also $j(D_\xi)$. Of course, it is also in N' the case that D_α (which is the same as $j(\vec{D})_\alpha \cap \mathrm{V}^{N'}_\kappa$) is predense in $\mathbb{Q}_{<\kappa}$. This shows that $X' \in S_{j(a_0),j(\vec{D})\cap\mathrm{V}_\kappa,\kappa}$. Moreover, $\kappa \in j(C)$, as $j(C) \cap \kappa = C$, so that κ is a limit point of $j(C)$ and hence a member of $j(C)$, as $j(C)$ is closed in $j(\kappa)$.

[8] I wrote $\vec{D} \cap \mathrm{V}_\gamma$ here for the sequence $\langle D_\alpha \cap \mathrm{V}_\gamma \mid \alpha < \eta \rangle$.

So this means that N' believes that there is an inaccessible $\gamma < j(\kappa)$ such that $\gamma \in j(C)$, $j(\vec{D}) \cap \mathrm{V}_\gamma$ is a sequence of predense subsets of $\mathbb{Q}_{<\gamma}$ and $S_{a_0, j(\vec{D})\cap \mathrm{V}_\gamma, \gamma}$ is a stationary subset of $[\mathrm{V}_{\gamma+1}]^\omega$. Pulling this statement back to N shows that there is a $\gamma \in C$ which satisfies (1) there, and since $\mathrm{V}^N_\kappa = \mathrm{V}_\kappa$, it holds in V as well.

Using (1), the lemma can now be proved using the original argument, so I only sketch how to argue. Let a_0 be a condition in $\mathbb{Q}_{<\kappa}$ forcing that τ is a name for an η-sequence of ordinals in the generic tower ultrapower of V, $\eta < \kappa$. It has to be shown that there is a stronger condition a and a function $f \in \mathrm{V}^{\Gamma_a}$ which is forced by a to represent τ. For $\alpha < \eta$, let A_α be a maximal antichain of conditions p in $\mathbb{Q}_{<\kappa}$ such that $p \Vdash \tau(\check{\alpha}) = [\check{f}]_\Gamma$, where Γ is the canonical name for the generic filter. By (1), there is a $\gamma < \kappa$ which is inaccessible, such that $a_0 \in \mathrm{V}_\gamma$, $\eta < \gamma$ and $A_\alpha \cap \mathbb{Q}_{<\gamma}$ is predense and semi-proper, and $a := S_{a_0, \vec{D}\cap \mathrm{V}_\gamma, \gamma}$ is stationary in $\mathrm{V}_{\gamma+1}$. It follows that a extends a_0. The function which will be forced to represent τ is defined as follows. Given $X \in a$ and $\alpha \in X \cap \eta$, there is a unique $b_X \in X \cap A_\alpha \cap \mathrm{V}_\gamma$ such that $X \cap (\cup b_X) \in b_X$. Since $b_X \in A_\alpha$, there is a function $f_{(b_X,\alpha)} : b \longrightarrow \mathrm{On}$ such that $b_X \Vdash [\check{f}_{(b_X,\alpha)}]_\Gamma = \tau(\check{\alpha})$. Defining $f(X)(\alpha) = f_{(b_X,\alpha)}(X \cap (\cup b))$ will do.

□

Theorem 4.7. *Assume that there is a $<\kappa$-closed-generic* V*-normal measure on $\mathcal{P}_\kappa(2^\kappa)$. Then the $\mathbb{Q}_{<\kappa}$-generic ultrapower is $<\kappa$-closed in the $\mathbb{Q}_{<\kappa}$-generic extension. In particular, this is true if κ is weakly compact with* $\mathsf{ID}(\kappa) > \Omega(2^\kappa)$.

Proof. Note that by Lemma 2.8, κ is weakly compact. The theorem follows by putting Lemma 4.6 and Lemma 4.5 together. □

Recall the notion of a countably completely ω_1-Jónsson cardinal introduced in Definition 3.17. The following is the *raison d'être* of this notion. The proof of [Lar04, Theorem 2.7.7] goes through.

Theorem 4.8. *If κ is a limit of completely ω_1-Jónsson cardinals and j is the generic embedding added by $\mathbb{Q}_{<\kappa}$, then $j(\omega_1^{\mathrm{V}}) = \kappa$. This is the case, in particular, if κ is indestructibly weakly compact, by Theorem 3.23 and Observation 3.18.*

Proof. It is a general fact that forcing with $\mathbb{Q}_{<\kappa}$ collapses every ordinal less than κ to be countable in the generic ultrapower. So it is enough to show that $j(\omega_1) \le \kappa$. Given $a \in \mathbb{Q}_{<\kappa}$ and $f : a \longrightarrow \omega_1$, I want to find $b \le a$ in $\mathbb{Q}_{<\kappa}$ and a $\gamma \in \cup b$ such that b forces that $[f] \le \gamma$. This will be the case if $f(X \cap (\cup a)) \le \mathrm{otp}(X \cap \gamma)$ for all $X \in b$, since $\gamma = [X \mapsto \mathrm{otp}(X \cap \gamma)]$. Now let $\gamma < \kappa$ be a completely ω_1-Jónsson cardinal with $a \in \mathrm{V}_\gamma$. Let b be the set of countable $X \subseteq \mathrm{V}_\gamma$ with $a \in X$, $X \cap (\cup a) \in a$ and $f(X \cap (\cup a)) \le \mathrm{otp}(X \cap \gamma)$. The point is that b is stationary in $[\mathrm{V}_\gamma]^\omega$: Let $H : [\mathrm{V}_\gamma]^\omega \longrightarrow \mathrm{V}_\gamma$. Since γ is completely ω_1-Jónsson, there is a $Y \subseteq \mathrm{V}_\gamma$ closed under H, with $a \in Y$, $Y \cap (\cup a) \in a$ and $\overline{\overline{Y \cap \gamma}} \ge \omega_1$. Let z be a countable subset of $Y \cap \gamma$ of order type greater than $f(Y \cap (\cup a)) < \omega_1$. Let z' be the closure of $(Y \cap (\cup a)) \cup z$ under H. Then as z' is countable, z' is in b, showing that b is stationary. By design, $b \leq a$, so the proof is complete. $\square$

Definition 4.9. Let κ be inaccessible. Then $C^\omega(\kappa)$ is the set of all countable $X \prec \mathrm{V}_{\kappa+1}$ that capture every predense $D \subseteq \mathbb{Q}_{<\kappa}$ with $D \in X$.

The importance of $C^\omega(\kappa)$ is the following Lemma, which is a slight reformulation of [Lar04, Lemma 2.7.14]:

Lemma 4.10. *Let $\delta_1 < \delta_2$, both inaccessible cardinals. If $C^\omega(\delta_1)$ is stationary, then $C^\omega(\delta_1)$ forces in $\mathbb{Q}_{<\delta_2}$ that $\Gamma \cap \check{\mathbb{Q}}_{<\delta_1}$ is $\check{\mathbb{Q}}_{<\delta_1}$-generic over $\check{\mathrm{V}}$, where Γ is a canonical name for the $\mathbb{Q}_{<\delta_2}$-generic.*

Proof. Let $G \subseteq \mathbb{Q}_{<\delta_2}$ be generic with $C^\omega(\delta_1) \in G$. Let $D \subseteq \mathbb{Q}_{<\delta_1}$ be predense. It has to be shown that $G \cap D \neq \emptyset$. Let $a_D = \{X \in [\mathrm{V}_{\delta_1+1}]^\omega \mid D \in X\}$. Then a_D is club in $[\mathrm{V}_{\delta_1+1}]^\omega$. Since $C^\omega(\delta_1)$ is stationary in $[\mathrm{V}_{\delta_1+1}]^\omega$ and is in G, it follows by a standard forcing argument that $a_D \cap C^\omega(\delta_1) \in G$. Let $F_D : a_D \cap C^\omega(\delta_1) \longrightarrow D$ be such that $F_D(X)$ witnesses that X captures D. So $X \cap (\cup F_D(X)) \in F_D(X)$ and $F_D(X) \in X$. By normality and genericity, there is some $d \in D$ such that $a_{D,d}$, the set of all $X \in a_D \cap C^\omega(\delta_1)$ with $F_D(X) = d$, is in G and $\cup a_{D,d} = \mathrm{V}_{\delta_1+1}$. But then $d \geq a_{D,d} \in G$, so that $d \in G$. $\square$

The following is a slight modification of Corollary 2.7.12, according to Remark 2.7.13, both of [Lar04]:

Lemma 4.11. *Assume that δ is inaccessible and every predense subset of $\mathbb{Q}_{<\delta}$ is semi-proper. Let ζ be an ordinal and κ a limit ordinal with $\zeta < \delta < \kappa$. Let $Y \prec \mathrm{V}_\kappa$ be countable with ζ, $\delta \in Y$, and either $Y \cap \mathrm{On}$ is cofinal in V_κ, or $\mathrm{cf}(\kappa) > \delta$. Then there exists a countable $Y' \prec \mathrm{V}_\kappa$ such that*

1. *$Y \subseteq Y'$,*
2. *$Y' \cap \mathrm{V}_\zeta = Y \cap \mathrm{V}_\zeta$,*
3. *$Y' \cap \mathrm{V}_{\delta+1} \in C^\omega(\delta)$.*

Proof. One builds an ω-chain of elementary submodels of V_κ, such that the $n+1$st one captures a predense subset in the nth model, using a straightforward bookkeeping device to ensure that in the end all predense subsets in the union of the models are captured. In the induction step one can use Lemma 2.5.4 of [Lar04]. $\square$

Lemma 4.12. *If κ is weakly compact and every predense subset of $\mathbb{Q}_{<\kappa}$ is semi-proper, and C is club in κ, then the set*

$$\{C^\omega(\delta) \mid \delta \in C,\ \delta \text{ is inaccessible and } C^\omega(\delta) \text{ is stationary}\}$$

is predense in $\mathbb{Q}_{<\kappa}$.

Proof. Let $p \in \mathbb{Q}_{<\kappa}$ be given, w.l.o.g. $\cup p = \mathrm{V}_\zeta$, for some $\zeta < \kappa$. Let N be a transitive model of ZFC – (Replacement) that's closed under $<\kappa$-sequences (so that $\mathrm{V}_\kappa \subseteq N$), is of size κ, and that has $C \in N$. Let $j : N \longrightarrow N'$ be a weakly compact embedding which is an ultrapower of N by some N-normal ultrafilter. Then N' is also closed under $<\kappa$-sequences.

The point is now: $N' \models C^\omega(\kappa)$ is stationary in $[\mathrm{V}_{\kappa+1}]^\omega$ and compatible with p. To see this, let $F \in N'$, $F : [\mathrm{V}^{N'}_{\kappa+1}]^{<\omega} \longrightarrow \mathrm{V}^{N'}_{\kappa+1}$ such that, if $Y \subseteq \mathrm{V}^{N'}_{\kappa+1}$ is closed under F, then $Y \prec \mathrm{V}^{N'}_{\kappa+1}$. I have to show that there is some $Z \in (C^\omega(\kappa))^{N'}$ that's closed under F and that satisfies $Z \cap \mathrm{V}_\zeta \in p$. Pick λ such that $\mathrm{cf}(\lambda) > \kappa$. Pick $Y \prec \mathrm{V}_\lambda$ so that $Y \cap \mathrm{V}_\zeta \in p$, $F \in Y$ and Y is countable. By the previous Lemma, there is some $Y' \prec \mathrm{V}_\lambda$ such that $Y \subseteq Y'$, $Y \cap \mathrm{V}_\zeta = Y' \cap \mathrm{V}_\zeta$ and $Y' \cap \mathrm{V}_{\kappa+1} \in C^\omega(\kappa)$. Set $Z = Y' \cap \mathrm{V}_{\kappa+1} \cap N'$. I claim that $Z \in (C^\omega(\delta))^{N'}$.

First, note that $Z \in N'$, since this is a countable subset of N' and N' is closed even under $<\kappa$-sequences. Moreover, Z is closed under F, so that $Z \prec \mathrm{V}^{N'}_{\kappa+1}$. To see that $Z \in C^\omega(\kappa)^{N'}$, let $D \in Z$ be such that N' believes that D is predense in $\mathbb{Q}_{<\kappa}$. Then D is really predense in $\mathbb{Q}_{<\kappa}$, since $\mathrm{V}_\kappa = \mathrm{V}^{N'}_\kappa$. So since $Y' \cap \mathrm{V}_{\kappa+1} \in C^\omega(\kappa)$ (in V), there is some $d \in D \cap (Y' \cap \mathrm{V}_{\kappa+1})$ such that $\cup d \cap Y' \cap \mathrm{V}_{\kappa+1} \in d$. But then this d is in $D \cap Z$, too, and since $\cup d \cap Y' \cap \mathrm{V}_{\kappa+1} = \cup d \cap Y' \cap \mathrm{V}^{N'}_{\kappa+1} = \cup d \cap Z$, this shows that $Z \in C^\omega(\kappa)$ in N'.

So in N', the statement that there is an inaccessible $\gamma \in C$ such that $C^\omega(\gamma)$ is stationary and compatible with $p = j(p)$, is true (as witnessed by κ). So the same must be true in N of C. Since $\mathrm{V}^N_\kappa = \mathrm{V}_\kappa$, this shows that the same is true in V. □

Theorem 4.13. *If κ is weakly compact with $\mathsf{ID}(\kappa) > \Omega(2^\kappa)$, and G is $\mathbb{Q}_{<\kappa}$-generic over* V*, then the set of inaccessible $\delta < \kappa$ such that $G \cap \mathbb{Q}_{<\delta}$ is $\mathbb{Q}_{<\delta}$-generic over* V *is unbounded in κ.*

Proof. Fix $\xi < \kappa$. Then by Lemma 4.12, there is some inaccessible $\delta > \xi$ with $C^\omega(\delta) \in G$. By Lemma 4.10, this implies that $G \cap \mathbb{Q}_{<\delta}$ is $\mathbb{Q}_{<\delta}$-generic over V. □

The conclusion of the following theorem is proven in [Lar04] under the assumption of a Woodin limit of Woodin cardinals.

Theorem 4.14. *If κ is indestructibly weakly compact, then every set of reals in the Chang model is Lebesgue-measurable, has the Baire Property, the Perfect Set Property and the Ramsey property. In fact, writing $L(\mathrm{On}^\omega)$ for the Chang model, it follows that*

$$L(\mathrm{On}^\omega) \equiv L(\mathrm{On}^\omega)^{\mathrm{Col}(\omega,<\kappa)}.$$

Proof. (Sketch.) I follow the argument of [Lar04, Theorem 3.1.2]. Writing $L(\mathrm{On}^\omega)$ for the Chang model, it suffices to show that there is a filter G which is $\mathrm{Col}(\omega, <\kappa)$-generic over V and an elementary embedding $j : L(\mathrm{On}^\omega) \longrightarrow L(\mathrm{On}^\omega)^{\mathrm{V}[G]}$ in a forcing extension. Since by [Sol70] and [Mat77], every set of reals in $L(\mathrm{On}^\omega)^{\mathrm{V}[G]}$ has the desired regularity properties, it follows that the same is true in $L(\mathrm{On}^\omega)^{\mathrm{V}}$. Let H be $\mathbb{Q}^{\mathrm{V}}_{<\kappa}$-generic over V. The main claim is that there is a filter G in $\mathrm{V}[H]$ which is $\mathrm{Col}(\omega, {<}\kappa)$-generic over V, such that

$$({}^\omega\mathrm{On})^{\mathrm{V}[H]} = \bigcup\{\mathrm{On}^\omega \cap \mathrm{V}[G \restriction \alpha] \mid \alpha < \kappa\}.$$

By [Lar04, Lemma 3.1.5], it suffices to show that each $x \in ({}^\omega\mathrm{On})^{\mathrm{V}[H]}$ is V-generic for some forcing in V_κ and $\kappa = \sup\{\omega_1^{\mathrm{V}[x]} \mid x \in ({}^\omega\mathrm{On})^{\mathrm{V}[H]}\}$.

Let $j : \mathrm{V} \longrightarrow M$ be the stationary tower embedding added by H. We know M is ${<}\kappa$-closed in $\mathrm{V}[H]$ under the current assumptions by Lemma 4.6, and $j(\omega_1) = \kappa$, by Theorem 4.8. It follows from these two properties that $\kappa = \omega_1^M = \omega_1^{\mathrm{V}[H]}$, in other words, $\kappa = \sup\{\omega_1^{\mathrm{V}[x]} \mid x \in \mathrm{V}[H] \cap {}^\omega\mathrm{On}\}$, which is one of the two requirements needed to prove. For the other one, let $x \in (\mathrm{On}^\omega)^{\mathrm{V}[H]}$. Pick a sequence of antichains $\langle A_i \mid i < \omega\rangle$ in $\mathbb{Q}_{<\kappa}$ (in V) deciding the members of x. By Theorem 4.13, pick an inaccessible $\delta < \kappa$ in such a way that $A_i \cap \mathbb{Q}_{<\delta}$ is a maximal antichain in $\mathbb{Q}_{<\delta}$, for every $i < \omega$, and such that $H \cap \mathbb{Q}_{<\delta}$ is $\mathbb{Q}_{<\delta}$-generic over V. So $H \cap \mathbb{Q}_{<\delta}$ meets every antichain A_i, and hence decides the i-th member of x, so that $x \in \mathrm{V}[H \cap \mathbb{Q}_{<\delta}]$. This proves the existence of G as above. It follows that $\mathrm{V}({}^\omega\mathrm{On})^{\mathrm{V}[H]} = \mathrm{V}({}^\omega\mathrm{On})^{\mathrm{V}[G]}$, since κ is regular, so that $j \restriction L({}^\omega\mathrm{On})^{\mathrm{V}} : L({}^\omega\mathrm{On})^{\mathrm{V}} \longrightarrow_{\Sigma_\omega} L({}^\omega\mathrm{On})^{\mathrm{V}[G]}$. $\square$

I would like to close the paper with the following question.

Question 4.15. If κ is indestructibly weakly compact, does it follow that the theory of the Chang model is invariant under set forcing in V_κ?

The reason why the usual proof from a Woodin limit of Woodin cardinals does not go through is that the indestructibility of κ is destructible by small forcing. See [Ham98] for more on this phenomenon.

It would be a promising line of research to investigate the effect of indestructible weak compactness on certain internally approachable towers. I leave an elaboration of these matters for a future project. There is a wide range of possibilities for using indestructible weak compactness in supercompactness arguments.

References

[AH99] Arthur W. Apter and Joel David Hamkins. Universal indestructibility. *Kobe Journal of Mathematics*, 16(2):119–130, 1999.

[AH01] Arthur W. Apter and Joel David Hamkins. Indestructible weakly compact cardinals and the necessity of supercompactness for certain proof schemata. *Mathematical Logic Quarterly*, 47(4):563–571, 2001.

[Apt05] Arthur Apter. Removing laver functions from supercompactness arguments. *Mathematical Logic Quarterly*, 51:154–156, 2005.

[FMS88] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and non-regular ultrafilters. Part I. *Annals of Mathematics*, 127(1):1–47, 1988.

[For] Matthew Foreman. *Handbook of Set Theory*, chapter Ideals and Generic Elementary Embeddings. Springer.

[Fuc] Gunter Fuchs. Combined maximality principles up to large cardinals. pages 1–40. Accepted for publication in the Journal of Symbolic Logic.

[Fuc08] Gunter Fuchs. Closed maximality principles: Implications, separations and combinations. *Journal of Symbolic Logic*, 73(01):276–308, March 2008.

[Ham] Joel David Hamkins. A class of strong diamond principles. arXiv:math.LO/0211419.

[Ham98] Joel David Hamkins. Small forcing makes any cardinal superdestructible. *Journal of Symbolic Logic*, 63(1):51–58, 1998.

[Jec03] Thomas Jech. *Set Theory. The Third Millenium Edition*. Springer monographs in mathematics. Springer, Berlin, 2003.

[JSSS07] Ronald B. Jensen, Ernest Schimmerling, Ralf Schindler, and John R. Steel. Stacking mice. 2007. In preparation.

[Kan03] Akihiro Kanamori. *The Higher Infinite*. Springer Monographs in Mathematics. Springer, 2 edition, 2003.

[Kun80] Kenneth Kunen. *Set Theory. An Introduction To Independence Proofs.* North Holland, 1980.

[Lar04] Paul Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin*, volume 32 of *University Lecture Notes*. American Mathematical Society, 2004.

[Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.

[Mat77] Adrian R. D. Mathias. Happy families. *Annals of Mathematical Logic*, 12:59–111, 1977.

[She98] Saharon Shelah. *Proper and Improper Forcing.* Perspectives in Mathematical Logic. Springer, 1998.

[Sol70] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics. Second Series*, 92:1–56, 1970.

[Vel92] Boban Veličković. Forcing axioms and stationary sets. *Advances in Mathematics*, 94(2):256–284, August 1992.