Knots, graphs and Khovanov homology II

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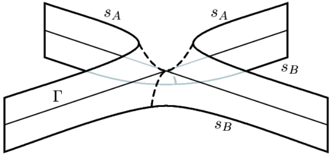
Outline

Turaev surface

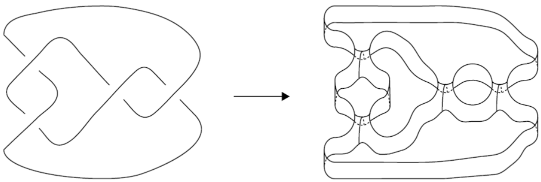
Let D be a link diagram and let s_A and s_B be the all–A and all–B states of D.

Turaev constructed a cobordism between s_A and s_B:

Let $\Gamma \subset S^2$ be the 4–valent projection of D at height 0.
Put s_A at height 1, and s_B at height -1, joined by saddles:



Turaev surface



Turaev surface $F(D)$: Attach $|s_A| + |s_B|$ discs to all boundary circles above.

Turaev genus of D, $g_T(D) := g(F) = (c(D) + 2 - |s_A| - |s_B|)/2$.

Turaev genus of non-split link $L := g_T(L) = \min_D \; g_T(D)$.

Properties

- Non-split link L is alternating iff $g_T(L) = 0$.
- D is alternating on the Turaev surface.
- $g_T(L) \leq dalt(L) =$ min number of crossing changes to make L alternating.
- Measures "distance" from alternating.
- Turaev surface can be constructed for any two complementary states of D.

Turaev's Proof of Tait's conjecture

Conjecture (Tait) A reduced alternating diagram D has minimal crossing number among all diagrams for the alternating link L.

The proof follows from three claims:

- Although defined for diagrams, the Jones polynomial $V_L(t)$ is a link invariant.
- s_A and s_B contribute the extreme terms $\pm t^\alpha$ and $\pm t^\beta$ of $V_L(t)$.
- span $V_\ell(t) = \alpha - \beta \leq c(\ell) - g_T(\ell)$, with equality if ℓ is alternating (generally, *adequate*).

Graphs on surfaces

Since D is alternating on the Turaev surface, we can generalize the Tait graph construction to get graphs on surfaces.

The Turaev surface $F(D)$ can be checkerboard colored with $|s_A|$ white regions (height > 0), and $|s_B|$ black regions (height < 0).

Let G_A, $G_B \subset F(D)$ be the adjacency graphs for respective regions. G_A and G_B are embedded and

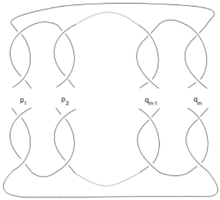
$$v(G_A) = |s_A|, \quad e(G_A) = c(D), \quad f(G_A) = |s_B|$$

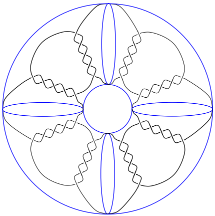
G_A (and G_B) give a cell decomposition of $F(D)$.

If D is alternating, G_A and G_B are dual Tait graphs on $F(D) = S^2$.

Example: Pretzel links

Let $p_i, q_j \in \mathbb{N}$. The pretzel link $P(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$, is a link with diagram of the form



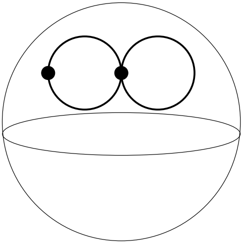


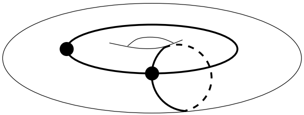
- If $m = 0$, then the pretzel link is alternating and $g_T = 0$.
- If $m > 0$ then the pretzel link is non-alternating and can be embedded on the torus. Hence $g_T = 1$.

Ribbon graphs

An (oriented) ribbon graph G is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface F, such that its complement is a union of 2-cells. The genus $g(G) := g(F)$.

Example





Algebraic definition

G can also be described by a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, 2, \ldots, 2n\}$ such that

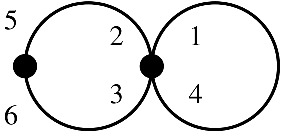
- σ_1 is a fixed-point-free involution.
- $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \mathrm{Identity}$

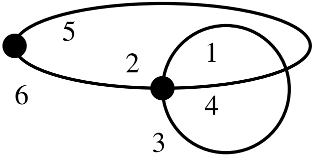
This triple gives a cell complex structure for the surface of G such that

- Orbits of σ_0 are vertices.
- Orbits of σ_1 are edges.
- Orbits of σ_2 are faces.

The genus $g(G) = (2 - v(G) + e(G) - f(G))/2$.

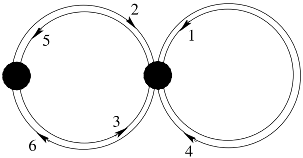
Ribbon graph example

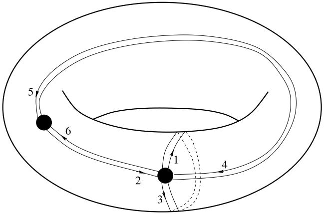




$$\sigma_0 = (1234)(56)$$
$$\sigma_1 = (14)(25)(36)$$
$$\sigma_2 = (1)(246)(35)$$

$$\sigma_0 = (1234)(56)$$
$$\sigma_1 = (13)(26)(45)$$
$$\sigma_2 = (152364)$$

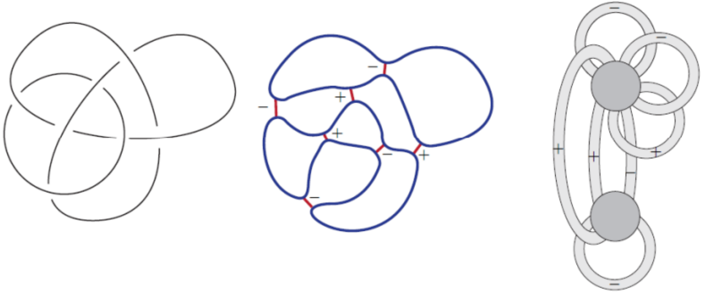




Ribbon graph from any state of a link diagram

G_A, G_B defined earlier as checkerboard graphs on Turaev surface $F(D)$ were ribbon graphs. We can construct the ribbon graph G_s directly from any state s of D:

1. For each crossing of D, attach an edge between state circle(s).
2. Collapse each state circle of s to a vertex of G_s.

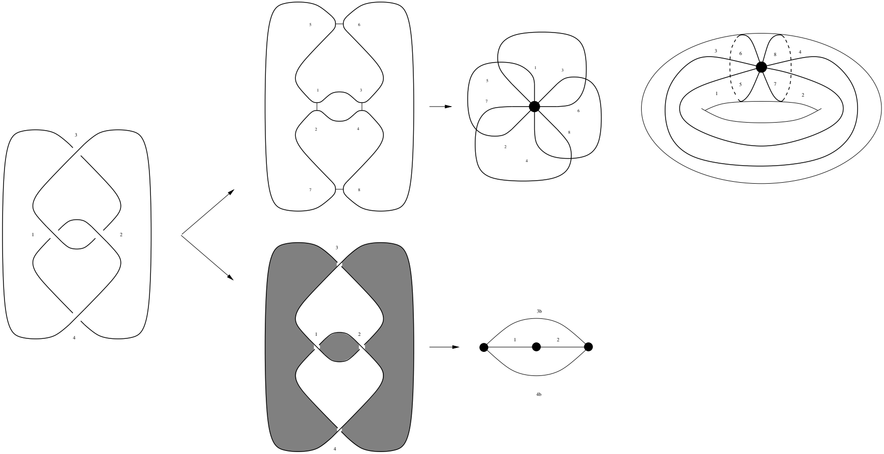


Ribbon graph polynomials

1. (2001) Bollobás and Riordan extended the Tutte polynomial to an invariant of oriented ribbon graphs, Bollobás–Riordan–Tutte polynomial.

2. (2006) Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that $V_L(t)$ can be recovered from BRT polynomial of G_A.

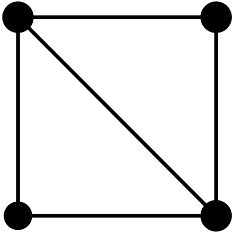
Example: From diagram to Tait graph and ribbon graph

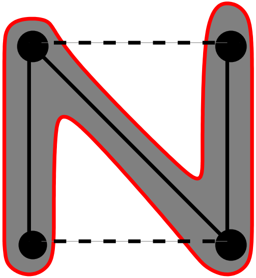


From spanning trees to quasi-trees

For a planar graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example

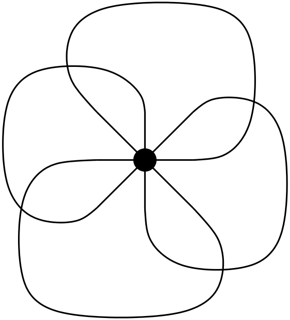


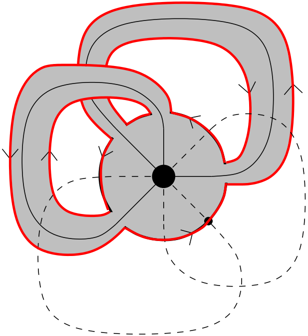


Quasi-trees

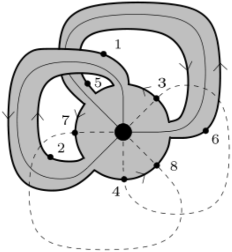
A quasi-tree of a ribbon graph is a spanning ribbon subgraph with one face. The genus of a quasi-tree is its genus as a ribbon graph.

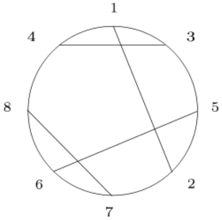
Example: Genus one quasi-tree of a genus two ribbon graph.





Every quasi-tree corresponds to an ordered chord diagram,





Spanning trees and Quasi-trees

Let D be a connected link diagram, G its Tait graph, G_A its all–A ribbon graph.

Theorem. (C-Kofman-Stoltzfus) Quasi-trees of G_A are in one-one correspondence with spanning trees of G:

$$\mathbb{Q}_j \leftrightarrow T_v \qquad \text{where} \qquad v + j = (V(G) + E_+(G) - V(G_A))/2$$

$\mathbb{Q}_j$ is quasi-tree of genus j, and T_v is spanning tree with v positive edges.

Graphs and Ribbon graphs

Graphs	$\longleftrightarrow$	Ribbon graphs
Spanning trees	$\longleftrightarrow$	Quasi-trees
Tutte polynomial	$\longleftrightarrow$	BRT polynomial
Activity w.r.t. spanning trees	$\longleftrightarrow$	Activity w.r.t. quasi-trees
Spanning tree expansion of the Tutte polynomial	$\longleftrightarrow$	Quasi-tree expansion of the BRT polynomial

Quasi-trees and Khovanov homology

Theorem (C-Kofman-Stoltzfus) For a knot diagram D, there exists a quasi-tree complex $\mathbb{C}(G_A) = \{\mathbb{C}^u_v(G_A), \partial\}$ that is a deformation retract of the reduced Khovanov complex, where

$$\mathbb{C}^u_v(G_A) = \mathbb{Z}\langle \mathbb{Q} \subset G_A |\ u(\mathbb{Q}) = u,\ -g(\mathbb{Q}) = v \rangle.$$

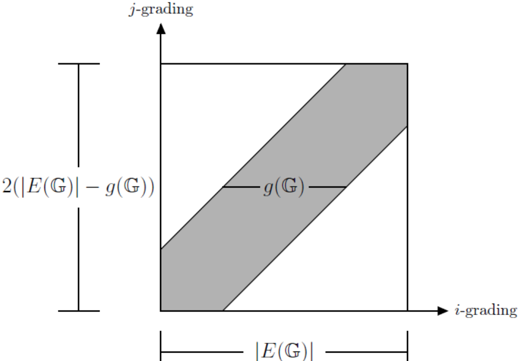
From above, if $\mathbb{Q}_j$ is quasi-tree of genus j, and T_v is spanning tree with v positive edges,

$$v + j = (V(G) + E_+(G) - V(G_A))/2$$

Turaev genus and homological width

Corollary (C-Kofman-Stoltzfus) For any knot K, the width of its reduced Khovanov homology $w_{KH}(K) \leq 1 + g_T(K)$.

Proof. For any ribbon graph G, $g(G) = \max_{\mathbb{Q} \subset G} g(\mathbb{Q})$. Therefore, the quasi-tree complex $\mathbb{C}(G_A)$ has at most $1 + g(G_A)$ rows.



Turaev genus and homological width

- Using $w_{KH}(K)$, we get lower bounds for $g_T(K)$. In particular, $g_T(T(3,q)) \underset{q\to\infty}{\longrightarrow} \infty$.

- For an adequate knot K with an adequate diagram D, T. Abe showed $g_T(K) = g_T(D) = w_{KH}(K) - 1 = c(K) - \mathrm{span}\, V_K(t)$.

- Dasbach and Lowrance also proved bounds in terms of $g_T(K)$ for the Ozsváth-Szabó τ invariant and the Rasmussen's s invariant.

- Similar bounds for homological width of knot Floer homology in terms of $g_T(K)$ were obtained by Adam Lowrance.

Related open problems

1. Find families of homologically thin knots with $g_T(K) > 1$? Generally, are there any lower bounds independent of knot homology?

2. Which operations on knots preserve or increase Turaev genus? For e.g. for adequate knots $g_T(K \# K') = g_T(K) + g_T(K')$ and g_T is preserved under mutation. How about non-adequate knots?

3. How is the Turaev genus related to the topology and hyperbolic geometry of knot complements ?

Computer programs to study Knots & Links

- SnapPy (study hyperbolic knots, links and 3-manifolds) by Weeks, Culler and Dunfield.
- Knotscape (old program to study knots) by Thistlethwaite.
- knot by Kodama.
- LinKnot by Jablan and Sazdanovic.
- KnotTheory by Dror Bar-Natan.
- KhoHo by Shumakovitch to compute Khovanov Homology.
- KnotAtlas (database) by Dror Bar-Natan.
- Table of Knot Invariants (database) by Livingston.

References

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Questions

Thank You

Slides available from :

`http://www.math.csi.cuny.edu/abhijit/`